数理解析研究所講究録
917巻 1995年 52-54

On Another Easy Proof of Owa's Result for Starlikeness

Mamoru Nunokawa(布川 護, 群馬大学)
Osamu Takei(武井 修, 群馬大学)
Kaoru Kosugi(小杉 薫, 群馬大学)

Abstract——Only recently, Owa[1] proved the following theorem.

If $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ is analytic in $|z|<1$,

$$|f'(z)-1|<\alpha \quad \text{for } |z|<1$$

and

$$\left|\arg\frac{f(z)}{z}\right| \leqq \mathrm{Tan}^{-1}\left(\frac{\sqrt{1-\alpha^2}}{\alpha}\right) \quad \text{for } |z|<1$$

where $\frac{2}{\sqrt{5}}<\alpha\leqq 1$, then $f(z)$ is starlike in $|z|<1$.

It is the purpose of the present paper to give an easy proof of the above result.

Keywords——Starlike and convex functions.

1. Intoroduction.

Let A denote the set of functions $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ that are analytic in $E=\{z:|z|<1\}$. A function $f(z)\in A$ is called starlike if and only if

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)}>0 \quad \text{in } E.$$

On the other hand, a function $f(z)\in A$ is called convex if and only if

$$1+\mathrm{Re}\,\frac{zf''(z)}{f'(z)}>0 \quad \text{in } E$$

It is easily proved that the necessary and sufficient condition for f(z) to be convex in E is that zf′(z) is starlike in E.

2. Main theorem.

Theorem 1

Let $f(z)\in A$ and suppose that

$$|f'(z)-1|<\alpha \quad \text{in } E \tag{1}$$

and

$$\left|\arg\frac{f(z)}{z}\right|\leqq \mathrm{Tan}^{-1}\left(\frac{\sqrt{1-\alpha^2}}{\alpha}\right) \quad \text{in } E \tag{2}$$

where $0<\alpha\leqq 1$.

Then f(z) is starlike in E.

Proof.

For the case $\alpha=1$, it is trivial. For the case $0<\alpha<1$, from the hypothesis (1), we easily have

$$|\arg f'(z)|\leqq \mathrm{Tan}^{-1}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right) \quad \text{in } E. \tag{3}$$

Then, from (2) and (3), we have

$$\begin{aligned}\left|\arg\frac{zf'(z)}{f(z)}\right| &\leqq \left|\arg\frac{z}{f(z)}\right| + |\arg f'(z)| \\ &< \left|\mathrm{Tan}^{-1}\left(\frac{\sqrt{1-\alpha^2}}{\alpha}\right)\right| + \left|\mathrm{Tan}^{-1}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\right)\right| \\ &= \frac{\pi}{2}\end{aligned}$$

Thus,

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)}>0 \quad \text{in } E.$$

This completes the proof.

Remark. Owa[1] proved Theorem 1 by using 3 Lemmas and with the restriction of α, $\frac{2}{\sqrt{5}}<\alpha\leqq 1$.

Our proof is simple, and generalized the result of [1, Theorem 2] for the case $0<\alpha\leqq 1$.

Applying Theorem 1, we have the following corollaries.

Corollary 1.
Let $f(z)\in A$ and suppose that

$$|f'(z)-1|<\alpha \quad \text{in } E$$

and

$$\left|\frac{f(z)}{z}-1\right|<\sqrt{1-\alpha^2} \quad \text{in } E$$

where $0<\alpha\leqq 1$.
Then $f(z)$ is starlike in E.

Corollary 2.
Let $f(z)\in A$ and suppose that

$$|f'(z)+zf''(z)-1|<\alpha \quad \text{in } E$$

and

$$|\arg f'(z)|\leqq \mathrm{Tan}^{-1}\left(\frac{\sqrt{1-\alpha^2}}{\alpha}\right) \quad \text{in } E$$

where $0<\alpha\leqq 1$.
Then $f(z)$ is convex in E.

Corollary 3.
Let $f(z)\in A$ and suppose that

$$|f'(z)+zf''(z)-1|<\alpha \quad \text{in } E$$

and

$$|f'(z)-1|<\sqrt{1-\alpha^2} \quad \text{in } E$$

where $0<\alpha\leqq 1$.
Then $f(z)$ is convex in E.

REFERENCE

1. S. Owa, On the conditions of starlikeness for analytic functions, Sugaku (Math. Soc. of Japan) Vol. 45(2), 180-182, (1994), (Japanese).