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ON CERTAIN INTEGRAL TRANSFORMATIONS

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ABSTRACT

The object of the present paper is to derive some subordination properties of certain integral transformations of functions which are analytic in the open unit disk.

I. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathbb{U} = \{z: |z| < 1\}$. For functions $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$, we say that $f(z)$ is subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{U}$ which satisfies $w(0) = 0$, $|w(z)| < 1$ $(z \in \mathbb{U})$, and $f(z) = g(w(z))$. We denote this subordination by $f(z) \prec g(z)$. If $g(z)$ is univalent in $\mathbb{U}$, then this subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

For a function $f(z)$ belonging to $\mathcal{A}$, we define the following integral transformation $I(f(z))$ by

$$I(f(z)) = \left\{\frac{\alpha+\beta}{z^{\beta}} \int_0^z t^{\beta-1} f(t)^{\alpha} dt\right\} \qquad (z \in \mathbb{U}),$$

where $\alpha \in \mathbb{C}$, $\alpha \neq 0$, and $\beta \in \mathbb{C}$.

To derive some subordination properties of the integral transformations $I(f(z))$, we have to recall here the following lemmas.

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LEMMA 1 ([1]). Let $f(z) \in \mathcal{A}$, $g(z) \in \mathcal{A}$, and $g(z)$ be univalent in $\overline{U} = U \cup \partial U$. If there exists a point $z_0 \in U$ such that $f(|z|<|z_0|) \subset g(U)$ and $f(z_0) = g(\zeta_0)$ for $\zeta_0 \in \partial U$, then $z_0 f'(z_0) = m\zeta_0 g'(\zeta_0)$, where m is real and $m \geqq 1$.

LEMMA 2 ([2]). Let $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ be analytic in U with $p(z) \not\equiv 1$. Let the function $\Psi(u,v): \mathbb{C}^2 \longrightarrow \mathbb{C}$ $(u = u_1 + iu_2,$ $v = v_1 + iv_2)$ satisfy the following conditions

(i) $\Psi(u,v)$ is continuous in $D \subset \mathbb{C}^2$,

(ii) $(1,0) \in D$ and $\mathrm{Re}(\Psi(1,0)) > 0$,

(iii) for all $(iu_2, v_1) \in D$ such that $v_1 \leqq -(1+u_2^2)/2$, $\mathrm{Re}(\Psi(iu_2,v_1)) \leqq 0$.

If $(p(z), zp'(z)) \in D$ for $z \in U$ and $\mathrm{Re}(\Psi(p(z),zp'(z))) > 0$ for $z \in U$, then $\mathrm{Re}(p(z)) > 0$ for all $z \in U$.

A function $L(z,t)$ defined on $U \times [0,\infty)$ is said to be a subordination chain (or Loewner chain) if it satisfies

(i) $L(z,t)$ is analytic and univalent in U for all $t \geqq 0$,

(ii) $L(z,t)$ is continuously differentiable on $t \geqq 0$ for all $z \in U$,

(iii) $L(z,s) \prec L(z,t)$ for $0 \leqq s \leqq t$.

LEMMA 3 ([4]). The function $L(z,t)$ given by

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \dots \qquad (a_1(t) \neq 0;\ t \geqq 0)$$

is a subordination chain if and only if

$$\mathrm{Re}\left\{ \frac{z \dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}} \right\} > 0 \qquad (z \in U;\ t \geqq 0).$$

Further, we need

LEMMA 4 ([3]). Let $\alpha \in \mathbb{C}$, $\alpha \neq 0$, $\beta \in \mathbb{C}$, and let the function

$$h(z) = c + h_1 z + h_2 z^2 + \dots$$

be analytic in U. If the function $h(z)$ satisfies

$$\operatorname{Re}(\alpha h(z) + \beta) > 0 \qquad (z \in \mathbb{U}),$$

then the solution of the Briot-Bouquet differential equation

$$q(z) + \frac{zq'(z)}{\alpha q(z) + \beta} = h(z) \qquad (h(0) = q(0) = c)$$

is analytic in $\mathbb{U}$ and $\operatorname{Re}(\alpha q(z) + \beta) > 0$ $(z \in \mathbb{U})$.

2. Main Theorem

We begin with the statement and the proof of the following main result.

THEOREM. Let $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{A}$. If

(i) $\operatorname{Re}(\alpha+\beta) > 0$,

(ii) $g(z)/z \neq 0$ $(z \in \mathbb{U})$, and $I(g(z))/z \neq 0$ $(z \in \mathbb{U})$ for $\alpha \neq 1$,

(iii) $\phi(z) = (g(z)/z)^{\alpha}$ satisfies

$$\operatorname{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\delta \qquad (z \in \mathbb{U})$$

with

$$-1 < \delta \leq \frac{1 + |\alpha+\beta|^2 - \sqrt{(1 + |\alpha+\beta|^2)^2 - 4(\operatorname{Re}(\alpha+\beta))^2}}{4\operatorname{Re}(\alpha+\beta)},$$

then the subordination

$$\frac{f(z)}{z} \prec \frac{g(z)}{z}$$

implies

$$\frac{I(f(z))}{z} \prec \frac{I(g(z))}{z}.$$

PROOF. We define $F(z) = (I(f(z))/z)^{\alpha}$ and $G(z) = (I(g(z))/z)^{\alpha}$. Without loss of generality, we may assume that $G(z)$ is analytic and univalent in $\overline{\mathbb{U}} = \mathbb{U} \cup \partial\mathbb{U}$. Otherwise, we consider $F(rz)/r$ and $G(rz)/r$ $(0 < r < 1)$ instead of $F(z)$ and $G(z)$, respectively.

We first prove that if $q(z) = 1 + zG''(z)/G'(z)$, then $\operatorname{Re}(q(z)) > 0$ $(z \in \mathbb{U})$.

Since

$$I(g(z)) = \left\{ \frac{\alpha+\beta}{z^{\beta}} \int_0^z t^{\beta-1} g(t)^{\alpha} dt \right\}^{1/\alpha},$$

we have

$$\alpha \frac{z(I(g(z)))'}{I(g(z))} = -\beta + (\alpha+\beta) \frac{\phi(z)}{G(z)} .$$

Also we have

$$\alpha \frac{z(I(g(z)))'}{I(g(z))} = \alpha + \frac{zG'(z)}{G(z)} .$$

Thus

$$(\alpha+\beta)\phi(z) = (\alpha+\beta)G(z) + zG'(z) .$$

Differentiating both sides the above, we see that

$$\beta z\phi'(z) = zG'(z)\left\{\alpha + \beta + 1 + \frac{zG''(z)}{G'(z)}\right\}$$

$$= zG'(z)(q(z) + \alpha + \beta) .$$

Further, making the logarithmic differentiation of the above, we get

$$1 + \frac{z\phi''(z)}{\phi'(z)} = 1 + \frac{zG''(z)}{G'(z)} + \frac{zq'(z)}{q(z) + \alpha+\beta}$$

$$= q(z) + \frac{zq'(z)}{q(z) + \alpha+\beta}$$

$$\equiv h(z) .$$

Note that $q(0) = h(0) = 1$ and $2\delta \leqq \mathrm{Re}(\alpha+\beta)$. Therefore,

$$\mathrm{Re}(h(z)+\alpha+\beta) = \mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)} + \alpha + \beta\right\}$$

$$\geqq -\delta + \mathrm{Re}(\alpha+\beta)$$

$$\geqq \frac{1}{2} \mathrm{Re}(\alpha+\beta)$$

$$> 0.$$

Using Lemma 4, we have that $q(z)$ is analytic in $\mathbb{U}$ and $\mathrm{Re}(q(z)+\alpha+\beta) > 0$ $(z \in \mathbb{U})$.

Let us define the function $\Psi(u,v)$ by

$$\Psi(u,v) = u + \frac{v}{u+\alpha+\beta} + \delta$$

with $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Then $\Psi(u,v)$ satisfies

(i) $\Psi(u,v)$ is continuous in $\mathbb{D} = (\mathbb{C}-\{-\alpha-\beta\}) \times \mathbb{C} \subset \mathbb{C}^2$,

(ii) $(1,0) \in \mathbb{D}$ and $\mathrm{Re}(\Psi(1,0)) = 1 + \delta > 0$,

(iii) for all $(iu_2, v_1) \in \mathbb{D}$ such that $v_1 \leqq -(1+u_2^2)/2$,

$$\begin{aligned}
\mathrm{Re}(\Psi(iu_2,v_1)) &= \mathrm{Re}\left\{\frac{v_1}{iu_2+\alpha+\beta}\right\} + \delta \\
&= \delta - \frac{v_1 \mathrm{Re}(\alpha+\beta)}{|\alpha+\beta|^2 + 2\mathrm{Im}(\alpha+\beta)u_2 + u_2^2} \\
&\leqq \delta - \frac{(1+u_2^2)\mathrm{Re}(\alpha+\beta)}{2(|\alpha+\beta|^2 + 2\mathrm{Im}(\alpha+\beta)u_2 + u_2^2)} \\
&= \frac{(2\delta - \mathrm{Re}(\alpha+\beta))u_2^2 + 4\delta\mathrm{Im}(\alpha+\beta)u_2 + 2\delta|\alpha+\beta|^2 - \mathrm{Re}(\alpha+\beta)}{2(|\alpha+\beta|^2 + 2\mathrm{Im}(\alpha+\beta)u_2 + u_2^2)}
\end{aligned}$$

Define $E_\delta(u_2)$ by

$$E_\delta(u_2) = (2\delta - \mathrm{Re}(\alpha+\beta))u_2^2 + 4\delta\mathrm{Im}(\alpha+\beta)u_2 + 2\delta|\alpha+\beta|^2 - \mathrm{Re}(\alpha+\beta).$$

Then, it is easy to see that $2\delta - \mathrm{Re}(\alpha+\beta) < 0$ and $2\delta|\alpha+\beta|^2 - \mathrm{Re}(\alpha+\beta) \leqq 0$. The discrimination Δ of $E_\delta(u_2)$ is

$$\begin{aligned}
\Delta &= 4(\mathrm{Im}(\alpha+\beta))^2\delta^2 - (2\delta - \mathrm{Re}(\alpha+\beta))(2\delta|\alpha+\beta|^2 - \mathrm{Re}(\alpha+\beta)) \\
&= -4(\mathrm{Re}(\alpha+\beta))^2\delta^2 + 2\mathrm{Re}(\alpha+\beta)(1+|\alpha+\beta|^2)\delta - (\mathrm{Re}(\alpha+\beta))^2 \\
&\leqq 0
\end{aligned}$$

because

$$-1 < \delta \leqq \frac{1 + |\alpha+\beta|^2 - \sqrt{(1+|\alpha+\beta|^2)^2 - 4(\mathrm{Re}(\alpha+\beta))^2}}{4\mathrm{Re}(\alpha+\beta)}$$

This implies that $E_\delta(u_2) \leqq 0$, that is, that $\mathrm{Re}(\Psi(iu_2, v_1)) \leqq 0$. Further, we see that

$$\mathrm{Re}(\Psi(q(z), zq'(z))) = \mathrm{Re}\left\{q(z) + \frac{zq'(z)}{q(z)+\alpha+\beta} + \delta\right\}$$

$$= \mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)} + \delta\right\}$$

$$> 0.$$

Therefore, an application of Lemma 2 gives us that $\mathrm{Re}(q(z)) > 0$ $(z \in \mathbb{U})$.

Next, we prove that if $f(z)/z \prec g(z)/z$, then $F(z) \prec G(z)$.

Let us define the function $L(z,t)$ by

$$L(z,t) = G(z) + \frac{1+t}{\alpha+\beta} zG'(z) \qquad (z \in \mathbb{U};\ t \geqq 0).$$

Noting that $G'(0) = 1$, we see that

$$\left.\frac{\partial L(z,t)}{\partial z}\right|_{z=0} = G'(0)\left\{1 + \frac{1+t}{\alpha+\beta}\right\}$$

$$= 1 + \frac{1+t}{\alpha+\beta}$$

$$\neq 0.$$

This implies that if

$$L(z,t) = a_1(t)z + a_2(t)z^2 + \dots \qquad (z \in \mathbb{U},\ t \geqq 0),$$

then $a_1(t) \neq 0$ for all $t \geqq 0$. Further, we know that

$$\mathrm{Re}\left\{\frac{z\dfrac{\partial L(z,t)}{\partial z}}{\dfrac{\partial L(z,t)}{\partial t}}\right\} = \mathrm{Re}\left\{\alpha + \beta + (1+t)\left[1 + \frac{zG''(z)}{G'(z)}\right]\right\}$$

$$= \mathrm{Re}(q(z) + \alpha + \beta) + t\mathrm{Re}(q(z))$$

$$> 0$$

for all $z \in \mathbb{U}$. Therefore, it follows from Lemma 3 that $L(z,t)$ is the subordination chain. Thus we have

$$\phi(z) = L(z,0) \prec L(z,t) \qquad (t \geqq 0)$$

by the definition of the subordination chain.

Suppose that $F(z) \not\prec G(z)$. Then there exists a point $z_0 \in U$ such that $F(|z|<|z_0|) \subset G(U)$ and $F(z_0) = G(\zeta_0)$ $(\zeta_0 \in \partial U)$. This means that $L(\zeta_0,t) \notin L(U,t)$. Since, by Lemma 1,

$$z_0F'(z_0) = (1+t)\zeta_0G'(\zeta_0) \qquad (t \geqq 0),$$

we have

$$L(\zeta_0,t) = G(\zeta_0) + \frac{1+t}{\alpha+\beta}\zeta_0G'(\zeta_0)$$

$$= F(z_0) + \frac{1}{\alpha+\beta}z_0F'(z_0)$$

$$= \left(\frac{f(z_0)}{z_0}\right)^{\alpha},$$

so $L(\zeta_0,t) \in \phi(U)$, for $f(z)/z \prec g(z)/z$. This contradicts that $L(\zeta_0,t) \notin L(U,t)$. Thus we prove $F(z) \prec G(z)$.

Finally, note that

$$\frac{I(g(z))}{z} = 1 + c_1z + c_2z^2 + \ldots \neq 0 \quad (z \in U)$$

for $\alpha \neq 1$. This proves that if $F(z) \prec G(z)$, then $I(f(z))/z \prec I(G(z))/z$. Thus we complete the proof of our main theorem.

Making $\alpha+\beta = 1$ in Theorem, we have

COROLLARY 1. Let $f(z)$ and $g(z)$ be in the class A. If

(i) $g(z)/z \neq 0$ $(z \in U)$, and $I(g(z))/z \neq 0$ $(z \in U)$ when $\alpha \neq 1$,

(ii) $\phi(z) = (g(z)/z)^{\alpha}$ satisfies

$$\mathrm{Re}\left\{1 + \frac{z\phi''(z)}{\phi'(z)}\right\} > -\frac{1}{2} \qquad (z \in U),$$

then $f(z)/z \prec g(z)/z$ implies $I(f(z))/z \prec I(g(z))/z$, where

$$I(f(z)) = \left\{ \frac{1}{z^{1-\alpha}} \int_0^z \left(\frac{f(t)}{t} \right)^\alpha dt \right\}^{1/\alpha} .$$

Letting $\alpha+\beta = 1+i$ in our theorem,

COROLLARY 2. Let $f(z)$ and $g(z)$ be in the class A. If

(i) $g(z)/z \neq 0$ ($z \in U$), and $I(g(z))/z \neq 0$ ($z \in U$) when $\alpha \neq 1$,

(ii) $\phi(z) = (g(z)/z)^\alpha$ satisfies

$$\mathrm{Re}\left\{ 1 + \frac{z\phi''(z)}{\phi'(z)} \right\} > \frac{\sqrt{5} - 3}{4} \qquad (z \in U),$$

then $f(z)/z \prec g(z)/z$ implies $I(f(z))/z \prec I(g(z))/z$, where

$$I(f(z)) = \left\{ \frac{1+i}{z^{1-\alpha+i}} \int_0^z t^{i-\alpha} f(t)^\alpha dt \right\}^{1/\alpha} .$$

COROLLARY 3. If $f(z) \in A$ satisfies

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^\lambda} ,$$

then

$$\frac{I(f(z))}{z} \prec {}_2F_1(\alpha+\beta, \lambda\alpha; \alpha+\beta+1; z)^{1/\alpha} ,$$

where $\alpha > 0$,

$$1 - \frac{1 + |\alpha+\beta|^2 - \sqrt{(1+|\alpha+\beta|^2)^2 - 4(\mathrm{Re}(\alpha+\beta))^2}}{2\mathrm{Re}(\alpha+\beta)} \leqq \lambda\alpha < 3,$$

and ${}_2F_1(a,b;c;z)$ means the hypergeometric function.

PROOF. Let $g(z) = z/(1-z)^\lambda$ in Theorem, then

$$I(g(z)) = \left\{ \frac{\alpha+\beta}{z^\beta} \int_0^z t^{\beta-1} t^\alpha (1-t)^{-\lambda\alpha} dt \right\}^{1/\alpha}$$

$$= \left\{ (\alpha+\beta) z^\alpha \int_0^1 u^{\alpha+\beta-1} (1-zu)^{-\lambda\alpha} du \right\}^{1/\alpha}$$

$$= z\,{}_2F_1(\alpha+\beta,\lambda\alpha;\alpha+\beta+1;z)^{1/\alpha}.$$

Therefore, we have

$$\frac{I(f(z))}{z} \prec {}_2F_1(\alpha+\beta,\lambda\alpha;\alpha+\beta+1;z)^{1/\alpha}.$$

Taking $\alpha+\beta = 1$ in Corollary 3, we have

EXAMPLE 1. If $f(z) \in \mathcal{A}$ satisfies

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{\lambda}},$$

then

$$\frac{I(f(z))}{z} \prec {}_2F_1(1,\lambda\alpha;2;z)^{1/\alpha},$$

where $\alpha > 0$ and $0 \leqq \lambda\alpha < 3$.

If we make $\alpha+\beta = 1+i$ in Corollary 3, then we have

EXAMPLE 2. If $f(z) \in \mathcal{A}$ satisfies

$$\frac{f(z)}{z} \prec \frac{1}{(1-z)^{\lambda}},$$

then

$$\frac{I(f(z))}{z} \prec {}_2F_1(1+i,\lambda\alpha;2+i;z)^{1/\alpha},$$

where $\alpha > 0$ and $(\sqrt{5}-1)/2 \leqq \lambda\alpha < 3$.

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