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On uniformly convex functions

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Abstract

A.W.Goodman[1] introduced the geometrically defined class UCV of uniformly convex functions on the unit disk; he established some theorems for this class. Recently, some mathematicians showed one-variable characterization for function in UCV which are closely related to Goodman's characterizations, for example Ma and Minda[2], Ronning[3].

In this short paper, we give a examples and conjectured necessary and sufficient condition of the class UCV_p.

1 Introduction

Let A denote the class of functions of the form

(1.1) $$f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

which are analytic in the open unit disk $U=\{z:|z|<1\}$. Let S denote the class of normalized analytic and univalent functions in U. We denote the subclass of S as follows :

(1.2) $$K=\left\{f(z)\in A:\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0,\qquad z\in U\right\}.$$

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K is called the usual class of convex functions.

Definition 1. *A function f is said to be uniformly convex in U if $f(z)$ is a normalized $(f(0) = f'(0) - 1 = 0)$ convex function and has the property that for every circular arc γ contained in U, with center also in U, the image arc $f(\gamma)$ is a convex arc.*

(1.3)
$$UCV = \left\{ f(z) \in K : \Re\left\{1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right\} \geq 0, \qquad ((z,\zeta) \in U \times U) \right\}$$

So far this two-variable characterization has not led to any sharp estimates for the class UCV. Ma and Minda introduced one-variable characterization for functions in UCV which are closely related to Goodman's characterizations.

Theorem A. *Assume that $f(z)$ is holomorphic and locally univalent in U with $f(0) = f'(0) - 1 = 0$. Then the following are equivalent:*

(i) $f(z) \in UCV$.

(ii) $\Re\left\{1 + \dfrac{zf''(z)}{f'(z)}\right\} > \left|\dfrac{zf''(z)}{f'(z)}\right| \qquad (z \in U)$.

2 A example

The following example will be useful. Goodman proved that

Theorem B. *The function*

$$f(z) = \frac{z}{1-Az} = z + \sum_{n=2}^{\infty} A^{n-1} z^n \tag{2.1}$$

is in UCV iff $|A| \leq \frac{1}{3}$.

It is clear from theorem B that the function $f(z) = \dfrac{z}{1-z}$ is not in UCV, but it is in K. We have

Example 1. *If* $0 < r \leq \frac{2}{7} \approx 0.2857 < 0.29$, *then the function*

$$f(z) = \frac{z}{1-z} = \sum_{n=1}^{\infty} z^n \tag{2.2}$$

is in UCV .

proof. A simple computation shows that for this function

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} - \left|\frac{zf''(z)}{f'(z)}\right| = \Re\left\{1 + \frac{2z}{1-z}\right\} - \left|\frac{2z}{1-z}\right|. \tag{2.3}$$

We set $z = re^{i\theta}$. For $0 < r \leq \frac{2}{7} \approx 0.2857 < 0.29$, we have

$$\frac{1 - r^2 - 2r\sqrt{2 - 2r\cos\theta}}{1 + r^2 - 2r\cos\theta} > 0$$

or

$$1 - r^2 - 2r\sqrt{2 - 2r\cos\theta} > 0. \tag{2.4}$$

∎

It is possible that in this example $\frac{2}{7}$ can be replaced by the larger constant.

Example 3. *The function(2.2) is in* UCV *iff* $0 < r \leq 1 - 2\rho, 0 < \rho \leq \frac{1}{2}$.

proof. A simple computation shows that for this function

$$\begin{aligned} Q(z,\zeta) &= 1 + \frac{(z-\zeta)f''(z)}{f'(z)} \\ &= \frac{1 + z - 2\zeta}{1-z}. \end{aligned} \tag{2.5}$$

We set $z = re^{i\theta}$ and $\zeta = \rho e^{i\phi}$. Then $\Re Q(z,\zeta) \geq 0$ iff

$$\Re(1 + re^{i\theta} - 2\rho e^{i\phi})(1 - re^{-i\theta}) \geq 0$$

or

$$1 - 2\rho\cos\phi - r^2 + 2r\rho\cos(\phi - \theta) \geq 0. \tag{2.6}$$

It is clear that the minimum of the expression on the left side of (2.6) occurs when $\phi = 0, \theta = \pi$.(Thus, $\zeta = \rho, z = -r$.) These values yield $1 - 2r\rho - 2\rho - r^2 \geq 0$, and this is true for $0 < r \leq 1 - 2\rho, 0 < \rho \leq \frac{1}{2}$. Thus, the condition is sufficient for (2.2) to be in *UCV*. By a limit argument, the condition is also necessary. ∎

3 Conjecture

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \qquad (p \in N = 1, 2, 3, \cdots) \tag{3.1}$$

which are analytic in $U = \{z : |z| < 1\}$.

A function $f(z) \in A_p$ is said to be $p-$valently convex iff

$$1 + \Re \frac{zf''(z)}{f'(z)} > 0 \qquad (z \in U). \tag{3.2}$$

We denote by K_p the subclass of A_p consisting of all $p-$ valently convex functions in U.

Using the idea contained in Theorem A, we pose the following conjecture.

Conjecture. *A function $f(z) \in A_p$ is said to be $p-$valent uniformly convex iff*

$$\Re \left\{ \frac{zf''(z)}{f'(z)} - (p-2) \right\} - \left| \frac{zf''(z)}{f'(z)} - (p-1) \right| > 0. \qquad (z \in U). \tag{3.3}$$

Also we denote by UCV_p the subclass of A_p consisting of all $p-$valent uniformly convex functions in U.

References

[1] A.W.Goodman. On uniformly convex functions. Ann. Polon. Math. **56**(1991), 87-92.

[2] W.Ma and D.Minda Uniformly convex functions. Ann. Polon. Math. **57**(1992), 165-175.

[3] F.Ronning. Uniformly convex functions and a corresponding class of starlike functions. Proc. Amer. Math. Soc. **118**(1993), 189-196.