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Initial and Initial-Boundary Value Problems for the Vortex Filament Equations

慶応大理工学部数理科学科 西山 高弘 (Takahiro NISHIYAMA)
谷 温之 (Atusi TANI)

1. Introduction

The system of equations

$$x_t = x_s \times x_{ss} + a\{x_{sss} + (3/2)x_{ss} \times (x_s \times x_{ss})\} \tag{1.1}$$

approximately describes the deformation of a vortex filament with or without axial velocity in its thin core, in a perfect fluid. Here $x = x(s,t)$ denotes the position of a point on the filament in $\mathbb{R}^3$ as a vector-valued function of arclength $s(\in \mathbb{R})$ and time $t(>0)$, and a real constant a represents the magnitude of the effect of the axial flow. In particular, (1.1) with $a=0$, from which the axial-flow effect is absent, is called the localized induction equation (LIE).

Since in 1906 Da Rios [1] formulated LIE, many authors have studied it from various points of view (see [9],[10] and the references therein). In [8] we proved the weak solvability of some initial and initial-boundary value problems for LIE, although the expected uniqueness and smoothness of the solution were not found. On the other hand, (1.1) with $a \neq 0$ were originally derived by Fukumoto and Miyazaki [2] as a generalization of LIE from the Moore-Saffman equation in [7].

Differentiating (1.1) with respect to s and setting $v=x_s$, we have

$$v_t=v\times v_{ss}+a\{v_{sss}+(3/2)v_{ss}\times(v\times v_s)+(3/2)v_s\times(v\times v_{ss})\}. \qquad (1.2)$$

Impose the initial condition

$$v(s,0)=v_0(s), \quad |v_0|=1 \qquad (1.3)$$

on (1.2) for $s\in\mathbb{R}$. One of our aims in this paper is to establish the unique solvability of the initial value problem (1.2) with (1.3) in the space where the curvature of the vortex filament $|v_s|$ tends to zero as $s\to\pm\infty$, on the time interval $[0,T]$ with any $T>0$. In order to achieve it we first investigate the parabolic regularization

$$\begin{aligned} v_t=&v\times v_{ss}+a\{v_{sss}+(3/2)v_{ss}\times(v\times v_s)+(3/2)v_s\times(v\times v_{ss})\} \\ &-\varepsilon\{v_{ssss}+4(v_s\cdot v_{sss})v+3|v_{ss}|^2v\} \end{aligned} \qquad (1.4)$$

for $\varepsilon>0$. After that, we let $\varepsilon\to0$. Recently, we proved the extension of T to ∞ in [12],[13].

The other aim is to obtain the unique and smooth solvability of an initial-boundary value problem for (1.2) by the above method. At this time, we treat the case $a=0$ only.

By the way, (1.1) or (1.2) can be transformed into the Hirota equation (or the nonlinear Schrödinger equation if $a=0$),

$$i\Psi_t+\Psi_{ss}+(1/2)|\Psi|^2\Psi-ia(\Psi_{sss}+(3/2)|\Psi|^2\Psi_s)=0 \qquad (1.5)$$

for $\Psi=\kappa(s,t)\exp(i\int_0^s\tau(s,t)ds+i\eta(t))$, where $\kappa(s,t)$ and $\tau(s,t)$ are the curvature and the torsion of the filament respectively, and $\eta(t)$ is a real function of t ([2],[3],[5]). But, as in [8], we should remark that (1.5) is always equivalent to neither (1.1) nor (1.2). In fact, if the filament has a segment where $|x_{ss}|$ vanishes and τ is indefinite, then ArgΨ is not well-defined even outside there.

We introduce the notation and a result for a linear parabolic system in section 2. Then a solution of (1.4) with (1.3) is obtained uniquely on $[0, T]$ with ε small enough in section 3. In section 4, we establish the theorem for (1.2), (1.3) and obtain a corollary on the vanishing axial flow. In section 5, an initial-boundary value problem is discussed.

The long version of this paper [11] will soon be published.

2. Preliminaries

Let us introduce the notation which we use. The letter m denotes an arbitrary nonnegative integer unless we particularly note it. The norms of vector-valued functions in $L^2(\Omega)$ and in the Sobolev space $W_2^m(\Omega)$ are denoted by $\|\cdot\|_\Omega$ and $\|\cdot\|_\Omega^{(m)}$, respectively. Then $\|\cdot\|_\Omega^{(0)} = \|\cdot\|_\Omega$. When $\Omega = \mathbb{R}$, we write the norms simply as $\|\cdot\|$ and $\|\cdot\|^{(m)}$. The set of all continuous (resp. once continuously differentiable) functions in a Hilbert space X on a finite time interval $[0, T]$ is denoted by $C(0, T; X)$ (resp. $C^1(0, T; X)$). The class of Hölder continuous X-valued functions on $[0, T]$ is written as $C^\beta(0, T; X)$, $0 < \beta < 1$. The norm $\langle\cdot\rangle_T$ (resp. $\langle\cdot\rangle_T^{(\beta)}$) represents the supremum (resp. the Hölder norm) over $[0, T]$. Positive constants, denoted by c, c_* and c_a, change from line to line but the second is independent of both ε and a, the third is monotonically increasing in $|a|$ and independent of ε. The operator D is equal to $\partial/\partial s$.

Next, consider a linear equation

$$u_t = -\varepsilon u_{ssss} + f(s,t), \tag{2.1}$$

$$u(s,0) = u_0(s) \tag{2.2}$$

for $s \in \mathbb{R}$. Then we get

Lemma 2.1. If $\varepsilon > 0$, $u_0 \in W_2^{4+m}(\mathbb{R})$ and $f \in C^\beta(0,T;W_2^m(\mathbb{R}))$ for $T>0$, $0<\beta<1$, then there exists a unique solution of (2.1), (2.2) in $C(0,T;W_2^{4+m}(\mathbb{R})) \cap C^1(0,T;W_2^m(\mathbb{R}))$. Moreover the following estimate is valid:

$$\langle \|u\|^{(4+m)} \rangle_T + \langle \|u_t\|^{(m)} \rangle_T \leqq c(\|u_0\|^{(4+m)} + \langle \|f\|^{(m)} \rangle_T^{(\beta)}), \tag{2.3}$$

where c is independent of u_0 and f.

This lemma was proved by the theory of analytic semigroups in [6].

3. Solvability of (1.4) with (1.3)

Noting that v is a tangential vector and is not square integrable over $\mathbb{R}$, we obtain

Proposition 3.1. Let $\varepsilon > 0$, $a \in \mathbb{R}$ and $v_{0s} \in W_2^{3+m}(\mathbb{R})$. Then on some time interval $[0, T_0]$, $T_0 > 0$ there exists a unique solution v of (1.4) with (1.3) such that $(v - v_0) \in C(0,T_0;W_2^{4+m}(\mathbb{R})) \cap C^1(0,T_0;W_2^m(\mathbb{R}))$.

In its proof, Lemma 2.1 and the standard iteration scheme are used.

Next, we prove the following lemma, which implies that the length of the vortex filament is conserved.

Lemma 3.1. Let υ be a solution of (1.3) and (1.4) such that $(\upsilon-\upsilon_0)\in C(0,T;W_2^{4+m}(\mathbb{R}))\cap C^1(0,T;W_2^m(\mathbb{R}))$, $T>0$. Then

$$|\upsilon|=1 \tag{3.1}$$

holds for any $(s,t)\in\mathbb{R}\times[0,T]$.

<u>Proof</u>. Define the function $h(s,t)$ by

$$h(s,t)=|\upsilon|^2-1$$

for $s\in\mathbb{R}$, $0\leqq t\leqq T$. And from (1.3) and (1.4) we obtain

$$h_t=a\{h_{sss}-3(\upsilon\cdot\upsilon_{ss})h_s+6(\upsilon_s\cdot\upsilon_{ss})h\}-\varepsilon\{h_{ssss}+8(\upsilon_s\cdot\upsilon_{sss})h+6|\upsilon_{ss}|^2h\},$$

$$h(s,0)=0.$$

For this linear system we conclude that $h=0$ is an only solution because of $\|h\|=0$ yielded by the estimate $(d/dt)\|h\|^2\leqq c\|h\|^2$, where c depends on $\langle\|\upsilon-\upsilon_0\|^{(4)}\rangle_T$ and $\|\upsilon_{0s}\|^{(3)}$. Hence (3.1) follows. □

Utilizing Lemma 3.1, we derive an a priori estimate for (1.4).

Lemma 3.2. Let υ be as in Lemma 3.1. Then there exists a positive constant ε_0 depending only on T and $\|\upsilon_{0s}\|$ such that υ for any $\varepsilon\in(0,\varepsilon_0]$ satisfies the estimate

$$\langle\|\upsilon-\upsilon_0\|^{(4+m)}\rangle_T+\langle\|\upsilon_t\|^{(m)}\rangle_T\leqq c_a, \tag{3.2}$$

where c_a depends only on $\|\upsilon_{0s}\|^{(3+m)}$, ε_0, T and $|a|$.

<u>Proof</u>. From (3.1) we have

$$\upsilon\cdot\upsilon_s=0,\quad \upsilon\cdot D^n\upsilon=-\frac{1}{2}\sum_{k=1}^{n-1}{}_nC_k\,D^k\upsilon\cdot D^{n-k}\upsilon\quad(n\geqq2). \tag{3.3}$$

It also follows from (3.1) that on the point where $|v_s|$ is nonzero the vectors v, $v_s/|v_s|$, $v\times v_s/|v_s|$ are the orthonormal ones in $\mathbb{R}^3$. Then

$$v_s\times D^n v = v_s\times\{(v\cdot D^n v)v+((v\times v_s)\cdot D^n v)v\times v_s/|v_s|^2\}$$

holds for $n\geqq 2$ and it leads to

$$v_s\times D^n v = -(v\cdot D^n v)v\times v_s+((v\times v_s)\cdot D^n v)v. \tag{3.4}$$

Clearly, (3.4) is also valid where $v_s=0$.

Multiplying (1.4) by v_{ss}, integrating over $\mathbb{R}$ and using (3.3), we obtain

$$\frac{d}{dt}\|v_s(\cdot,t)\|^2 = -2\varepsilon(\|v_{sss}\|^2+4\int_{\mathbb{R}}|v_s|^2 v_s\cdot v_{sss}ds+3\||v_s||v_{ss}|\|^2)$$

$$\leqq -\varepsilon\|v_{sss}\|^2+\varepsilon c_0\|v_s\|^{10},$$

where c_0 is a positive constant yielded by use of the multiplicative inequality and Young's. Let $r(t)$ be a solution of the scalar equation $dr/dt=\varepsilon c_0 r^5$ with $r(0)=\|v_{0s}\|^2$. Then we solve it as $r(t)=(\|v_{0s}\|^{-8}-4\varepsilon c_0 t)^{-1/4}$ when $4\varepsilon c_0 t<\|v_{0s}\|^{-8}$. Choosing ε_0 so small that

$$0<\varepsilon_0<(4c_0 T\|v_{0s}\|^8)^{-1}, \tag{3.5}$$

we have

$$\|v_s(\cdot,t)\|\leqq r(t)^{1/2}\leqq c_* \tag{3.6}$$

on $[0,T]$ for all $\varepsilon\in(0,\varepsilon_0]$.

Next, by (3.3), (3.4), (3.6), the multiplicative and Young's inequalities, we obtain

$$\frac{d}{dt}(\|v_{ss}(\cdot,t)\|^2-(5/4)\||v_s(\cdot,t)|^2\|^2)=-\int_{\mathbb{R}}(2v_{sss}\cdot v_{st}+5|v_s|^2v_s\cdot v_{st})ds$$
$$\leqq\int_{\mathbb{R}} 3\{|v_s|^2v_{ss}\cdot(v\times v_s)\}_s\, ds$$
$$+a\int_{\mathbb{R}}\{|v_s|^2|v_{ss}|^2+8(v_s\cdot v_{ss})^2-5|v_s|^2v_s\cdot v_{sss}\}_s\, ds$$
$$-2\varepsilon\|v_{ssss}\|^2+\varepsilon c_*(\|v_{ssss}\|^{5/3}+\|v_{ssss}\|^{4/3})$$
$$\leqq c_*.$$

It yields

$$\|v_{ss}(\cdot,t)\|^2\leqq\|v_{0ss}\|^2-(5/4)\||v_{0s}|^2\|^2+(5/4)\||v_s(\cdot,t)|^2\|^2+c_*t$$
$$\leqq c_*+(1/2)\|v_{ss}(\cdot,t)\|^2+c_*\|v_s(\cdot,t)\|^6+c_*t,$$

from which

$$\langle\|v_{ss}\|\rangle_T\leqq c_* \tag{3.7}$$

follows.

In the same way, by boring calculation we can verify

$$\frac{d}{dt}\{\|v_{sss}\|^2-(7/2)\||v_s||v_{ss}|\|^2-14\|v_s\cdot v_{ss}\|^2+(21/8)\||v_s|^3\|^2\}\leqq c_*,$$

which yields

$$\langle\|v_{sss}\|\rangle_T\leqq c_*. \tag{3.8}$$

Let $j=4,5,\cdots,4+m$. Then, using (3.3), (3.4) and integration by parts, we can derive $(d/dt)\|D^jv\|^2\leqq c_a\|D^jv\|^2+c_a$ if $\langle\|v_s\|^{(j-2)}\rangle_T\leqq c_a$ is given. This fact, together with (3.6), (3.7), (3.8) and Gronwall's inequality, yields $\langle\|v_s\|^{(3+m)}\rangle_T\leqq c_a$. Hence we have $\langle\|(v-v_0)_s\|^{(3+m)}\rangle_T\leqq c_a$. The estimates $\langle\|v_t\|^{(m)}\rangle_T\leqq c_a$ and $\langle\|v-v_0\|\rangle_T\leqq c_a$ are easily obtained. □

From Proposition 3.1 and Lemmas 3.1, 3.2 by the standard continuation argument we have

Theorem 3.1. Let $T>0$, $v_{0s}\in W_2^{3+m}(\mathbb{R})$ and $a\in\mathbb{R}$. Then for each $\varepsilon\in(0,\varepsilon_0]$ with ε_0 satisfying (3.5) there exists a unique solution v of (1.3), (1.4) such that $(v-v_0)\in C(0,T;W_2^{4+m}(\mathbb{R}))\cap C^1(0,T;W_2^m(\mathbb{R}))$, (3.1) and (3.2) hold.

4. Solvability of (1.2) with (1.3)

Considering the limit $\varepsilon\to 0$, we establish the following theorem. Its proof is based mainly on the method in [4, Section 3].

Theorem 4.1. Let $v_{0s}\in W_2^{3+m}(\mathbb{R})$ and $a\in\mathbb{R}$. Then there exists a unique solution v of (1.2), (1.3) such that (3.1) is satisfied, $(v-v_0)\in C(0,T;W_2^{4+m}(\mathbb{R}))\cap C^1(0,T;W_2^{1+m}(\mathbb{R}))$ if $a\neq 0$, and $(v-v_0)\in C(0,T;W_2^{4+m}(\mathbb{R}))\cap C^1(0,T;W_2^{2+m}(\mathbb{R}))$ if $a=0$ with any $T>0$.

Since we have $c_a\leqq c_*$ if $|a|\leqq 1$ is assumed, the limit $a\to 0$ can be discussed in the same way as $\varepsilon\to 0$:

Corollary. In Theorem 4.1 the difference between the solution v for $a\neq 0$ and that for $a=0$ converges to zero strongly in $W_2^1(\mathbb{R})$ and weakly in $W_2^{4+m}(\mathbb{R})$, uniformly in t as $a\to 0$.

It should be noted that our method is also applicable when $a\in\mathbb{R}$ and the spatially periodic condition $v(s,t)=v(s+1,t)$ is imposed.

5. Initial-Boundary Value Problem

In this section the domain of s is restricted to $J \equiv (-1, 1)$ and a is assumed to be equal to zero. As a boundary condition imposed on (1.2) we take

$$v_s(\pm 1, t) = 0. \tag{5.1}$$

Let V^m be the completion with respect to $\|\cdot\|_J^{(m)}$ of the space where every element g belongs to $C^\infty([-1,1])$ and satisfies $D^{2j-1}g(\pm 1)=0$ for $j=1,2,\cdots$. Then, using the theory on the initial-boundary value problem for (2.1), we prove the following theorem for (1.4) with (1.3), (5.1) and

$$v_{sss}(\pm 1, t) = 0. \tag{5.2}$$

Theorem 5.1. Let $T>0$, $v_0 \in V^{4+m}$ and $a=0$. Then for each $\varepsilon \in (0, \varepsilon_0]$ with $0<\varepsilon_0<(4c_0T\|v_{0s}\|_J^8)^{-1}$ there exists a unique solution of (1.3), (1.4), (5.1), (5.2) such that $v \in C(0, T; V^{4+m}) \cap C^1(0, T; V^m)$ and (3.1) holds. Moreover, $\langle \|v\|_J^{(4+m)} \rangle_T + \langle \|v_t\|_J^{(m)} \rangle_T \leqq c_*$ is valid, where c_* depends only on v_0, T and ε_0.

Proof. The proof is divided into two parts. One is to establish the existence of a temporally local solution. It is done as in the proof of Proposition 3.1 because the s-derivatives of any odd order for $v \times v_{ss}$, $(v_s \cdot v_{sss})v$, $|v_{ss}|^2 v$ are equal to zero at $s=\pm 1$ if $D^{2j-1}v(\pm 1, t)=0$ for $j=1,2,\cdots$. The other is to derive (3.1) and the a priori estimate in the theorem, and we do by the method in the proofs of lemmas 3.1 and 3.2. □

In the same manner as in the proof of Theorem 4.1 we establish

Theorem 5.2. Let $v_0 \in V^{4+m}$ and $a=0$. Then there exists a unique solution of (1.2),(1.3),(5.1) such that $v \in C(0,T;V^{4+m}) \cap C^1(0,T;V^{2+m})$ with any $T>0$ and (3.1) is satisfied.

Here we noted that (5.2) is formally derived from (1.2) with $a=0$, (1.3) and (5.1), irrespective of the class of v. In fact, (3.1) is formally obtained because of $v \cdot v_t = 0$, and $v_{sss} = (v_{st} - v_s \times v_{ss}) \times v - 3(v_s \cdot v_{ss}) v$ follows.

Remark. Our method is also useful to another initial-boundary value problem given by (1.2) with $a=0$ for $s>0$, (1.3) and the condition $v_s(0,t)=0$.

References

[1] L.S.Da Rios, On the motion of an unbounded fluid with a vortex filament of any shape, (in Italian), Rend.Circ.Mat.Palermo, 22 (1906), pp.117-135.

[2] Y.Fukumoto and T.Miyazaki, Three-dimensional distortions of a vortex filament with axial velocity, J.Fluid Mech., 222 (1991), pp.369-416.

[3] H.Hasimoto, A soliton on a vortex filament, J.Fluid Mech., 51 (1972), pp.477-485.

[4] T.Kato, Nonstationary flows of viscous and ideal fluids in R^3, J.Funct.Anal., 9 (1972), pp.296-305.

[5] G.L.Lamb,Jr., Solitons on moving space curves, J.Math.Phys., 18 (1977), pp.1654-1661.

[6] S. Mizohata, Theory of Partial Differential Equations, Cambridge Univ. Press, Cambridge, 1973.

[7] D. W. Moore and P. G. Saffman, The motion of a vortex filament with axial flow, Phil. Trans. Roy. Soc. London A, 272 (1972), pp. 403-429.

[8] T. Nishiyama and A. Tani, Solvability of the localized induction equation for vortex motion, Commun. Math. Phys., 162 (1994), pp. 433-445.

[9] R. L. Ricca, Rediscovery of Da Rios equations, Nature, 352 (1991) pp. 561-562.

[10] R. L. Ricca, Physical interpretation of certain invariants for vortex filament motion under LIA, Phys. Fluids A, 4 (1992), pp. 938-944.

[11] T. Nishiyama and A. Tani, Initial and initial-boundary value problems for a vortex filament with or without axial flow, SIAM J. Math. Anal. (to appear).

[12] T. Nishiyama, On the initial value problems for the vortex filament equations, preprint.

[13] A. Tani and T. Nishiyama, Solvability of equations for motion of a vortex filament with or without axial flow, preprint.