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ON FAIR PARAMETRIC RATIONAL CURVES

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1. Introduction.

For given data $(x_i^{(k)}, y_i^{(k)})$ ($i = 0, 1; k = 0, 1$), we consider the problem of finding a fair parametric rational curve with a parameter p (> -1):

$$(1)\qquad \begin{aligned} x(t) &= a_0 t + a_1 u + a_2 t^3/(1+pt) + a_3 u^3/(1+pu) \\ & \qquad\qquad (0 \le t \le 1, u = 1-t) \\ y(t) &= b_0 t + b_1 u + b_2 t^3/(1+pt) + b_3 u^3/(1+pu) \end{aligned}$$

or

$$(1)'\qquad \begin{aligned} x(t) &= a_0 t + a_1 u + a_2 t^3/(1+pu) + a_3 u^3/(1+pt) \\ & \qquad\qquad (0 \le t \le 1, u = 1-t) \\ y(t) &= b_0 t + b_1 u + b_2 t^3/(1+pu) + b_3 u^3/(1+pt) \end{aligned}$$

so that

$$(2)\qquad (x^{(k)}(i), y^{(k)}(i)) = (x_i^{(k)}, y_i^{(k)}) \quad (i = 0, 1; k = 0, 1).$$

It is well-known that a drawback of a parametric cubic curve ($p = 0$) is indicated by the fact that unwanted inflection points or singularities may occur on its segment. In what follows, the adjective "*parametric*" on the curve is usually suppressed since all the curves in this paper are parametric ones and a *fair* curve means a one free of unwanted inflection points and singularities. Techniques for eliminating the unwanted ones have been developed ([1], [2], [5], [6]). One of them is to use the rational curve of the form (1) or a different looking rational curve of the form (1)'. Note there is no difference between use of the two rational ones since letting $1/p + 1/q = -1$ ($p \to -1+ \Leftrightarrow q \to \infty$, $-1 < p < 0 \Leftrightarrow q > 0$), then $t^3/(1+pt) = (1+q)t^3/(1+qu)$ and $u^3/(1+pu) = (1+q)u^3/(1+qt)$. Hence, all the subsequent results for the curve of the form (1) are still valid for the one of the form (1)' with p (> -1) replaced by $-p/(1+p)$ ($= q$). The first object of Section 2 is to show that we can find a fair curve segment of the form (1) interpolating to (2) if λ ($= C_0/D$), μ ($= -C_1/D$) $\ge (1+p)/(3+2p)$ where $\Delta x = x_1 - x_0$, $\Delta y = y_1 - y_0$, $C_i = x_i'\Delta y - y_i'\Delta x$ ($i = 0, 1$) and $D = x_0'y_1' - x_1'y_0'$. Note that we can find a fair curve of the form (1)' if $\lambda, \mu \ge 1/(3+p)$. That is, if (λ, μ) is in the interior of the first quadrant of the (λ, μ) plane, a suitable choice of the parameter p gives a fair curve of the form (1) interpolating to (2), strictly speaking, according to ρ ($= \min(\lambda, \mu)$) $\ge 1/3$ and $0 < \rho < 1/3$ we can take $p = 0$ and $(3\rho - 1)/(1 - 2\rho)$, respectively since a smaller value of $|p|$ would make the truncation error be smaller provided that the data arise from a function.

For the curve of the form (1)', we choose $p = \max(0, 1/\rho - 3)$. The second object of Section 2 is to show that the region for a fair curve also contains the whole third quadrant in addition to the whole first quadrant (theoretically for p sufficiently close to -1). Accordingly, we can find a C^2 fair interpolatory rational curve to data $S = \{(x_i, y_i), 0 \le i \le n\}$ by a suitable choice of the parameter p if

$$x[t_i, t_{i+1}]y[t_j, t_{j+1}, t_{j+2}] - y[t_i, t_{i+1}]x[(t_j, t_{j+1}, t_{j+2}] > 0 \ (j = i-1, i)$$

where $x[t_i, t_{i+1}]$ ($x[t_i, t_{i+1}, t_{i+2}]$) is the first (second) divided difference and $t_0 = 0$, $t_{i+1} = t_i + h_i \ (= \sqrt{(x_{i+1} - x_i)^2 + (y_{i+1} - y_i)^2}$ ($0 \le i \le n-1$). In Section 3, we derive a theorem concerning the distribution of inflection points and singularities for the cubic curve segment ($p = 0$) which has given another technique for finding a C^1 (not C^2) fair cubic curve interpolating to S ([6]). In Section 3, some numerical examples are given.

2. On segments of parametric rational curves.

We shall show that the curve segment (1) is fair for $\lambda, \mu \ge (1+p)/(3+2p)$ in the first quadrant of the (λ, μ) plane.

Inflection points. First we obtain sufficient conditions for the curve segment (1) not to contain an inflection point. Defining $\phi(t) = t^3/(1+pt) - t/(1+p)$, the segment (1) interpolating to (2) is expressed by equations

(3) $\quad x(t) = x_1 t + x_0 u + c_1\phi(t) + d_1\phi(u),\ y(t) = y_1 t + y_0 u + c_2\phi(t) + d_2\phi(u)$

in which

(4) $$(3+2p)/(1+p)^2(c_1, d_1) = (\ (1+p)x_0' + (2+p)x_1' - (3+2p)\Delta x,\\ -(2+p)x_0' - (1+p)x_1' + (3+2p)\Delta x\ )$$

and (c_2, d_2) is given by (4) with y replacing x.

Inflection points of the curve (1) are determined by the equation:

(5) $\quad x'(t)y''(t) - x''(t)y'(t) = 0 \quad (0 < t < 1)$.

We assume for the moment $D\ (= x_0'y_1' - x_1'y_0') \ne 0$. Then the equation (5) can be equivalently rewritten as

(6) $w(t)\ (= \lambda\theta(t) + \mu\theta(u) + (1+p)^2\{\phi'(t)\phi''(u) + \phi''(t)\phi'(u)\}\ ) = 0 \ (0 < t < 1)$

where

(7) $\quad \theta(t) = (2+p)\phi''(u) - (1+p)\phi''(t) - (1+p)^2\{\phi'(t)\phi''(u) + \phi''(t)\phi'(u)\}$.

By a direct calculation, $\theta(t) > 0$ ($0 < t < 1$) since

(8) $$\theta(1) = 0,\quad \theta'(t) = (1+p)^2\phi^{(3)}(u)\{\phi'(t) - (2+p)/(1+p)^2\}\\ -(1+p)^2\phi^{(3)}(t)\{\phi'(u) + 1/(1+p)\} < 0$$

where $\phi'(t) = (3t^2 + 2pt^3)/(1+pt)^2 - 1/(1+p)$, $\phi''(t) = 2t(3 + 3pt + p^2t^2)/(1+pt)^3$ and $\phi^{(3)}(t) = 6/(1+pt)^4$. Therefore, if $\lambda, \mu \ge (1+p)/(3+2p)$, then no inflection point occurs on the curve segment (1) since

(9) $w(t) \ge \{(1+p)^2/(3+2p)\}[\phi''(t)\{\phi'(u) + 1/(1+p)\} + \phi''(u)\{\phi'(t) + 1/(1+p)\}] > 0$.

Singularities. Next we shall obtain a sufficient condition for the curve segment (1) not to contain a singularity. To this end, we have to get an equation of the image of the curve (1) by eliminating the parameter t from (1) (or (3)). From (3),

(10) $\quad c_1\phi(t) + d_1\phi(u) = x - x_0u - x_1t,\ c_2\phi(t) + d_2\phi(u) = y - y_0u - y_1t.$

The case $\Delta \neq 0$ is considered since the other case $\Delta = 0$ is easily treated where $\Delta\ (= c_1d_2 - c_2d_1) = \{(1+p)^4/(3+2p)\}(D - C_0 + C_1)\ (= \{(1+p)^4/(3+2p)\}(1-\lambda-\mu)D)$. A combination of the two equations in (10) gives

(11) $\ t^3/(1+pt) = (d_2x - d_1y)/\Delta - \alpha_1u - \beta_1t,\ u^3/(1+pu) = (c_1y - c_2x)/\Delta - \alpha_2u - \beta_2t$

where

(12)
$$\alpha_1 - \beta_1 = \frac{\{(3+2p)C_0 - (1+p)D\}}{(1+p)^2(C_0 - C_1 - D)} = \frac{\{(3+2p)\lambda - (1+p)\}}{(1+p)^2(\lambda + \mu - 1)}$$
$$-(\alpha_2 - \beta_2) = \frac{\{(3+2p)(-C_1) - (1+p)D\}}{(1+p)^2(C_0 - C_1 - D)} = \frac{\{(3+2p)\mu - (1+p)\}}{(1+p)^2(\lambda + \mu - 1)}.$$

Further we rewrite (11) as

(13) $\quad t^3 = p(\alpha_1 - \beta_1)t^2 + q_1t + r_1,\ u^3 = p(\beta_2 - \alpha_2)u^2 + q_2u + r_2$

in which

(14)
(i) $\quad r_1 = (d_2x - d_1y)/\Delta - \alpha_1,\ r_2 = (c_1y - c_2x)/\Delta - \beta_2,$

(ii) $\quad q_1 = pr_1 + \alpha_1 - \beta_1,\ q_2 = pr_2 - \alpha_2 + \beta_2.$

Defining

(15)
$$k = 3 - p(\alpha_1 - \beta_1) + p(\alpha_2 - \beta_2) = \frac{(3+3p+p^2)(C_0 - C_1) - (1+p)(3+p)D}{(1+p)^2(C_0 - C_1 - D)}$$
$$= \frac{(3+3p+p^2)(\lambda + \mu) - (1+p)(3+p)}{(1+p)^2(\lambda + \mu - 1)}\ (\neq 0)\ \text{for } \lambda, \mu \geq (1+p)/(3+2p),$$

the summation of the two equations in (13) gives

(16) $\quad kt^2 - \{3 + q_1 - q_2 + 2p(\alpha_2 - \beta_2)\}t + \{1 - q_2 - r_1 - r_2 + p(\alpha_2 - \beta_2)\} = 0.$

Rewriting a quadratic equation (16) as $(t+\alpha)^2 = \beta$, easy calculation gives

(17)
(i) $\quad 2\alpha k = -3 - p(r_1 - r_2) - (\alpha_1 - \beta_1) - (1+2p)(\alpha_2 - \beta_2)$

(ii) $\quad \beta k = -1 + r_1 + (1+p)r_2 - (1+p)(\alpha_2 - \beta_2) + \alpha^2k.$

Here we consider (α, β) as functions of (r_1, r_2) instead of (x, y), and then

(18) $\quad \kappa(\alpha_{r_1}, \alpha_{r_2}) = (-p/2, p/2),\ \kappa(\beta_{r_1}, \beta_{r_2}) = (1 - p\alpha,\ 1 + p + p\alpha)$

Use a change of variable $t = t^* - \alpha$ and eliminate the parameter t^* ($(t^*)^2 = \beta$) from the first equation in (13) to give the required equation $\psi(x, y) = 0$ of the image of the curve (1):

(19)
$$\psi(x, y) = \{\beta + 3\alpha^2 + (2\alpha p - 1)(\alpha_1 - \beta_1) - pr_1\}^2\beta$$
$$- [\alpha^3 + 3\alpha\beta + \{p(\alpha^2 + \beta) - \alpha\}(\alpha_1 - \beta_1) + (1 - p\alpha)r_1]^2.$$

For simplicity, we can consider ψ as a function of (r_1, r_2) instead of (x, y). Then, the singularity of ψ is determined by the system of equations

(20) $\qquad \psi(r_1, r_2) = \psi_{r_1}(r_1, r_2) = \psi_{r_2}(r_1, r_2) = 0.$

Writing ψ as $A^2\beta - C^2$, $\psi_{r_1}(r_1, r_2) = \psi_{r_2}(r_1, r_2) = 0$ are equivalent to

(21)
$$2\{-p-\frac{p}{2k}A_\alpha+\frac{1-p\alpha}{k}A_\beta\}A\beta+\frac{1-p\alpha}{k}A^2=2\{1-p\alpha-\frac{p}{2k}C_\alpha+\frac{1-p\alpha}{k}C_\beta\}C$$
$$2\{\frac{p}{2k}A_\alpha+\frac{1+p+p\alpha}{k}A_\beta\}A\beta+\frac{1+p+p\alpha}{k}A^2=2\{\frac{p}{2k}C_\alpha+\frac{1+p+p\alpha}{k}C_\beta\}C$$

On eliminating A^2 from (21), we obtain

(22)
$$\{p(1+p+p\alpha)+(p/k)(1+p/2)A_\alpha\}A\beta$$
$$=\{-(1-p\alpha)(1+p+p\alpha)+(p/k)(1+p/2)C_\alpha\}C.$$

Since

(23)
$$\begin{vmatrix} A_{r_1} \\ A_{r_2} \end{vmatrix} = \begin{vmatrix} \frac{-p}{2k} & \frac{1-p\alpha}{k} \\ \frac{p}{2k} & \frac{1+p+p\alpha}{k} \end{vmatrix} \begin{vmatrix} A_\alpha \\ A_\beta \end{vmatrix},$$

by (23) and the similar relations for β and C we get

(24)
$$(1+p+p\alpha)A_{r_1}-(1-p\alpha)A_{r_2}=-(2p+p^2)/(2k)A_\alpha$$
$$(1+p+p\alpha)\beta_{r_1}-(1-p\alpha)\beta_{r_2}=0$$
$$(1+p+p\alpha)C_{r_1}-(1-p\alpha)C_{r_2}=-(2p+p^2)/(2k)C_\alpha$$

Making a linear combination of $\psi_{r_1}(r_1, r_2)$ and $\psi_{r_2}(r_1, r_2)$, i.e., $(1+p+p\alpha)\psi_{r_1}(r_1, r_2) - (1-p\alpha)\psi_{r_2}(r_1, r_2)$, from $\psi_{r_1}(r_1, r_2) = \psi_{r_2}(r_1, r_2) = 0$ we get another equation in addition to (22):

(25) $\qquad A_\alpha A\beta = C_\alpha C.$

From (22) and (25),

(26)
$$C\{-(1-p\alpha)(1+p+p\alpha)+(p/k)(1+p/2)C_\alpha\}A_\alpha A\beta$$
$$=C_\alpha C\{p(1+p+p\alpha)+(p/k)(1+p/2)A_\alpha\}A\beta.$$

Two cases $C \neq 0$ and $C = 0$, will be considered separately. (a) $C \neq 0$: Then, $A, \beta \neq 0$ since $A^2\beta = C^2$. From (26) we have

(27) $\qquad (1+p+p\alpha)(1+p\alpha)\{pC_\alpha - (p\alpha-1)A_\alpha\} = 0.$

In this case, we show that the curve segment (1) (or (19)) does not have a singularity. If there were any singularity (loop or cusp), then we note that two values of the parameter t defined by the quadratic equation $(t+\alpha)^2 = \beta$ must belong to (0, 1), i.e.,

(28) $\quad 0 < t\,(= t^* - \alpha = \pm\sqrt{\beta} - \alpha) < 1$ or $-1 < \alpha < 0, 0 \leq \beta < \alpha^2, \beta < (1+\alpha)^2$

where $\beta > 0$ and $\beta = 0$ correspond to a loop and cusp, respectively.

Since $1+p+p\alpha, 1+p\alpha > 0$ for $p > -1$, (27) gives $pC_\alpha = (p\alpha - 1)A_\alpha$ from which $A_\alpha C_\alpha\{p\beta A - (p\alpha-1)C\} = 0$ with the aid of (25). Hence we have

(29) $\qquad p\beta A - (p\alpha - 1)C = 0$

where if A_α or $C_\alpha = 0$, then we obtain $p\beta A - (p\alpha-1)C = 0$ from (22) and (27).

Combining (29) with $A^2\beta = C^2$, we have

(30) $\beta = (\alpha - 1/p)^2$

which can not satisfy the required inequalities in (28), i.e., $\beta < \alpha^2$ ($-1/2 \le \alpha < 0$) or $\beta < (1+\alpha)^2$ ($-1 < \alpha < -1/2$). That is, if $C \neq 0$, the curve segment (1) (or (19)) does not have a singularity. (b) $C = 0$: From (24) we get $\beta = 0$ or $A = 0$. Since $\beta = 0$ gives $A = 0$, we have only to consider the case when $A = C = 0$. Then

(31)
(i) $\beta + 3\alpha^2 + (2p\alpha - 1)(\alpha_1 - \beta_1) = pr_1$
(ii) $2\alpha^3 - 2\alpha\beta + p(\alpha^2 - \beta)(\alpha_1 - \beta_1) = r_1.$

In addition, eliminating r_2 from (17),

(32) $k\{p\alpha^2 + 2\alpha(1+p) - p\beta\} + 3 + 2p + (1+p)(\alpha_1 - \beta_1) + (1+p)^2(\alpha_2 - \beta_2)$
$= -(2p + p^2)r_1.$

Eliminate r_1 from 31(i)-(ii) and 31(1)-(32) to give two equations, respectively

(33)
(i) $$\alpha_1 - \beta_1 = \frac{3\alpha^2 - 2p\alpha^3 + \beta(1 + 2p\alpha)}{(1 - p\alpha)^2 - \beta p^2}$$
(ii) $$-(\alpha_2 - \beta_2) = \frac{3(1+\alpha)^2 + 2p(1+\alpha)^3 + \beta\{1 - 2p(1+\alpha)\}}{(1 + p + p\alpha)^2 - \beta p^2}.$$

Note that for $p \neq 0$, it is be difficult to find the solution (α, β) of (33) (the singular point (x, y) of (19) by 14(i)-(17)). Therefore we show that the existence of the solution (α, β) satisfying the required inequalities (28) brings λ, $\mu < (1+p)/(3+2p)$, as implies that the curve segment (19) is free of a singularity for λ, $\mu \ge (1+p)/(3+2p)$. Since the both right hand sides of (33) are monotone increasing in β ($0 \le \beta < \alpha^2$, $-1/2 \le \alpha < 0$ and $0 \le \beta < (1+\alpha)^2$, $-1 < \alpha < -1/2$),

(34) $$0 < \alpha_1 - \beta_1 < \frac{4\alpha^2}{1 - 2p\alpha}, \quad 0 < -(\alpha_2 - \beta_2) < \frac{4\alpha^2 + 6\alpha + (3+2p)/(1+p)}{1 + p + 2p\alpha}$$
$$(-1/2 \le \alpha < 0)$$

or

(35) $$0 < \alpha_1 - \beta_1 < \frac{4\alpha^2 + 2\alpha + 1/(1+p)}{1 - p - 2p\alpha}, \quad 0 < -(\alpha_2 - \beta_2) < \frac{4(1+\alpha)^2}{1 + 2p + 2p\alpha}$$
$$(-1 < \alpha < -1/2).$$

In addition, use (34)-(35) to obtain

(36) $$(\alpha_1 - \beta_1) - (\alpha_2 - \beta_2) < \frac{3 + 2p}{(1+p)^2}$$

which is easily checked since the above inequality (36) is equivalent to

(37) $\alpha\{2(2 + 2p + p^2)\alpha + (3 + 3p + p^2)\} < 0 \quad (-1/2 \le \alpha < 0)$

or

(38) $(1+\alpha)\{2(2 + 2p + p^2)\alpha + (1 + p + p^2)\} < 0 \quad (-1 < \alpha < -1/2).$

Combining (12) with the first inequalities of (34)-(35) and (36), the necessary conditions for the curve segment (19) to contain a singularity are given by

$$\{(3+2p)\lambda - (1+p)\}/(\lambda + \mu - 1) > 0, \quad \{(3+2p)\mu - (1+p)\}/(\lambda + \mu - 1) > 0$$

(39)
$$\{(3+2p)(\lambda+\mu)-2(1+p)\}/(\lambda+\mu-1) < (3+2p) \quad (\text{ i.e., } \lambda+\mu<1\text{ })$$
from which follow

(40) $$\lambda\text{ , }\mu\ < (1+p)/(3+2p).$$

Therefore, if λ , $\mu\ \geq (1+p)/(3+2p)$, then the curve segment (19) has no singularity. In summary, a theorem concerning the inflection points and singularity is obtained:

Theorem 1. The curve segment of the form (1) interpolating to (2) is fair for λ, $\mu\ \geq (1+p)/(3+2p)$ in the first quadrant of the (λ, μ) plane.

Remark. For the case $D=0$ in which the tangent vectors at the two end points are parallel provided that $(x_i')^2+(y_i')^2 \neq 0$ ($i=0, 1$). The curve segment (1) does not contain an inflection point if $C_0(-C_1)\ >0$, i.e.,

(41) $$(x_0'\Delta y - y_0'\Delta x)(x_1'\Delta y - y_1'\Delta x)\ <0$$

since for $D=0$, $w(t)=C_0\theta(t)+(-C_1)\theta(u)$. In addition, the curve segment (1) is free of a singularity since $\alpha_1-\beta_1-(\alpha_2-\beta_2)=(3+2p)/(1+p)^2$ from (12), i.e., the necessary inequalities (36) for the existence of the singularities does not hold where note that all the argument in case I is still valid for $D=0$ with (12) and (15) having C_0, C_1 for λ, μ. Therefore, the curve segment (1) is fair for $C_0(-C_1)\ >0$.

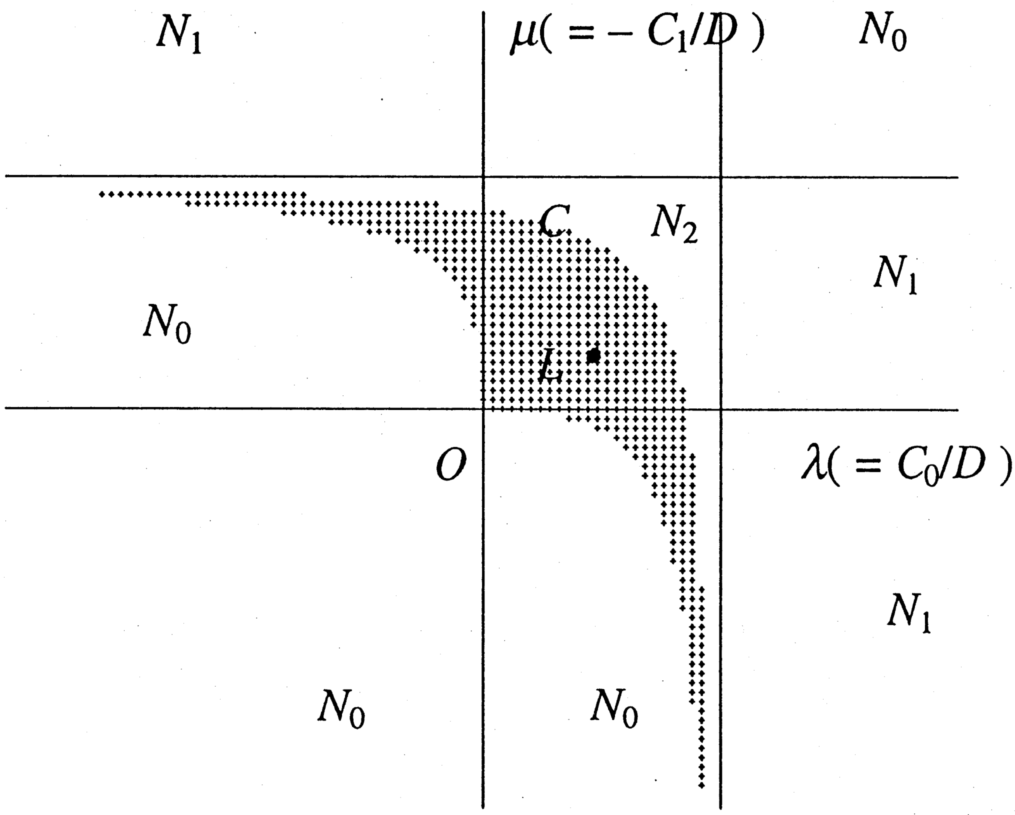

Fig. 1. Distribution of inflection points and singularities ($p=-0.5$)

Fig. 1 gives the numerically determined distribution of inflection points and singularities for $p = -0.5$ where N_i ($i = 0, 1, 2$) or L (shown by dots) represent the regions for the curve segment (1) to contain i inflection points and no singularity or a loop and no inflection point. The upper boundary C of L represents the region for the curve segment (1) to have a cusp and passes through (λ, μ) with $\lambda = \mu = (1+p)/(6+6p+p^2)$, since then ($\alpha$, β) = ($-$ 1/2, 0) is the solution of (33). As p decreases to $-$ 1, the region for a fair curve rapidly increases to the whole first and third quadrants; refer to Figs 1-2 and 4. By means of Theorems 1, generally speaking, we see that if the curve segment ($p = 0$) contains a loop, according as p decreases to $-$ 1, it contains a loop and no inflection point $\Rightarrow$ a cusp and no inflection point $\Rightarrow$ two inflection points and no singularity $\Rightarrow$ one inflection point and no singularity $\Rightarrow$ no inflection point and no singularity, by turns, provided that (C_0/D, $-$ C_1/D) is in the first quadrant. In numerical determination of the distribution, we repeat the process that first give (λ, μ), and then count the number of the roots of (6) belonging to ($-$ 1, 0) and check if the solution (α, β) of (33) satisfies (28). At the end of this section, we remark that if the interval of the parameter t is [α, β] instead of [0, 1] in the end conditions (2), i.e.,

(2)' $\quad (x^{(k)}(\alpha), y^{(k)}(\alpha)) = (x_0^{(k)}, y_0^{(k)}),\ (x^{(k)}(\beta), y^{(k)}(\beta)) = (x_1^{(k)}, y_1^{(k)})\ (k = 0, 1),$

then we use the parametric rational curve of the form (1) with $t^3/(1+pt)$ and $u^3/(1+pu)$ replaced by $s^3/(1+ps)$ and $r^3/(1+pr)$ with $s = (t-\alpha)/(\beta-\alpha)$ and $r = 1-s$. In this case, letting $C_i = y[\alpha, \beta]x_i' - x[\alpha, \beta]y_i'$ ($i = 0, 1$) and $D = x_0'y_1' - x_1'y_0'$, we can find a fair curve if $C_0/D, -C_1/D \geq (1+p)/(3+2p)$. With help of this remark, we can easily check that the curve $(x(t), y(t)) \equiv \{(x_i(t), y_i(t)), 0 \leq i \leq n-1 : x_i(t)$ and $y_i(t)$ of the above modified form with (t_i, t_{i+1}) replacing (α, β) } interpolating to the data $S = \{(x_i, y_i), 0 \leq i \leq n\}$ is fair on the same assumption given in [6] where he gave a numerical example to demonstrate the automatic removal of spurious singularities (loops or cusps) without theoretical analysis which would be required since inflection points and singularities occur under different conditions. The curve $(x(t), y(t))$ is given by

$$x(t) := x_i(t) = x_{i+1}s + x_ir + h_i\{c_1(i)\phi(s) + d_1(i)\phi(r)\}$$

(3)' $\qquad (t_i \leq t \leq t_{i+1})$

$$y(t) := y_i(t) = y_{i+1}s + y_ir + h_i\{c_2(i)\phi(s) + d_2(i)\phi(r)\}$$

where

(4)' $$(3+2p)/(1+p)^2(c_1(i), d_1(i)) = (\ (1+p)x_i' + (2+p)x_{i+1}' - (3+2p)x[t_i, t_{i+1}], \ -(2+p)x_i' - (1+p)x_{i+1}' + (3+2p)x[t_i, t_{i+1}]\ )$$

and $(c_2(i), d_2(i))$ is given by (4)' with y replacing x.

Here z_i' ($= x_i', y_i'$) are determined by the consistency relation for $z \in C^2[t_0, t_n]$:

(42) $$\frac{z_{i+1}'}{h_i} + \frac{(2+p)}{(1+p)}\left(\frac{1}{h_i} + \frac{1}{h_{i-1}}\right)z_i' + \frac{z_{i-1}'}{h_{i-1}} = \frac{(3+2p)}{(1+p)}\left\{\frac{z[t_i, t_{i+1}]}{h_i} + \frac{z[t_{i-1}, t_i]}{h_{i-1}}\right\} (1 \leq i \leq n-1)$$

and the boundary conditions $z_0'' = z_n'' = 0$ which are equivalent to

(43) $(2+p)z_0' + (1+p)z_1' = (3+2p)z[t_0, t_1]$, $(2+p)z_n' + (1+p)z_{n-1}' = (3+2p)z[t_{n-1}, t_n]$.

It was proved in [1] or from (42)-(43) that on letting $p \to -1$

$$z_i' \to (h_{i-1}z[t_j, t_{i+1}] + h_i z[t_{i-1}, t_i])/(h_i + h_{i-1}) \quad (1 \le i \le n-1)$$

(44)

$$z_0' \to z[t_0, t_1], \quad z_n' \to z[t_{n-1}, t_n] .$$

Use (44) to obtain on $[t_i, t_{i+1}]$ ($1 \le i \le n-2$)

$$C_0 \; (= y[t_i, t_{i+1}]x_i' - x[t_i, t_{i+1}]y_i')$$
$$\to h_i\{x[t_i, t_{i+1}]y[t_{i-1}, t_i, t_{i+1}] - y[t_i, t_{i+1}]x[t_{i-1}, t_i, t_{i+1}]\}$$

(45)

$$-C_1 \; (= -y[t_i, t_{i+1}]x_{i+1}' + x[t_i, t_{i+1}]y_{i+1}')$$
$$\to h_i\{x[t_i, t_{i+1}]y[t_i, t_{i+1}, t_{i+2}] - y[t_i, t_{i+1}]x[t_i, t_{i+1}, t_{i+2}]\}.$$

Therefore, if $x[t_i, t_{i+1}]y[t_j, t_{j+1}, t_{j+2}] - y[t_i, t_{i+1}]x[(t_j, t_{j+1}, t_{j+2}] > 0$ ($j = i-1, i$) or < 0 ($j = i-1, i$), then (λ, μ) ($= (C_0/D, -C_1/D)$) is in the first or third quadrants for p sufficiently close to -1, i.e., the curve segment $(x_i(t), y_i(t))$ is fair for p sufficiently close to -1 on $[t_i, t_{i+1}]$ ($1 \le i \le n-2$). On $[t_0, t_1]$, from (43)-(44) $(2+p)C_0 = (1+p)(-C_1)$ and $-C_1 \to h_0\{x[t_0, t_1]y[t_0, t_1, t_2] - y[t_0, t_1]x[t_0, t_1, t_2]\}$. Therefore, if $x[t_0, t_1]y[t_0, t_1, t_2] - y[t_0, t_1]x[t_0, t_1, t_2] \ne 0$, then the curve segment $(x_0(t), y_0(t))$ is also fair for p sufficiently close to -1. Similarly the curve segment $(x_{n-1}(t), y_{n-1}(t))$ is fair if $x[t_{n-1}, t_n]y[t_{n-2}, t_{n-1}, t_n] - y[t_{n-1}, t_n]x[t_{n-2}, t_{n-1}, t_n] \ne 0$ for p sufficiently close to -1.

Suppose that the tangent directions are fixed at the two end points of a segment, and only the magnitudes of the tangents are allowed to be varied in scalar multiples η and κ (η , $\kappa > 0$), respectively. Then $C_0 \to \eta C_0$, $C_1 \to \kappa C_1$, $D \to \eta\kappa D$, i.e., $\lambda \to \lambda/\kappa$, $\mu \to \mu/\eta$. Therefore, if C_0/D, $-C_1/D > 0$, then Theorem 1 enables us to find a fair curve segment (1) by a suitable choice of η and κ, strictly speaking, if λ/κ , $\mu/\eta \ge 1/3$, i.e., $0 < \kappa \le 3\lambda$ ($= 3C_0/D$), $0 < \eta \le 3\mu$ ($= -3C_1/D$) where for another proof, see [6, p. 54].

3. Numerical examples.

In this section, we consider two numerical examples. First we consider the different shapes of the curve segments with different values of the parameter p; see Fig. 2 where the data are given by $(x_0^{(k)}, y_0^{(k)}) = (0, 1), (5, 6)$, $(x_1^{(k)}, y_1^{(k)}) = (1, 1), (8, -4)$, i.e., $(\lambda, \mu) = (2/17, 1/17)$ and the values of the parameter p are 0, -0.5 and -0.83 (= " an approximate value when a cusp occurs") and $-14/15$ (= " the proposed one in this paper"). Point (2/17, 1/17) is denoted by solid circles in Fig 1, as implies that the curve segment with $p = -0.5$ contains a loop. Numerical determination of the distribution assured that (2/17, 1/17) denoted by a solid circle is nearly on the boundary C of L with $p = -0.83$ in Fig. 3, i.e., then the curve segment contains a cusp. Next $S = \{(0, 0), (2, 4), (4, 0), (6, 2), (8, 10), (10, 2), (10.5, 2)\}$(Clements [1]) . The signs of $C_0(-C_1) < 0$ on $[t_i, t_{i+1}]$ ($i = 3, 4$) change from + to – as p decreases to – 1, and so one inflection point which does not occur with $p = 0$ appears on each $[t_i, t_{i+1}]$ ($i = 3, 4$) when letting $p \to -1$ in order to eliminate a loop on $[t_3, t_5]$, i.e., in this case no loop would be more desirable than two inflection points. Generally speaking, a loop on two or more consecutive curve segments could be eliminated by letting $p \to -1$ since then each curve segment reduces to a straight line one by (4) or (4)'. Note that all the discussion in Sections 2-3 is concerned with a singularity (loop and cusp) on a *single* segment. In Fig 4, we give a graph of the rational curves ($p = 0, -0.3, -0.5$) on $[t_4, t_5]$ of interest. The algorithm proposed by Clements [1] is sufficient in practical computation of the curve segment of the form (1) or (1)' interpolating to (2).

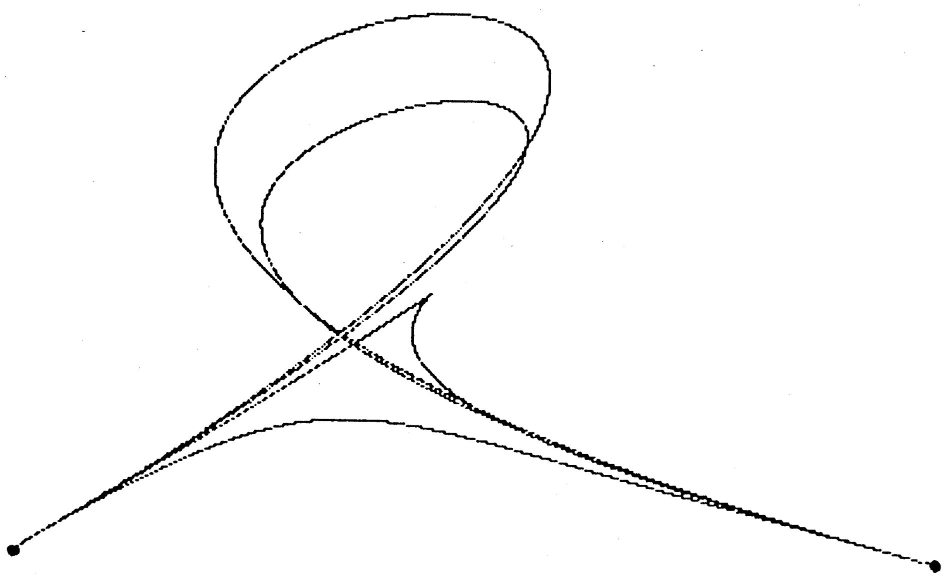

Fig. 2

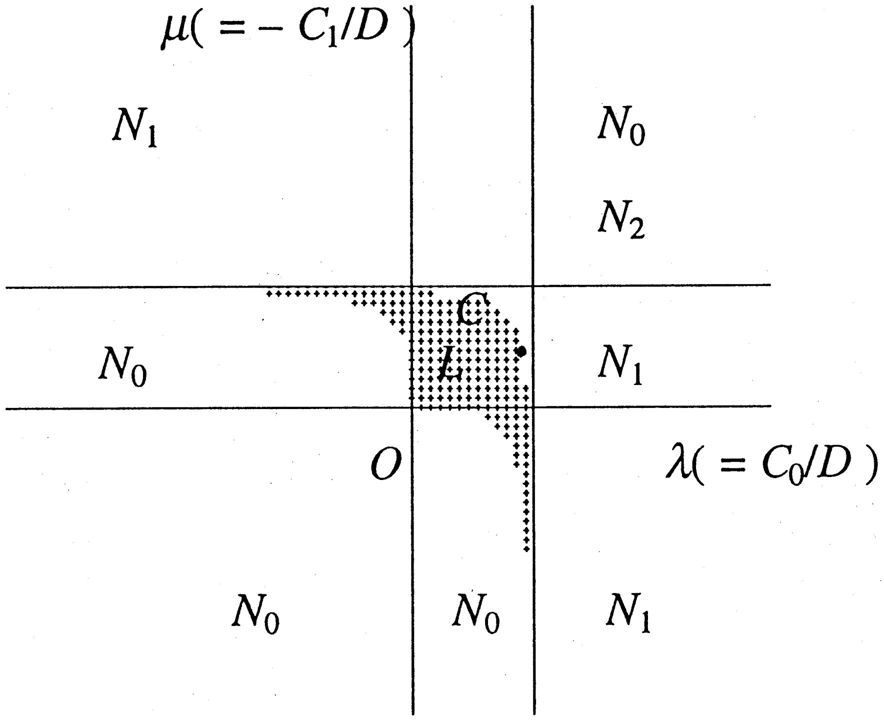

Fig. 3

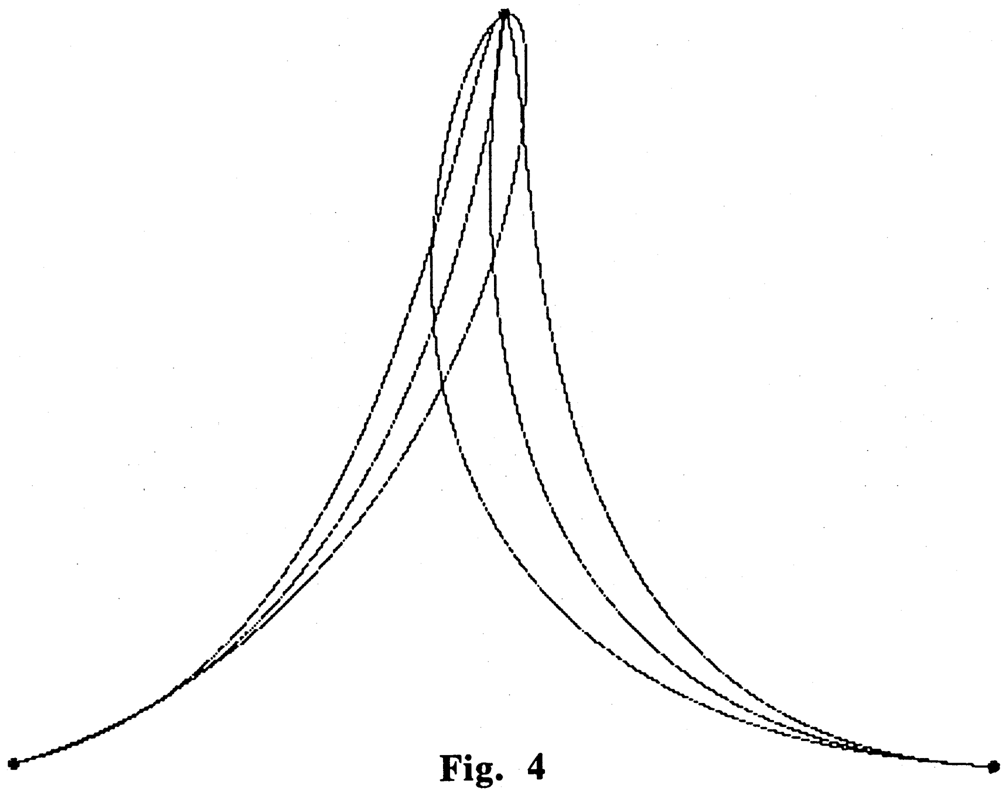

Fig. 4

References

1. Clements, J. C.: A convexity-preserving C^2 parametric rational interpolation, Numer. Math. **63**, 165-171, 1992.
2. M. Sakai and R,. Usmani: Shape preserving approximation by rational splines, WSSIAA **2**, 345-354, World Scientific, Singapore, 1993.
3. M. Sakai and J. W. Schmidt: Positive interpolation with rational splines, BIT **29**, 140-147, 1989.
4. M. Sakai and R,. Usmani: On orders of approximation of plane curves by rational splines, BIT **30**, 735-741, 1990.
5. H. Späth: Eindimensionale Spline-Interpolations-Algorithmen, R. Oldenbourg Verlag Munchen, 1990.
6. Su, B-O and Liu, D-Y: Computational Geometry-Curve and Surface Modeling, Academic Press, New York, 1989.