Perturbation of the Navier-Stokes flow
in an annular domain
with the non-vanishing outflow condition

MORIMOTO, Hiroko   UKAI, Seiji
Meiji University   Tokyo Institute of Technology

Abstract. The boundary value problem of the Navier-Stokes equations has been studied so far only under the vanishing outflow condition due to Leray. We consider this problem in an annular domain $D = \{x \in \mathbf{R}^2; R_1 < |x| < R_2\}$, under the boundary condition with non-vanishing outflow. In a previous paper of the first author, an exact solution is obtained for a simple boundary condition of non-vanishing outflow type: $u = \dfrac{\mu}{R_i}e_r + b_i e_\theta$ on Γ_i, $i = 1, 2$, where μ, b_1, b_2 are arbitrary constants. In this paper, we show the existence of solutions satisfying the boundary condition: $u = \{\dfrac{\mu}{R_i} + \varphi_i(\theta)\}e_r + \{b_i + \psi_i(\theta)\}e_\theta$ on Γ_i, $i = 1, 2$, where $\varphi_i(\theta), \psi_i(\theta)$ are 2π-periodic smooth function of θ, under some additional condition.
1991 Mathematics Subject Classification 35Q30, 76D05.

Let D be an annular domain in $\mathbf{R}^2$:

$$D = \{x \in \mathbf{R}^2; R_1 < |x| < R_2\},$$

where $0 < R_1 < R_2$, and Γ_i its boundary:

$$\Gamma_i = \{x \in \mathbf{R}^2; |x| = R_i\}, \quad i = 1, 2.$$

We consider the boundary value problem of the Navier-Stokes equations:

$$(1) \quad \begin{cases} -\nu\Delta u + (u \cdot \nabla)u + \dfrac{1}{\rho}\nabla p &= 0 \quad \text{in} \quad D, \\ \operatorname{div} u &= 0 \quad \text{in} \quad D, \\ u &= b \quad \text{on} \quad \partial D, \end{cases}$$

where u is the fluid velocity, p is the pressure, ν(the kinematic viscosity) and ρ (the density) are given positive constants, and the vector b is a given boundary value of the velocity.

For the general bounded domain $D \subset \mathbf{R}^n, n \geq 2$, J. Leray [6] showed the existence of the solution to this problem under the following condition:

$$(H) \quad \int_{\Gamma_i} \boldsymbol{b} \cdot \boldsymbol{n} \, ds = 0, \quad 1 \le i \le k,$$

where $\partial D = \bigcup_{i=1}^{k} \Gamma_i$, Γ_i is the connected component of ∂D and $\boldsymbol{n}$ is the unit outward normal to the boundary ∂D. The condition (H) is stronger than the condition

$$(H)_0 \quad \int_{\partial \Omega} \boldsymbol{b} \cdot \boldsymbol{n} \, ds = \sum_{i=1}^{k} \int_{\Gamma_i} \boldsymbol{b} \cdot \boldsymbol{n} \, ds = 0,$$

which is to be satisfied by the boundary value $\boldsymbol{b}$ of a solenoidal vector $\boldsymbol{u}$.

We are concerned with the problem whether does exist a solution to (1) under the non-vanishing outflow condition $(H)_0$, even if the boundary value does not satisfy the vanishing outflow condition (H) ([2], [6]). In the previous paper [7], the first author showed an exact solution to this equation in an annular domain under the boundary condition with non-vanishing outflow given by,

$$\boldsymbol{b} = \frac{\mu}{R_i} \boldsymbol{e}_r + b_i \boldsymbol{e}_\theta \quad \text{on } \Gamma_i, \quad i = 1, 2,$$

where μ, b_1, b_2 are given constants and $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ are the unit vectors in the polar coordinates representation $\{r, \theta\}$.

In this paper, we study the case where the boundary value depends on θ variable, more precisely, the vector $\boldsymbol{b}$ is given as follows:

$$(2) \quad \boldsymbol{b} = \{a_i + \varphi_i(\theta)\} \boldsymbol{e}_r + \{b_i + \psi_i(\theta)\} \boldsymbol{e}_\theta \quad \text{on } \Gamma_i, \quad i = 1, 2.$$

Remark 1. *Since the condition $(H)_0$ has to be satisfied,*

$$(A1) \quad a_1 R_1 = a_2 R_2$$

should hold. We denote this common value by μ. If $\mu \ne 0$, the condition (H) does not hold.

On the other hand, without loss of generality, we can suppose the following:

$$(A2) \quad \varphi_i(\theta), \psi_i(\theta) \text{ be } 2\pi\text{-periodic smooth function of } \theta, \text{satisfying}$$

$$\int_0^{2\pi} \varphi_i(\theta) d\theta = 0, \quad \int_0^{2\pi} \psi_i(\theta) d\theta = 0, \quad i = 1, 2.$$

Finally we put

$$\omega_i = \frac{b_i}{R_i}, \quad i = 1, 2.$$

If the absolute value $|\mu|$ of μ is small, we can show the existence of a solution to (1) (2) by the usual method (c.f. [5], [7]). We show, in the following, the existence of a solution to (1) (2) even for large $|\mu|$.

Theorem 1. *Suppose (A1), A(2) hold. If the inequality*

$$|\omega_1 - \omega_2| \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \left(\log \frac{R_2}{R_1}\right)^2 < 2\nu$$

is satisfied and $\varphi_i(\theta)$, $\psi_i(\theta)$ $(i = 1, 2)$ are sufficiently small, then there exists at least one strong solution to (1) (2) for any $\mu \in \mathbf{R}$ with at most countable exception.

Remark 2. *Let μ, ω_1, ω_2 be constants and $\varphi_i(\theta) = \psi_i(\theta) = 0 \, (i = 1, 2)$. Then, we have the following exact solution $\boldsymbol{u}_0$ to (1) (2) of the form:*

$$\boldsymbol{u}_0 = \frac{\mu}{r} \boldsymbol{e}_r + b(\mu, r) \boldsymbol{e}_\theta.$$

(i) *For $\mu \neq -2\nu$,*

$$b(\mu, r) = \frac{c_1}{r} + c_2 r^{1 + \frac{\mu}{\nu}},$$

where

$$c_1 = \frac{\omega_1 R_1^2 R_2^{2 + \frac{\mu}{\nu}} - \omega_2 R_2^2 R_1^{2 + \frac{\mu}{\nu}}}{R_2^{2 + \frac{\mu}{\nu}} - R_1^{2 + \frac{\mu}{\nu}}}, \quad c_2 = \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{R_2^{2 + \frac{\mu}{\nu}} - R_1^{2 + \frac{\mu}{\nu}}}.$$

(ii) *For $\mu = -2\nu$,*

$$b(\mu, r) = \frac{1}{r}(c_1 + c_2 \log r),$$

where

$$c_1 = \frac{\omega_1 R_1^2 \log R_2 - \omega_2 R_2^2 \log R_1}{\log R_2 - \log R_1}, \quad c_2 = \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{\log R_2 - \log R_1}.$$

The pressure p_0 can be obtained from the equation. This solution is unique if $|\mu|$ and $|\omega_1 - \omega_2|$ (case(i)) ($|\omega_1|, |\omega_2|$ (case(ii))) are sufficiently small (c.f.[7],[8]).

Let us prove Theorem 1 in several steps. Let $C_{0,\sigma}^\infty(D)$ be all smooth solenoidal functions with compact support in the domain D, H_σ the closure of $C_{0,\sigma}^\infty(D)$ in $L^2(D)$, and V the closure of $C_{0,\sigma}^\infty(D)$ in the Sobolev space $H^1(D)$.

Let $\boldsymbol{u}_0, p_0$ be the solution as above. Let $\boldsymbol{v}_0$ satisfy the condition

(3) $\operatorname{div} \boldsymbol{v}_0 = 0$ in D, and $\boldsymbol{v}_0 = \varphi_i(\theta) \boldsymbol{e}_r + \psi_i(\theta) \boldsymbol{e}_\theta$ on Γ_i, $i = 1, 2$.

The existence of such function is known(c.f. [1]) but for our convenience, we choose:

$$
\begin{aligned}
\boldsymbol{v}_0 = & \left[R_1 r^{-1} \int_r^{R_2} \alpha(t) dt \, \varphi_1(\theta) + R_2 r^{-1} \int_{R_1}^r \alpha(t) dt \, \varphi_2(\theta) \right. \\
& \left. \quad -r^{-1} \int_{R_1}^r \beta_1(t) dt \, \psi_1'(\theta) - r^{-1} \int_{R_1}^r \beta_2(t) dt \, \psi_2'(\theta) \right] \boldsymbol{e}_r \\
& + \left[\int_0^\theta \{ R_1 \varphi_1(t) - R_2 \varphi_2(t) \} dt \, \alpha(r) + \beta_1(r) \psi_1(\theta) + \beta_2(r) \psi_2(\theta) \right] \boldsymbol{e}_\theta,
\end{aligned}
$$

(4)

where, $\alpha(r)$, $\beta_i(\theta)$ $(i = 1, 2)$ are smooth functions such that

$$\alpha(R_1) = \alpha(R_2) = 0, \quad \int_{R_1}^{R_2} \alpha(t) dt = 1,$$

$$\beta_i(R_j) = \delta_{ij} \, (i, j = 1, 2), \quad \int_{R_1}^{R_2} \beta_i(t) dt = 0 \, (i = 1, 2).$$

Then, we have the following estimate.

Lemma 1. *There exists a positive constant c_0 such that*

$$\|\boldsymbol{v}_0\|_{C^2(D)} \leq c_0 \sum_{i=1}^{2}(\|\varphi_i\|_{C^2(I)} + \|\psi_i\|_{C^3(I)})$$

holds, where I is the closed interval $[0, 2\pi]$.

Suppose $\boldsymbol{u} = \boldsymbol{w} + \boldsymbol{u}_0 + \boldsymbol{v}_0$ satisfy (1) with $\boldsymbol{b}$ given in (2). Then, the equation for $\boldsymbol{w}$ is as follows:

$$(5) \quad \begin{cases} -\nu\Delta\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + \dfrac{1}{\rho}\nabla q \\ \qquad\qquad\qquad +(\boldsymbol{w}\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{w} + \boldsymbol{f}_0 = 0 \quad \text{in} \quad D, \\ \operatorname{div}\boldsymbol{w} = 0 \quad \text{in} \quad D, \\ \boldsymbol{w} = 0 \quad \text{on} \quad \partial D, \end{cases}$$

where $\boldsymbol{f}_0 = -\nu\Delta\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{v}_0$.

Let P be the orthogonal projection from $L^2(D)$ onto H_σ and $A = -P\Delta$ be the Stokes operator. Applying the orthogonal projection P to the first equation in (5), we get:

$$(6) \quad \begin{cases} A\boldsymbol{w} + \dfrac{1}{\nu}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{w}\} \\ \qquad\qquad +\dfrac{1}{\nu}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{w} + \boldsymbol{f}_0\} = 0. \end{cases}$$

As is well known, A is a self-adjoint positive operator in H_σ and the inverse A^{-1} is a compact operator on H_σ (e.g., [5], [10]). Applying A^{-1} to the equation (6), we obtain:

$$(7) \quad \boldsymbol{w} - T(\mu)\boldsymbol{w} + \frac{1}{\nu}A^{-1}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + (\boldsymbol{w}\cdot\nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{w} + \boldsymbol{f}_0\} = 0,$$

where

$$T(\mu)\boldsymbol{w} \equiv -\frac{1}{\nu}A^{-1}P\{(\boldsymbol{w}\cdot\nabla)\boldsymbol{u}_0 + (\boldsymbol{u}_0\cdot\nabla)\boldsymbol{w}\}.$$

Since A^{-1} is compact operator in H_σ, and its range is the domain of the operator A which is compactly imbedded in V, we obtain the following:

Lemma 2. *The operator $T(\mu)$ is a compact linear operator on V.*

Let

$$\mu_n = -2\nu + \frac{2n\pi\nu}{\log R_2 - \log R_1}i, \quad n \in \mathbf{Z}, \quad i = \sqrt{-1}.$$

We define $b(\mu, r)$ for all $\mu \in \mathbf{C}$ letting $b(\mu, r) = b(-2\nu, r)$ for $\mu = \mu_n$. Then, $b(\mu, r)$ is continuous and holomorphic in $\mu \in \mathbf{C}$ even at the points $\mu = \mu_n$. Therefore, we have:

Lemma 3. *(c.f.Kato[4]) The compact operator $T(\mu)$ is an entire function of μ.*

Lemma 4. *If $|\omega_1 - \omega_2|$ is sufficiently small, then 1 is not the eigenvalue of the operator $T(0)$.*

Proof. Suppose that 1 is an eigenvalue of $T(0)$, i.e., there exists a nonzero $\boldsymbol{w} \in V$ such that $T(0)\boldsymbol{w} = \boldsymbol{w}$. Then,

$$-\frac{1}{\nu}A^{-1}P\{(\boldsymbol{w} \cdot \nabla)\tilde{\boldsymbol{u}}_0 + (\tilde{\boldsymbol{u}}_0 \cdot \nabla)\boldsymbol{w}\} = \boldsymbol{w}$$

holds, that is,

$$-\nu A\boldsymbol{w} = P\{(\boldsymbol{w} \cdot \nabla)\tilde{\boldsymbol{u}}_0 + (\tilde{\boldsymbol{u}}_0 \cdot \nabla)\boldsymbol{w}\}$$

holds, where $\tilde{\boldsymbol{u}}_0 = b(0, r)\boldsymbol{e}_\theta$ (See Remark 2). Taking the inner product with $\boldsymbol{w}$, we have

$$\nu||\nabla \boldsymbol{w}||^2 = ((\boldsymbol{w} \cdot \nabla)\tilde{\boldsymbol{u}}_0, \boldsymbol{w}).$$

The right hand side is equal to:

$$2c_1(0) \int_D \frac{w_r w_\theta}{r} dr d\theta, \quad \text{where} \quad c_1(0) = \frac{\omega_1 - \omega_2}{R_2^2 - R_1^2} R_1^2 R_2^2.$$

After some calculation, we have :

$$|((\boldsymbol{w} \cdot \nabla)\tilde{\boldsymbol{u}}_0, \boldsymbol{w})| \leq \frac{|\omega_1 - \omega_2|}{2} \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \left(\log \frac{R_2}{R_1}\right)^2 ||\nabla \boldsymbol{w}||^2.$$

If $|\omega_1 - \omega_2|$ is sufficiently small, then the inequality

$$\frac{|\omega_1 - \omega_2|}{2} \frac{R_1^2 R_2^2}{R_2^2 - R_1^2} \left(\log \frac{R_2}{R_1}\right)^2 < \nu$$

holds, and contradiction. c.q.f.d.

Let K be any compact subset of $\mathbf{C}$. According to Theorem 1.9 in Chapter VII §1 of [4], there exist a finite set $\{\mu_1^*, \mu_2^*, \ldots, \mu_k^*\}$ such that for any μ in $K \setminus \{\mu_1^*, \mu_2^*, \ldots, \mu_k^*\}$, $(T(\mu) - 1)^{-1}$ exists and is bounded on V. Let $\mu \in K \setminus \{\mu_1^*, \mu_2^*, \ldots, \mu_k^*\}$. From (7), we obtain:

$$(8) \quad \boldsymbol{w} = \frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\{(\boldsymbol{w} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{w} + \boldsymbol{f}_0\}.$$

Let us denote the right hand side of (8) by $N(\mu)\boldsymbol{w}$:

$$(9) \quad N(\mu)\boldsymbol{w} = \frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\{(\boldsymbol{w} \cdot \nabla)\boldsymbol{w} + (\boldsymbol{w} \cdot \nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{w} + \boldsymbol{f}_0\},$$

and

$$\sigma = \sum_{i=1}^{2}(||\varphi_i||_{C^2(I)} + ||\psi_i||_{C^3(I)}).$$

According to Lemma 1, we have:

(10) $\quad \|\frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\{(\boldsymbol{w} \cdot \nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0 \cdot \nabla)\boldsymbol{w}\}\|_V \leq \dfrac{C\sigma}{\nu}\|\boldsymbol{w}\|_V,$

(11) $\quad \|\frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\boldsymbol{f}_0\|_V \leq \dfrac{C\sigma}{\nu}\{(\nu + c_0\sigma)|D|^{1/2} + \|\boldsymbol{u}_0\|_V\},$

where c_0 is the constant in Lemma 1, $C = c_0\|(1 - T(\mu))^{-1}A^{-1/2}\|$ and $|D|$ is the measure of D.

Lemma 5. *There exists a positive constant c_D such that the estimate*

$$\|A^{-1}P(\boldsymbol{v} \cdot \nabla)\boldsymbol{w}\|_V \leq c_D\|\boldsymbol{v}\|_V\|\boldsymbol{w}\|_V, \quad \forall \boldsymbol{v}, \boldsymbol{w} \in V$$

holds.

It is known that there exists an absolute constant c such that

$$\|A^{-1/4}P(\boldsymbol{v} \cdot \nabla)\boldsymbol{w}\| \leq c\|A^{1/2}\boldsymbol{v}\|\|A^{1/2}\boldsymbol{w}\|, \quad \forall \boldsymbol{v}, \boldsymbol{w} \in C_{0,\sigma}^\infty(D)$$

holds. See, e.g., Fujita-Kato [3]. Using this inequality, we obtain Lemma 5 easily.

Put

(12) $\quad \rho_0 = \max\left[\dfrac{C}{\nu}, \dfrac{c_D}{\nu}\|(T(\mu) - 1)^{-1}\|, \dfrac{C}{\nu}\{(\nu + c_0)|D|^{1/2} + \|\boldsymbol{u}_0\|_V\} \right],$

where c_0, c_D, C are constants given in Lemma 1, in Lemma 5, and in (10), respectively. Now we have the following estimate for the nonlinear operator $N(\mu)$:

(13) $\quad \|N(\mu)\boldsymbol{w}\|_V \leq \rho_0(\|\boldsymbol{w}\|_V^2 + \sigma\|\boldsymbol{w}\|_V + \sigma).$

Let $\sigma_0 = \dfrac{1}{\rho_0}\{1 + 2\rho_0 - \sqrt{(1 + 2\rho_0)^2 - 1}\}.$

Remark 3. $\rho_0\sigma_0$ *is the smallest positive root of the equation*

$$X^2 - 2(1 + 2\rho_0)X + 1 = 0.$$

The inequality $0 < \sigma_0 < 1$ follows easily. If $0 < \sigma < \sigma_0$, then the equation

$$\rho_0(X^2 + \sigma X + \sigma) = X$$

has two positive roots. Let r_σ be the smaller one:

$$r_\sigma = \dfrac{1}{2\rho_0}\left\{1 - \rho_0\sigma - \sqrt{(1 - \rho_0\sigma)^2 - 4\rho_0^2\sigma}\right\}.$$

Lemma 6. *If $0 < \sigma < \sigma_0$, then the operator $N(\mu)$ maps the ball*

$$B(r_\sigma) \equiv \{\boldsymbol{w} \in V \ ; \ \|\boldsymbol{w}\|_V \leq r_\sigma\}$$

into itself.

Proof. Let $\boldsymbol{w} \in B(r_\sigma)$. Then,

$$||N(\mu)\boldsymbol{w}||_V \leq \rho_0(||\boldsymbol{w}||_V^2 + \sigma||\boldsymbol{w}||_V + \sigma) \leq \rho_0(r_\sigma^2 + \sigma r_\sigma + \sigma) = r_\sigma.$$

<div align="right">c.q.f.d.</div>

Lemma 7. *If* $0 < \sigma < \sigma_0$, *the operator* $N(\mu)$ *is a contraction operator on* $B(r_\sigma)$.

Proof. Let $\boldsymbol{w}_1, \boldsymbol{w}_2$ be arbitrary elements in $B(r_\sigma)$. Then, we have:

$$N(\mu)\boldsymbol{w}_1 - N(\mu)\boldsymbol{w}_2$$
$$= \frac{1}{\nu}(T(\mu) - 1)^{-1}A^{-1}P\{(\boldsymbol{w}_1 \cdot \nabla)\boldsymbol{w}_1 - (\boldsymbol{w}_2 \cdot \nabla)\boldsymbol{w}_2$$
$$+((\boldsymbol{w}_1 - \boldsymbol{w}_2) \cdot \nabla)\boldsymbol{v}_0 + (\boldsymbol{v}_0 \cdot \nabla)(\boldsymbol{w}_1 - \boldsymbol{w}_2)\}.$$

Since

$$(\boldsymbol{w}_1 \cdot \nabla)\boldsymbol{w}_1 - (\boldsymbol{w}_2 \cdot \nabla)\boldsymbol{w}_2 = ((\boldsymbol{w}_1 - \boldsymbol{w}_2) \cdot \nabla)\boldsymbol{w}_1 + (\boldsymbol{w}_2 \cdot \nabla)(\boldsymbol{w}_1 - \boldsymbol{w}_2),$$

therefore,

$$||N(\mu)\boldsymbol{w}_1 - N(\mu)\boldsymbol{w}_2||_V$$
$$\leq \frac{c_D}{\nu}||(T(\mu) - 1)^{-1}||||\boldsymbol{w}_1 - \boldsymbol{w}_2||_V (||\boldsymbol{w}_1||_V + ||\boldsymbol{w}_2||_V) + \frac{C\sigma}{\nu}||\boldsymbol{w}_1 - \boldsymbol{w}_2||_V$$
$$\leq \rho_0(||\boldsymbol{w}_1||_V + ||\boldsymbol{w}_2||_V + \sigma)||\boldsymbol{w}_1 - \boldsymbol{w}_2||_V,$$

where we used (10) and Lemma 5. Since $\boldsymbol{w}_1, \boldsymbol{w}_2 \in B(r_\sigma)$ and $\sigma < \sigma_0$, we have:

$$\rho_0(||\boldsymbol{w}_1||_V + ||\boldsymbol{w}_2||_V + \sigma) \leq \rho_0(2r_\sigma + \sigma) = 1 - \sqrt{(1 - \rho_0\sigma)^2 - 4\rho_0^2\sigma} < 1.$$

Therefore the operator $N(\mu)$ is a contraction and has a fixed point in the ball $B(r_\sigma)$. Theorem 1 is thus proved.

References

[1] Bogovski, M. E., Solution of the first boundary value problem for the equation of continuity of an incompressible medium, Soviet Math. Dokl. 20(1979), pp.1094-1098.

[2] Fujita, H., On the existence and regularity of the steady-state solutions of the Navier-Stokes equation, J. Fac. Sci., Univ.Tokyo, Sec.I, 9(1961), pp.59-102.

[3] Fujita, H.,and Kato, T., On the Navier-Stokes initial value problem I, Arch. Rational Mech. Anal., 16(1964), pp.269-315.

[4] Kato, T., Perturbation Theory for Linear Operators, Springer-Verlag Berlin, 1984

[5] Ladyzhenskaya, O. A., The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1969.

[6] Leray, J., Etude de diverses équations intégrales nonlinéaires et de quelques problèmes que pose l'hydrodynamique, J. Math. Pure Appl. 12(1933) pp.1-82.

[7] Morimoto, H., A solution to the stationary Navier-Stokes equations under the boundary conditon with non-vanishing outflow, Memoirs of the Institute of Science and Technology, Meiji Univ., Vol.31(1992), pp.7-12.

[8] Morimoto, H., Stationary Navier-Stokes equations under general outflow conditon, preprint

[9] Takesita, A., A remark on Leray's inequality, Pacific J. Math., Vol 157, No.1, 1993 pp.151-158.

[10] Temam, R., Navier-Stokes Equations, North-Holland, Amsterdam, 1977.