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Global Existence and Asymptotic Behavior of Solutions for the Klein-Gordon Equations with Quadratic Nonlinearity in Two Space Dimensions

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1 Introduction and Results

We consider the global existence and asymptotic behavior of solutions for the Cauchy problem of the quadratically nonlinear Klein-Gordon equations in two space dimensions:

$$\partial_t^2 u - \Delta u + u = F(u, \partial_t u, \nabla u), \quad t > 0, \quad x \in \mathbf{R}^2, \tag{1.1}$$

$$u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \quad x \in \mathbf{R}^2, \tag{1.2}$$

where $\partial_t = \partial/\partial t$, $F(u,v,p) \in C^\infty(\mathbf{R} \times \mathbf{R} \times \mathbf{R}^2)$ and

$$F(u,v,p) = O(|u|^2 + |v|^2 + |p|^2) \quad \text{near} \quad (u,v,p) = (0,0,0). \tag{1.3}$$

We state the results concerning the global existence of solutions to (1.1)–(1.3) for small intial data, which have recently been obtained by the authors in [18].

There are many papers concerning the global existence and the asymptotic behavior of solutions for nonlinear Klein-Gordon equations (see, e.g., [1], [5]-[15]). Let N be the spatial dimensions. When $N \geq 5$, Klainerman and Ponce [9] and Shatah [11] showed that problem (1.1)-(1.2) has the unique global solution under (1.3) for small initial data and that the solution asymptotically approaches the free solution of the linear Klein-Gordon equation as $t \to \infty$. The proofs in [9] and [11] are based on the usual $L^p - L^q$ estimate of the linear Klein-Gordon equation. When $N \leq 4$ and F is quadratic, however, the usual $L^p - L^q$

estimate does not provide us with a sufficient time decay estimate. To overcome this difficulty, Klainerman [8] and Shatah [12] separately developed two new techniques. In [8] Klainerman uses the invariant Sobolev space with respect to the generators of the Lorentz group in order to prove the global existence of solution of (1.1)-(1.2) under (1.3) for small initial data, when $N = 3, 4$. Recently, Hörmander [7], Sideris [13] and Georgiev [4] have refined Klainerman's technique to show new time decay estimates of solution for the linear Klein-Gordon equation by combining the generators of the Lorentz group and the estimate of the fundamental solution of the linear Klein-Gordon equation. On the other hand, in [12] Shatah extends Poincaré's theory of normal forms for the ordinary differential equations to the case of nonlinear Klein-Gordon equations and proves the global existence of solution of (1.1)-(1.2) under (1.3) for small initial data, when $N = 3, 4$ (see also Simon [16] and Simon and Taflin [14,17], where they give a different transformation cancelling out quadratic terms). It is easily verified that the solution of (1.1)-(1.2) constructed in [8] and [12] approaches the free solution as $t \to \infty$. When $N = 2$ and the initial data are small, Georgiev and Popivanov [5] and Kosecki [10] prove the global existence of solution of (1.1)-(1.2) for a certain special form of quadratic nonlinearity by using Klainerman's technique and by combining the techniques of Klainerman and Shatah, respectively. In contrast to the papers [8], [9], [11] and [12], however, it seems unlikely to follow immediately from the proofs of [5] and [10] that the solution of (1.1)-(1.2) given by [5] and [10] approaches the free solution as $t \to \infty$. Recently, in [14] Simon and Taflin have shown that when $N = 2$ and F satisfies (1.3), for small initial data (1.1)-(1.2) has a unique global solution, which approaches the free solution as $t \to \infty$. The results obtained in [14] seem fairly satisfactory, as long as we consider solving (1.1)-(1.2) around the zero solution under (1.3). The proof in [14] consists of the construction of the wave operators and their asymptotic completeness in a neighborhood of zero. This leads to the global solvability of (1.1)-(1.2) under (1.3) for small initial data. However, the proof in [14] seems indirect, as far as the Cauchy problem with the initial data given at $t = 0$ is concerned. In fact, the proof in [14] seems rather involved even when restricted to the case of quadratic polynomial nonlinearity covariant under the Poincaré transformations. Actually the proof is described only for that special case and is briefly indicated for the general case. It is, therefore, of great interest to give a simple and direct proof without restrictions on the nonlinearity. In this note, by combining the method of normal forms due to Shatah [12] and the decay estimate of the linear Klein-Gordon equation due to Georgiev [4], we prove that when $N = 2$ and F satisfies (1.3), for small initial data there exists the unique global solution of (1.1)-(1.2). Our proof seems simpler than that in [14] and is based on the familiar arguments in nonlinear wave equations. Moreover, our proof naturally implies that the above solution of (1.1)-(1.2) asymptotically approaches the free solution as $t \to \infty$.

Before we state the main results in the present note, we give several notations. We put $\partial_j = \partial/\partial x_j$ for $j = 1, 2$. Let $\Gamma = (\Gamma_j; j = 1, \cdots, 6)$ denote the generators of the Poincaré group $(\partial_t, \partial_1, \partial_2, L_1, L_2, \Omega_{12})$, where

$$\begin{aligned} L_j &= x_j\partial_t + t\partial_j, \quad j = 1, 2, \\ \Omega_{12} &= x_1\partial_2 - x_2\partial_1, \end{aligned}$$

and we put

$$\partial = (\partial_t, \partial_1, \partial_2).$$

For a multi-index $\alpha = (\alpha_1, \alpha_2)$, we put

$$\partial_x^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}.$$

For a multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, we put

$$\partial^\alpha = \partial_t^{\alpha_1}\partial_1^{\alpha_2}\partial_2^{\alpha_3}.$$

For a multi-index $\alpha = (\alpha_1, \cdots, \alpha_6)$, we put

$$\Gamma^\alpha = \Gamma_1^{\alpha_1}\cdots\Gamma_6^{\alpha_6}.$$

For $1 \leq p \leq \infty$, let L^p denote the standard L^p space on $\mathbf{R}^2$. For $m \geq 0$ and $s \geq 0$, we define the weighted Sobolev space $H^{m,s}$ on $\mathbf{R}^2$ as follows:

$$H^{m,s} = \{v \in L^2; (1+|x|^2)^{s/2}(1-\Delta)^{m/2}v \in L^2\}$$

with the norm

$$\|v\|_{H^{m,s}} = \|(1+|x|^2)^{s/2}(1-\Delta)^{m/2}v\|_{L^2}.$$

We put $H^m \equiv H^{m,0}$ for $m \geq 0$. Let $\omega = (1-\Delta)^{1/2}$.

We have the following theorem concerning the global existence and asymptotic behavior of solutions for (1.1)-(1.2).

Theorem 1.1 *Assume that F satisfies (1.3). Let $k \geq 21$ and let $u_0 \in H^{k+16,k+15}$, $u_1 \in H^{k+15,k+15}$. Let $0 < \varepsilon \leq 1/2$. There exists a $\delta > 0$ such that if*

$$\|u_0\|_{H^{k+16,k+15}} + \|u_1\|_{H^{k+15,k+15}} \leq \delta, \tag{1.4}$$

then (1.1)-(1.2) has the unique global solution u satisfying

$$u \in \bigcap_{j=0}^{k+16} C^j([0,\infty); H^{k+16-j}), \tag{1.5}$$

$$\begin{aligned} &\sum_{|\alpha|=k+15} \sup_{t\geq 0}(1+t)^{-\varepsilon}\{\|\partial_t\Gamma^\alpha u(t)\|_{L^2} + \|\omega\Gamma^\alpha u(t)\|_{L^2}\} \\ &+ \sum_{|\alpha|\leq k+15} \sup_{t\geq 0}(1+t)^{-\varepsilon}\|\Gamma^\alpha u(t)\|_{L^2} \\ &+ \sum_{|\alpha|\leq k+10} \sup_{t\geq 0}\{\|\partial_t\Gamma^\alpha u(t)\|_{L^2} + \|\omega\Gamma^\alpha u(t)\|_{L^2}\} \\ &+ \sum_{|\alpha|\leq k} \sup_{\substack{t\geq 0 \\ x\in\mathbf{R}^2}} |(1+t+|x|)\Gamma^\alpha u(t,x)| < \infty. \end{aligned} \tag{1.6}$$

Furthermore, the above solution u has a free profile $(u_{+0}, u_{+1}) \in H^{k+10} \oplus H^{k+9}$ such that

$$\sum_{j=0}^{1} \|\partial_t^j\{u(t) - u_+(t)\}\|_{H^{k+10-j}} \longrightarrow 0 \tag{1.7}$$

as $t \to \infty$, where

$$u_+(t) = (\cos\omega t)u_{+0} + (\omega^{-1}\sin\omega t)u_{+1}.$$

Remark 1.1 The function u_+ in Theorem 1.1 is a free solution of the linear Klein-Gordon equation

$$\partial_t^2 u_+ - \Delta u_+ + u_+ = 0, \quad t > 0, \quad x \in \mathbf{R}^2$$

with initial condition

$$u_+(0,x) = u_{+0}(x), \quad \partial_t u_+(0,x) = u_{+1}(x), \quad x \in \mathbf{R}^2.$$

The relation (1.7) implies that the solution u of (1.1)-(1.2) given by Theorem 1.1 behaves like the free solution u_+ as $t \to \infty$.

Remark 1.2 If F is not smooth, (1.7) need not hold. In fact, for $F(u) = -|u|u$ (1.7) fails except the trivial case $u(t) = u_+(t) \equiv 0$ (see, e.g., [15]). The proof uses the positivity of the cubic term of energy functional, which is not expected in the present framework. An early contribution to the nonexistence result of this type is due to Glassey, Matsumura, and Strauss.

The following corollary follows easily from Theorem 1.1 and Proposition 3.1 in Section 3 of [18].

Corollary 1.2 *In addition to all the assumptions in Theorem 1.1, if $u_0 \in \bigcap_{m\geq 1} H^m$ and $u_1 \in \bigcap_{m\geq 1} H^m$, then the solution u given by Theorem 1.1 belongs to $C^\infty([0,\infty) \times \mathbf{R}^2)$.*

The unique existence of local solutions for (1.1)-(1.2) follows from the standard contraction argument (see, e.g., [9] and [11]). The crucial part of proof of Theorem 1.1 is to establish a priori estimates of the solution for (1.1)-(1.2) in order to extend the local solution globally in time. The global behavior of local solution for (1.1)-(1.2) with (1.3) is out of control in a direct estimate, since the quadratic nonlinearity in (1.1) does not provide the sufficient decay for the two dimensional case in connection with the integrability in time of the norm appearing as a coefficient of the energy norm associated with the Poincaré group. Here, we use the argument of normal forms of Shatah [12] to transform the quadratic nonlinearity into the cubic one. Still, the cubic nonlinearity in the two dimensional case leads to insufficient decay as long as the proof depends exclusively on the usual $L^p - L^q$ estimate (see, e.g., [9] and [11]). We show that Klainerman's technique works on the resulting equation with cubic nonlinearity. At this stage, we employ the decay estimate of the inhomogeneous linear Klein-Gordon equation due to Georgiev [4]. The generators of the Poincaré group operate on the local interaction nonlinearity properly like a differential operator. But it is not necessarily the case with the non-local interaction nonlinearity which appears in the transformed equation through the argument of normal forms. Since the resulting cubic nonlinearity is represented in terms of the integral operators, our main task in the proof of Theorem

1.1 is to handle the commutators between the generators of the Poincaré group and the integral operators in the resulting cubic nonlinearity so that every norm is reproduced in the decay and energy estimates. The rest part of the proof of Theorem 1.1 proceeds almost in the same way as in the previous papers (see, e.g., [5], [8] and [12]). Finally we should briefly state the relation between the paper [12] by Shatah and the papers [14,16,17] by Simon and Taflin. In both [12] and [14,16,17] they use the methods to transform the original equation with quadratic nonlinearity into the new one with cubic nonlinearity. However, the transformations constructed in [12] and [14,16,17] are different.

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