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ON CERTAIN SUBCLASSES OF UNIVALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. The object of the present paper is to derive several interesting properties of the class $P^*(n,\alpha,\beta)$ consisting of analytic and univalent functions with negative coefficients. Coefficient estimates, distortion theorems and closure theorems of functions in the class $P^*(n,\alpha,\beta)$ are determined. Also radii of close - to- convexity, starlikeness and convexity for the class $P^*(n,\alpha,\beta)$ are determined. Also modified Hadamard product of sevral functions belonging to the class $P^*(n,\alpha,\beta)$ are studied here.

KEY WORDS - Analytic, modified Hadamard product.

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1. Introduction

Let S denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic and univalent in the unit disc $U=\{z: |z|<1\}$. For a function f(z) in S, we define

$$D^0 f(z) = f(z), \tag{1.2}$$

$$D^1 f(z) = Df(z) = zf'(z), \tag{1.3}$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \qquad (n \in N = \{1,2,\dots\}). \tag{1.4}$$

The differential operator D^n was introduced by Salagean [3]. With the help of the differential operator D^n, we say that a function f(z) belonging to S is in the class $S(n,\alpha,\beta)$ if and only if

$$\left| \frac{\frac{D^n f(z)}{z} - 1}{\frac{D^n f(z)}{z} + 1 - 2\alpha} \right| < \beta \qquad (n \in N_0 = N \cup \{0\}) \tag{1.5}$$

for $0 \le \alpha < 1$, $0 < \beta \le 1$, and for all $z \in U$.

Let T denote the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \qquad (a_k \ge 0). \tag{1.6}$$

Further, we define the class $P^*(n,\alpha,\beta)$ by

$$P^*(n,\alpha,\beta) = S(n,\alpha,\beta) \cap T. \tag{1.7}$$

We note that, by specializing the parameters n, α, and β, we obtain the following subclasses studied by various authors:

(i) $P^*(0,\alpha,\beta) = P^*(\alpha,\beta)$ (Srivastava and Owa [6]);

(ii) $P^*(1,\alpha,\beta) = P^*(\alpha,\beta)$ (Gupta and Jain [2]);

(iii) $P^*(0,\alpha,0) = P^{**}(\alpha)$ (Sarangi and Uralegaddi [4]);

(iv) $P^*(1,\alpha,1) = T^{**}(\alpha)$ (Sarangi and Uralegaddi [4] and Al-Amiri [1]).

2. Coefficient Estimates

Theorem 1. Let the function $f(z)$ be defined by (1.6). Then $f(z) \in P^*(n,\alpha,\beta)$ if and only if

$$\sum_{k=2}^{\infty} (1+\beta)k^n a_k \le 2\beta(1-\alpha). \tag{2.1}$$

The result is sharp.

Proof. Assume that the inequality (2.1) holds true and let $|z| = 1$. Then, we have

$$\left| \frac{D^n f(z)}{z} - 1 \right| - \beta \left| \frac{D^n f(z)}{z} + 1 - 2\alpha \right|$$

$$= \left| - \sum_{k=2}^{\infty} k^n a_k z^{k-1} \right| - \beta \left| 2(1-\alpha) - \sum_{k=2}^{\infty} k^n a_k z^{k-1} \right|$$

$$\le \sum_{k=2}^{\infty} (1+\beta) k^n a_k - 2\beta(1-\alpha) \le 0 .$$

Hence, by the maximum modulus theorem, we have $f(z) \in P^*(n,\alpha,\beta)$.

For the converse, assume that

$$\left| \frac{\frac{D^n f(z)}{z} - 1}{\frac{D^n f(z)}{z} + 1-2\alpha} \right| = \left| \frac{-\sum_{k=2}^{\infty} k^n a_k z^{k-1}}{2(1-\alpha) - \sum_{k=2}^{\infty} k^n a_k z^{k-1}} \right| < \beta. \tag{2.2}$$

Since $|\mathrm{Re}(z)| \leq |z|$ for all z, we find from (2.2) that

$$\mathrm{Re} \left\{ \frac{\sum_{k=2}^{\infty} k^n a_k z^{k-1}}{2(1-\alpha) - \sum_{k=2}^{\infty} k^n a_k z^{k-1}} \right\} < \beta. \tag{2.3}$$

Choose values of z on the real axis so that $\frac{D^n f(z)}{z}$ is real. Upon clearing the denominator in (2.3) and letting $z \longrightarrow 1^-$ through real values, we have

$$\sum_{k=2}^{\infty} k^n a_k \leq 2\beta(1-\alpha) - \beta \sum_{k=2}^{\infty} k^n a_k , \tag{2.4}$$

which gives the required assertion (2.1).

Finally, we note that the assertion (2.1) of Theorem 1 is sharp, the extremal function being

$$f(z) = z - \frac{2\beta(1-\alpha)}{(1+\beta)k^n} z^k \quad (k \geq 2). \tag{2.5}$$

Corollary 1. Let the function $f(z)$ defined by (1.6) be in the class $P^*(\alpha,\beta,\gamma)$. Then we have

$$a_k \le \frac{2\beta(1-\alpha)}{(1+\beta)k^n} \qquad (k\ge 2). \tag{2.6}$$

The equality in (2.6) is attained for the function f(z) given by (2.5).

3. Further Properties of the Class $P^*(n,\alpha,\beta)$

Theorem 2. Let $0 \le \alpha < 1$, $0 < \beta \le 1$, and $n \in N_0$. Then

$$P^*(n,\alpha,\beta) = P^*\left(n, \frac{1-\beta+2\alpha\beta}{1+\beta}, 1\right). \tag{3.1}$$

More generally, if $0 \le \alpha' < 1$, $0 < \beta' \le 1$, and $n \in N_0$, then

$$P^*(n,\alpha,\beta) = P^*(n,\alpha',\beta') \tag{3.2}$$

if and only if

$$\frac{\beta(1-\alpha)}{1+\beta} = \frac{\beta'(1-\alpha')}{1+\beta'} . \tag{3.3}$$

Proof. Frist assume that the function f(z) is in the class $P^*(n,\alpha,\beta)$, and let the condition (3.3) holds true. Then, by using the assertion (2.1) of Theorem 1, we readily have

$$\sum_{k=2}^{\infty} k^n a_k \le \frac{2\beta(1-\alpha)}{1+\beta} = \frac{2\beta'(1-\alpha')}{1+\beta'},$$

which shows that $f(z) \in P^*(n,\alpha',\beta')$, again with the aid of Theorem 1.

Reversing the above steps,we can similarly prove the other part of the equivalence (3.2) which, for $\beta'=1$, immediately yields the special case (3.1).

Conversely, the assertion (3.2) can easily be shown to imply the condition (3.3), and the proof of Theorem 2 is thus completed.

Theorem 3. Let $0 \le \alpha_1 \le \alpha_2 < 1$, $0 < \beta \le 1$, and $n \in N_0$. Then

$$P^*(n,\alpha_2,\beta) \le P^*(n,\alpha_1,\beta). \tag{3.4}$$

The proof of Theorem 3 uses Theorem 1 in a straightforward manner. The details may be aomitted .

Theorem 4. Let $0 \le \alpha < 1$, $0 < \beta_1 \le \beta_2 \le 1$, and $n \in N_0$. Then

$$P^*(n,\alpha,\beta_1) \le P^*(n,\alpha,\beta_2) \ . \tag{3.5}$$

Proof. By using Theorem 2, we obtain

$$P^*(n,\alpha,\beta_1) = P^*(n, \frac{1-\beta_1+2\alpha\beta_1}{1+\beta_1},1) \tag{3.6}$$

and

$$P^*(n,\alpha,\beta_2) = P^*(n,\frac{1-\beta_2+2\alpha\beta_2}{1+\beta_2},1) \ . \tag{3.7}$$

Furthermore

$$0 \le \frac{1-\beta_2+2\alpha\beta_2}{1+\beta_2} \le \frac{1-\beta_1+2\alpha\beta_1}{1+\beta_1} < 1 \tag{3.8}$$

for $0 \le \alpha < 1$ and $0 < \beta_1 \le \beta_2 \le 1$.

Consequently, by using Theorem 3, we arrive at our assertion (3.5).

Corollary 2. Let $0 \le \alpha_1 \le \alpha_2 < 1$, $0 < \beta_1 \le \beta_2 \le 1$, and $n \in N_0$. Then

$$P^*(n,\alpha_2,\beta_1) \le P^*(n,\alpha_1,\beta_1) \le P^*(n,\alpha_1,\beta_2).$$

Corollary 3. $P^*(n+1,\alpha,\beta) \subset P^*(n,\alpha,\beta)$

for $0 \le \alpha < 1$, $0 < \beta \le 1$, and $n \in N_0$.

4. Distortion Theorem

Theorem 5. Let the function $f(z)$ defined by (1.6) be in the class $P^*(n,\alpha,\beta)$. Then we have

$$|z|-\frac{\beta(1-\alpha)}{2^{n-i-1}(1+\beta)}|z|^2\leq\left|D^i f(z)\right|\leq|z|+\frac{\beta(1-\alpha)}{2^{n-i-1}(1+\beta)}|z|^2 \quad (4.1)$$

for $z\in U$, where $0\leq i\leq n$. The result is sharp.

PROOF. Note that $f(z)\in P^*(n,\alpha,\beta)$ if and only if $D^i f(z) \in P^*(n-i,\alpha,\beta)$, and that

$$D^i f(z) = z-\sum_{k=2}^{\infty} k^i a_k z^k. \quad (4.2)$$

Using Theorem 1, we know that

$$2^{n-i}(1+\beta)\sum_{k=2}^{\infty} k^i a_k \leq \sum_{k=2}^{\infty}(1+\beta)k^n a_k \leq 2\beta(1-\alpha), \quad (4.3)$$

that is, that

$$\sum_{k=2}^{\infty} k^i a_k \leq \frac{\beta(1-\alpha)}{2^{n-i-1}(1+\beta)}. \quad (4.4)$$

It follows from (4.2) and (4.4) that

$$\left|D^i f(z)\right| \geq |z| - |z|^2\sum_{k=2}^{\infty} k^i a_k$$

$$\geq |z| - \frac{\beta(1-\alpha)}{2^{n-i-1}(1+\beta)}|z|^2 \quad (4.5)$$

and

$$\left|D^i f(z)\right| \leq |z| + |z|^2\sum_{k=2}^{\infty} k^i a_k$$

$$\leq |z| + \frac{\beta(1-\alpha)}{2^{n-i-1}(1+\beta)}|z|^2. \quad (4.6)$$

Finally, we note that the equality in (4.1) is attained by the function

$$D^i f(z) = z - \frac{\beta(1-\alpha)}{2^{n-i-1}(1+\beta)} z^2 \tag{4.7}$$

or by

$$f(z) = z - \frac{\beta(1-\alpha)}{2^{n-1}(1+\beta)} z^2. \tag{4.8}$$

Corollary 4. Let the function f(z) defined by (1.6) be in the class $P^*(n,\alpha,\beta)$. Then we have

$$|z| - \frac{\beta(1-\alpha)}{2^{n-1}(1+\beta)}|z|^2 \le |f(z)| \le |z| + \frac{\beta(1-\alpha)}{2^{n-1}(1+\beta)}|z|^2 \tag{4.9}$$

for $z \in U$. The result is sharp for the function f(z) given by (4.8).

Proof. Taking i=0 in Theorem 5, we can easily show (4.9).

Corollary 5. let the function f(z) defined by (1.6) be in the class $P^*(n,\alpha,\beta)$. Then we have

$$1 - \frac{\beta(1-\alpha)}{2^{n-1}(1+\beta)}|z| \le |f'(z)| \le 1 + \frac{\beta(1-\alpha)}{2^{n-1}(1+\beta)}|z| \tag{4.10}$$

for $z \in U$. The result is sharp for the function f(z) given by (4.8).

Proof. Note that $D^1 f(z) = zf'(z)$. Hence, taking i=1 in Theorem 5, we have the corollary.

Corollary 6. Let the function f(z) defined by (1.6) be in the class $P^*(n,\alpha,\beta)$. Then f(z) is included in a disc with its center at the origin and radius R_1 given by

$$R_1 = \frac{2^{n-1}(1+\beta)+\beta(1-\alpha)}{2^{n-1}(1+\beta)} . \tag{4.11}$$

Further, $f'(z)$ is included in a disc with its center at the origin and radius R_2 given by

$$R_2 = \frac{2^{n-2}(1+\beta)+\beta(1-\alpha)}{2^{n-2}(1+\beta)} . \tag{4.12}$$

The result is sharp with the extremal function $f(z)$ given by (4.8).

5. Closure Theorems

Let the functions $f_j(z)$ be defined, for $j=1,2,...,m$, by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{k,j} z^k \qquad (a_{k,j} \geq 0) \tag{5.1}$$

for $z \in U$.

We shall prove the following results for the closure of functions in the class $P^*(n,\alpha,\beta)$.

THEOREM 6. Let the functions $f_j(z)$ $(j=1,2,...,m)$ defined by (5.1) be in the class $P^*(n,\alpha,\beta)$. Then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=2}^{\infty} b_k z^k \tag{5.2}$$

also belongs to the class $P^*(n,\alpha,\beta)$, where

$$b_k = \frac{1}{m} \sum_{j=1}^{m} a_{k,j} . \tag{5.3}$$

PROOF. Since $f_j(z) \in P^*(n,\alpha,\beta)$, it follows from Theorem 1, that

$$\sum_{k=2}^{\infty} (1+\beta) k^n a_{k,j} \leq 2\beta(1-\alpha), \qquad j=1,2,...,m. \tag{5.4}$$

Therefore,

$$\sum_{k=2}^{\infty}(1+\beta)k^n b_k = \sum_{k=2}^{\infty}(1+\beta)k^n\left(\frac{1}{m}\sum_{j=1}^{m}a_{k,j}\right)$$

$$\leq 2\beta(1-\alpha). \tag{5.5}$$

Hence by Theorem 1, $h(z) \in P^*(n,\alpha,\beta)$. Thus we have the theorem.

Theorem 7. Let the functions $f_j(z)$ defined by (5.1) be in the classes $P^*(n,\alpha_j,\beta)$ for each $j=1,2,\ldots,m$. Then the function $h(z)$ defined by

$$h(z) = z - \frac{1}{m}\sum_{k=2}^{\infty}\left(\sum_{j=1}^{m}a_{k,j}\right)z^k \tag{5.6}$$

is in the class $P^*(n,\alpha,\beta)$, where

$$\alpha = \min_{1\leq j\leq m}\{\alpha_j\}. \tag{5.7}$$

Proof. Since $f_j(z) \in P^*(n,\alpha_j,\beta)$ for each $j=1,2,\ldots,m$, we observe that

$$\sum_{k=2}^{\infty}(1+\beta)k^n a_{k,j} \leq 2\beta(1-\alpha_j) \tag{5.8}$$

with the aid of Theorem 1. Therefore

$$\sum_{k=2}^{\infty}(1+\beta)k^n\left(\frac{1}{m}\sum_{j=1}^{m}a_{k,j}\right) = \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{k=2}^{\infty}(1+\beta)k^n a_{k,j}\right)$$

$$\leq \frac{1}{m}\sum_{j=}^{m}2\beta(1-\alpha_j) \leq 2\beta(1-\alpha). \tag{5.9}$$

Thus

$$\sum_{k=2}^{\infty}(1+\beta)k^n\left[\frac{1}{m}\sum_{j=1}^{m}a_{k,j}\right]\leq 2\beta(1-\alpha) \tag{5.10}$$

which shows that $h(z) \in P^*(n,\alpha,\beta)$, where α is given by (5.7).

Theorem 8. Let the functions $f_j(z)$ defined by (5.1) be in the class $P^*(n,\alpha,\beta)$ for every $j=1,2,...,m$. Then the function $h(z)$ defined by

$$h(z) = \sum_{j=1}^{m} c_j f_j(z) \qquad (c_j \geq 0) \tag{5.11}$$

is in the class $P^*(n,\alpha,\beta)$, where

$$\sum_{j=1}^{m} c_j = 1. \tag{5.12}$$

Proof. According to the definition of $h(z)$, we can write

$$h(z) = z - \sum_{k=2}^{\infty}\left(\sum_{j=1}^{m} c_j a_{k,j}\right)z^k. \tag{5.13}$$

Further, since $f_j(z)$ are in $P^*(n,\alpha,\beta)$ for every $j=1,2,...,m$, we get

$$\sum_{k=2}^{\infty}(1+\beta)k^n a_{k,j} \leq 2\beta(1-\alpha) \tag{5.14}$$

for every $j=1,2,,...,m$. Hence we can see that

$$\sum_{k=2}^{\infty}(1+\beta)k^n\left(\sum_{j=1}^{m} c_j a_{k,j}\right) = \sum_{j=}^{m} c_j\left(\sum_{k=2}^{\infty}(1+\beta)k^n a_{k,j}\right)$$

$$\leq \left(\sum_{j=1}^{m} c_j\right) 2\beta(1-\alpha) = 2\beta(1-\alpha). \tag{5.15}$$

with the aid of (5.12). This proves that the function $h(z)$ is in the class $P^*(n,\alpha,\beta)$ by means of Theorem 1. Thus we have the theorem.

Theorem 9. The class $P^*(n,\alpha,\beta)$ is closed under convex linear combination.

Proof. Let the functions $f_j(z)$ $(j=1,2)$ defined by (5.1) be in the class $P^*(n,\alpha,\beta)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z)=\mu f_1(z)+(1-\mu)f_2(z) \qquad (0 \leq \mu \leq 1) \tag{5.16}$$

is in the class $P^*(n,\alpha,\beta)$. Since, for $0 \leq \mu \leq 1$,

$$h(z) = z - \sum_{k=2}^{\infty} [\mu a_{k,1} + (1-\mu)a_{k,2}]z^k, \tag{5.17}$$

with the aid of Theorem 1, we have

$$\sum_{k=2}^{\infty} (1+\beta)k^n[\mu a_{k,1} + (1-\mu)a_{k,2}] \leq 2\beta(1-\alpha) \tag{5.18}$$

which implies that $h(z) \in P^*(n,\alpha,\beta)$.

As a consequence of Theorem 9, there exists the extreme points of the class $P^*(n,\alpha,\beta)$.

Theorem 10. Let $f_1(z) = z$ and

$$f_k(z)= z - \frac{2\beta(1-\alpha)}{(1+\beta)k^n} z^k \quad (k \geq 2) \tag{5.19}$$

for $0 \leq \alpha < 1$, $0 < \beta \leq 1$, and $n \in N_0$. Then $f(z)$ is in the class

$P^*(n,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) \tag{5.20}$$

where $\mu_k \geq 0 \quad (k \geq 1)$ and $\sum_{k=1}^{\infty} \mu_k = 1$.

PROOF. Suppose that

$$f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z) = z - \sum_{k=2}^{\infty} \frac{2\beta(1-\alpha)}{(1+\beta)k^n} \mu_k z^k. \tag{5.21}$$

Then we get

$$\sum_{k=2}^{\infty} \frac{(1+\beta\gamma)k^n}{2\beta(1-\alpha)} \cdot \frac{2\beta(1-\alpha)}{(1+\beta)k^n} \mu_k = \sum_{k=2}^{\infty} \mu_k = 1-\mu_1 \leq 1. \tag{5.22}$$

By virtue of Theorem 1, this shows that $f(z) \in P^*(n,\alpha,\beta)$.

On the other hand, suppose that the function f(z) defined by (1.6) is in the class $P^*(n,\alpha,\beta)$. Again, by using Theorem 1,we can show that

$$a_k \leq \frac{2\beta(1-\alpha)}{(1+\beta)k^n} \quad (k \geq 2). \tag{5.23}$$

Setting

$$\mu_k = \frac{(1+\beta)k^n}{2\beta(1-\alpha)} a_k \quad (k \geq 2) , \tag{5.24}$$

and

$$\mu_1 = 1- \sum_{k=}^{\infty} \mu_k . \tag{5.25}$$

Hence, we can see that f(z) can be expressed in the form (5.20). This completes the proof of Theorem 10.

COROLLARY 7. The extreme points of the class $P^*(n,\alpha,\beta)$ are the functions $f_k(z)$ ($k \geq 1$) given by Theorem 10.

6. Radii of Close-to-Convexity, Starlikeness and Convexity

Theorem 11. Let the function f(z) defined by (1.6) be in the classs $P^*(n,\alpha,\beta)$, then f(z) is close-to-convex of order ρ $(0\leq\rho<1)$ in $|z|<r_1(n,\alpha,\beta,\rho)$, where

$$r_1(n,\alpha,\beta,\rho) = \inf_k \left\{ \frac{(1-\rho)(1+\beta)k^{n-1}}{2\beta(1-\alpha)} \right\}^{\frac{1}{k-1}} \quad (k\geq 2). \tag{6.1}$$

The result is sharp with the extremal function f(z) given by (2.5).

Proof. We must show that $|f'(z)-1|\leq 1-\rho$ for $|z|<r_1(n,\alpha,\beta,\rho)$. We have

$$|f'(z)-1| \leq \sum_{k=2}^{\infty} ka_k |z|^{k-1} .$$

Thus $|f'(z)-1| \leq 1-\rho$ if

$$\sum_{k=2}^{\infty} \left(\frac{k}{1-\rho} \right) a_k |z|^{k-1} \leq 1. \tag{6.2}$$

According to Theorem 1, we have

$$\sum_{k=2}^{\infty} \frac{(1+\beta)k^n}{2\beta(1-\alpha)} a_k \leq 1. \tag{6.3}$$

Hence (6.2) will be true if

$$\frac{k|z|^{k-1}}{(1-\rho)} \leq \frac{(1+\beta)k^n}{2\beta(1-\alpha)}$$

or if

$$|z| \leq \left\{ \frac{(1-\rho)(1+\beta)k^{n-1}}{2\beta(1-\alpha)} \right\}^{\frac{1}{k-1}} \quad (k \geq 2). \tag{6.4}$$

The theorem follows easily from (6.4).

Theorem 12. Let the function f(z) defined by (1.6) be in the class $P^*(n,\alpha,\beta)$, then f(z) is starlike of order ρ $(0\le\rho<1)$ in $|z|<r_2(n,\alpha,\beta,\rho)$, where

$$r_2(n,\alpha,\beta,\rho)=\inf_k\left\{\frac{(1-\rho)(1+\beta)k^n}{2(k-\rho)\beta(1-\alpha)}\right\}^{\frac{1}{k-1}}\quad(k\ge 2).\qquad(6.5)$$

The result is sharp with the extremal function f(z) given by (2.5).

Proof. It is sufficient to show that $\left|\frac{zf'(z)}{f(z)}-1\right|\le 1-\rho$ for $|z|<r_2(n,\alpha,\beta,\rho)$. We have

$$\left|\frac{zf'(z)}{f(z)}-1\right|\le\frac{\sum_{k=2}^{\infty}(k-1)a_k|z|^{k-1}}{1-\sum_{k=2}^{\infty}a_k|z|^{k-1}}.$$

Thus $\left|\frac{zf'(z)}{f(z)}-1\right|\le 1-\rho$ if

$$\sum_{k=2}^{\infty}\frac{(k-\rho)a_k|z|^{k-1}}{(1-\rho)}\le 1.\qquad(6.6)$$

Hence, by using (6.3), (6.6) will be true if

$$\frac{(k-\rho)|z|^{k-1}}{(1-\rho)}\le\frac{(1+\beta)k^n}{2\beta(1-\alpha)}$$

or if

$$|z|\le\left\{\frac{(1-\rho)(1+\beta)k^n}{2(k-\rho)\beta(1-\alpha)}\right\}^{\frac{1}{k-1}}\quad(k\ge 2).\qquad(6.7)$$

The theorem follows easily from (6.7).

Corollary 8. Let the function f(z) defined by (1.6) be in the class $P^*(n,\alpha,\beta)$, then $f(z)$ is convex of order ρ $(0\le\rho<1)$ in $|z|< r_3(n,\alpha,\beta,\rho)$, where

$$r_3(n,\alpha,\beta,\rho) = \inf_k \left\{ \frac{(1-\rho)(1+\beta)k^n}{2(k-\rho)\beta(1-\alpha)} \right\}^{\frac{1}{k-1}} \quad (k \ge 2). \tag{6.8}$$

The result is sharp with the extremal function $f(z)$ given by (2.5).

7. Integral Operators

THEOREM 13. Let the function $f(z)$ defined by (1.6) be in the class $P^*(n,\alpha,\beta)$, and let c be a real number such that $c>-1$. Then the function $F(z)$ defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t)\,dt \tag{7.1}$$

also belongs to the class $P^*(n,\alpha,\beta)$.

PROOF. From the representation of $F(z)$, it follows that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k, \tag{7.2}$$

where

$$b_k = \left(\frac{c+1}{c+k} \right) a_k . \tag{7.3}$$

Therefore,

$$\sum_{k=2}^{\infty} (1+\beta)k^n b_k = \sum_{k=2}^{\infty} (1+\beta)k^n \left(\frac{c+1}{c+k} \right) a_k$$

$$\le \sum_{k=2}^{\infty} (1+\beta)k^n a_k \le 2\beta(1-\alpha) , \tag{7.4}$$

since $f(z) \in P^*(n,\alpha,\beta)$. Hence, by Theorem 1, $F(z) \in P^*(n,\alpha,\beta)$.

Theorem 14. Let c be a real number such that $c>-1$. If $F(z) \in P^*(n,\alpha,\beta)$, then the function defined by (7.1) is univalent in $|z| < R^*$, where

$$R^* = \inf_k \left[\frac{(1+\beta)k^{n-1}(c+1)}{2\beta(1-\alpha)(c+k)} \right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.5}$$

The result is sharp.

Proof. Let $F(z) = z-\sum_{k=2}^{\infty} a_k z^k$ $(a_k \geq 0)$. It follows from (7.1) that

$$f(z) = \frac{z^{1-c}[z^c F(z)]'}{(c+1)}, \quad (c>-1)$$

$$= z - \sum_{k=2}^{\infty} \left(\frac{c+k}{c+1}\right) a_k z^k. \tag{7.6}$$

In order to obtain the required result it suffices to show that

$$|f'(z)-1| < 1 \quad \text{in} \quad |z| < R^* .$$

Now

$$|f'(z)-1| \leq 1 \text{ if}$$

$$\sum_{k=2}^{\infty} \frac{k(c+k)}{(c+1)} a_k |z|^{k-1} \leq 1 . \tag{7.7}$$

Hence by using (6.3), (7.7) will be satisfied if

$$\frac{k(c+k)|z|^{k-1}}{(c+1)} < \frac{(1+\beta)k^n}{2\beta(1-\alpha)} \quad (k \geq 2)$$

or if

$$|z| < \left[\frac{(1+\beta)k^{n-1}(c+1)}{2\beta(1-\alpha)(c+k)}\right]^{\frac{1}{k-1}} \quad (k \geq 2). \tag{7.8}$$

Therefore $f(z)$ is univalent in $|z| < R^*$. Sharpness follows if we take

$$f(z) = z - \frac{2\beta(1-\alpha)(c+k)}{(1+\beta)(c+1)k^n} z^k \quad (k \geq 2). \tag{7.9}$$

8. Modified Hadamard Products

Let the function $f_j(z)$ $(j=1,2)$ defined by (5.1) The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$f_1*f_2(z) = z - \sum_{k=2}^{\infty} a_{k,1}a_{k,2}\, z^k . \tag{8.1}$$

Theorem 15. Let the functions $f_j(z)$ $(j=1,2)$ defined by (5.1) be in the class $P^*(n,\alpha,\beta)$. Then $f_1*f_2(z)$ belongs to the class $P^*(n,\gamma(n,\alpha,\beta),\beta)$, where

$$\gamma(n,\alpha,\beta) = 1 - \frac{\beta(1-\alpha)^2}{2^{n-1}(1+\beta)} . \tag{8.2}$$

The result is sharp.

<u>**Proof.**</u> Employing the technique used earlier by Schild and Silverman [5], we need to find the largest $\gamma = \gamma(n,\alpha,\beta)$ such that

$$\sum_{k=2}^{\infty} \frac{(1+\beta)k^n}{2\beta(1-\gamma)} a_{k,1}a_{k,2} \leq 1. \tag{8.3}$$

Since

$$\sum_{k=2}^{\infty} \frac{(1+\beta)k^n}{2\beta(1-\alpha)} a_{k,1} \leq 1 \tag{8.4}$$

$$\sum_{k=2}^{\infty} \frac{(1+\beta)k^n}{2\beta(1-\alpha)} a_{k,2} \leq 1 \tag{8.5}$$

by the Cauchy-Schwarz we have

$$\sum_{k=2}^{\infty} \frac{(1+\beta)k^n}{2\beta(1-\alpha)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{8.6}$$

Thus it is sufficient to show that

$$\frac{(1+\beta)k^n}{2\beta(1-\gamma)} a_{k,1}a_{k,2} \leq \frac{(1+\beta)k^n}{2\beta(1-\alpha)} \sqrt{a_{k,1}a_{k,2}} \quad (k \geq 2), \tag{8.7}$$

that is, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{1-\alpha}{1-\gamma} . \tag{8.8}$$

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{2\beta(1-\alpha)}{k^n(1+\beta)} \quad (k \geq 2). \tag{8.9}$$

Consequently, we need only to prove that

$$\frac{2\beta(1-\alpha)}{k^n(1+\beta)} \leq \frac{1-\alpha}{1-\gamma} \quad (k \geq 2), \tag{8.10}$$

or, equivalently, that

$$\gamma \leq 1 - \frac{2\beta(1-\alpha)^2}{k^n(1+\beta)} \quad (k \geq 2). \tag{8.11}$$

Since

$$A(k) = 1 - \frac{2\beta(1-\alpha)^2}{k^n(1+\beta)} \tag{8.12}$$

is an increasing function of k ($k \geq 2$), letting $k=2$ in (8.12), we obtain

$$\gamma \leq A(2) = 1 - \frac{\beta(1-\alpha)^2}{2^{n-1}(1+\beta)} , \tag{8.13}$$

which completes the proof of Theorem 15.

Finally, by taking the functions $f_j(z)$ given by

$$f_j(z) = z - \frac{\beta(1-\alpha)}{2^{n-1}(1+\beta)} z^2 \quad (j=1,2), \tag{8.14}$$

we can see that the result is sharp.

Corollary 9. For $f_1(z)$ and $f_2(z)$ as in Theorem 15, we have

$$h(z) = z - \sum_{k=2}^{\infty} \sqrt{a_{k,1}\, a_{k,2}}\; z^k \tag{8.15}$$

belongs to the class $P^*(n,\alpha,\beta)$.

This result follows from the Cauchy- inequality (8.6). It is sharp for the same functions $f_j(z)$ $(j=1,2)$ as in Theorem 15.

Theorem 16. Let the function $f_1(z)$ defined by (5.1) be in the class $P^*(n,\alpha,\beta)$ and the function $f_2(z)$ defined by (5.1) be in the class $P^*(n,\tau,\beta)$. Then $f_1 * f_2(z)$ belongs to the class $P^*(n,\zeta(n,\alpha,\beta,\tau),\beta)$, where

$$\zeta(n,\alpha,\beta,\tau) = 1 - \frac{\beta(1-\alpha)(1-\tau)}{2^{n-1}(1+\beta)}. \tag{8.16}$$

The result is sharp.

Proof. Proceeding as in the proof of Theorem 15, we get

$$\zeta \le B(k) = 1 - \frac{2\beta(1-\alpha)(1-\tau)}{k^n(1+\beta)} \quad (k \ge 2). \tag{8.17}$$

Since the function $B(k)$ is an increasing function of k $(k \ge 2)$, letting $k=2$ in (8.17), we obtain

$$\zeta \le B(2) = 1 - \frac{\beta(1-\alpha)(1-\tau)}{2^{n-1}(1+\beta)}, \tag{8.18}$$

which evidently proves Theorem 16.

Finally, the result is best possible for the functions

$$f_1(z) = z - \frac{\beta(1-\alpha)}{2^{n-1}(1+\beta} z^2 \tag{8.19}$$

and

$$f_2(z) = z - \frac{\beta(1-\tau)}{2^{n-1}(1+\beta} z^2 . \tag{8.20}$$

Corollary 10. Let the functions $f_j(z)$ $(j=1,2,3)$ defined by (5.1) be in the class $P^*(n,\alpha,\beta)$. Then $f_1 * f_2 * f_3(z)$ belongs to the class $P^*(n,\eta(n,\alpha,\beta),\beta)$, where

$$\eta(n,\alpha,\beta) = 1 - \frac{\beta^2(1-\alpha)^3}{2^{2(n-1)}(1+\beta)^2}. \tag{8.21}$$

The result is best possible for the functions

$$f_j(z) = z - \frac{\beta(1-\alpha)}{2^{n-1}(1+\beta)} z^2 \quad (j=1,2,3). \tag{8.22}$$

Proof. From Theorem 15 , we have $f_1 * f_2(z) \in P^*(n,\gamma(n,\alpha,\beta),\beta)$, where γ is given by (8.2) . We now use Theorem 16, we get $f_1 * f_2 * f_3(z) \in P^*(n,\eta(n,\alpha,\beta),\beta)$, where

$$\eta(n,\alpha,\beta) = 1 - \frac{\beta(1-\alpha)(1-\gamma)}{2^{n-1}(1+\beta)}$$

$$= 1 - \frac{\beta^2(1-\alpha)^3}{2^{2(n-1)}(1+\beta)^2}.$$

This completes the proof of Corollary 10.

Theorem 17. Let the functions $f_j(z)$ $(j=1,2)$ defined by (5.1) be in the class $P^*(n,\alpha,\beta)$. Then the function

$$h(z) = z - \sum_{k=2}^{\infty} [a_{k,1}^2 + a_{k,2}^2] z^k, \tag{8.23}$$

belongs to the class $P^*(n,\phi(n,\alpha,\beta),\beta)$, where

$$\phi(n,\alpha,\beta) = 1 - \frac{\beta(1-\alpha)^2}{2^{n-2}(1+\beta)} . \tag{8.24}$$

The result is sharp for the functions $f_j(z)$ $(j=1,2)$ defined by (8.14).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=2}^{\infty} \left[\frac{(1+\beta)k^n}{2\beta(1-\gamma)}\right]^2 a_{k,1} \le \left[\sum_{k=2}^{\infty} \frac{(1+\beta)k^n}{2\beta(1-\alpha)} a_{k,1}\right]^2 \le 1 \tag{8.25}$$

and

$$\sum_{k=2}^{\infty} \left[\frac{(1+\beta)k^n}{2\beta(1-\gamma)}\right]^2 a_{k,2} \le \left[\sum_{k=2}^{\infty} \frac{(1+\beta)k^n}{2\beta(1-\alpha)} a_{k,2}\right]^2 \le 1. \tag{8.26}$$

It follows from (8.25) and (8.26) that

$$\sum_{k=2}^{\infty} \frac{1}{2} \left[\frac{(1+\beta)k^n}{2\beta(1-\gamma)}\right]^2 (a_{k,1}^2 + a_{k,2}^2) \le 1. \tag{8.27}$$

Therefore, we need to find the largest $\phi = \phi(n,\alpha,\beta)$ such that

$$\frac{(1+\beta)k^n}{2\beta(1-\phi)} \le \frac{1}{2}\left[\frac{(1+\beta)k^n}{2\beta(1-\gamma)}\right]^2 \quad (k \ge 2) \tag{8.28}$$

that is,

$$\phi \le 1 - \frac{4\beta(1-\alpha)^2}{k^n(1+\beta)} \quad (k \ge 2) . \tag{8.29}$$

Since

$$D(k) = 1 - \frac{4\beta(1-\alpha)^2}{k^n(1+\beta)}$$

is an increasing function of k ($k \ge 2$), we readily have

$$\phi \le D(2) = 1 - \frac{\beta(1-\alpha)^2}{2^{n-2}(1+\beta)} ,$$

and Theorem 17 follows at once.

Theorem 18. Let the functions $f_1(z)$ defined by (5.1) be in the class $P^*(n_1,\alpha,\beta)$ and the functions $f_2(z)$ defined by (5.1) be in the class $P^*(n_2,\alpha,\beta)$. Then $f_1*f_2(z) \in P^*(n_1,\alpha,\beta) \cap P^*(n_2,\alpha,\beta)$.

Proof. Since $f_2(z) \in P^*(n_2,\alpha,\beta)$, we have

$$a_{k,2} \le \frac{2\beta(1-\alpha)}{c_{2,2}}, \tag{8.30}$$

where

$$c_{k,j} = (1+\beta)k^{n_j}. \qquad (j=1,2) \tag{8.31}$$

From Theorem 1, since $f_1(z) \in P^*(n_1,\alpha,\beta)$, we have

$$\sum_{k=2}^{\infty} c_{k,1}a_{k,1} \le 2\beta(1-\alpha). \tag{8.32}$$

Now, from (8.30) and (8.32), we have

$$\sum_{k=2}^{\infty} c_{k,1}a_{k,1}a_{k,2} \le \frac{2\beta(1-\alpha)}{c_{2,2}} \sum_{k=2}^{\infty} c_{k,1}a_{k,1}$$

$$\leq \frac{[2\beta(1-\alpha)]^2}{c_{2,2}} \leq 2\beta(1-\alpha).$$

Since $\frac{2\beta(1-\alpha)}{c_{2,2}} \leq 1$. Hence $f_1 * f_2(z) \in P^*(n_1,\alpha,\beta)$. Interchanging n_1 and n_2 by each other in the above, we get $f_1 * f_2(z) \in P^*(n_2,\alpha,\beta)$. Hence the theorem.

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