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On the General Index Transforms in L_p-Space

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Abstract

This paper is devoted to study index transforms under general constructions of kernels, which involve the known Kontorovich-Lebedev, the Mehler-Fock integral transforms and, further, the index transforms with Meijer's G-function and Fox's H-function as the kernels. Mapping properties and inversion theorem on the space $L_{\nu,p}(\mathbf{R}_+)\cap L_{\nu,1}(\mathbf{R}_+)$ with the norm

$$||f||_{\nu,p}=\left(\int_0^\infty t^{\nu p-1}|f(t)|^p dt\right)^{1/p}<\infty \quad (1\le p\le 2;\ \nu\in\mathbf{R})$$

are investigated. As an example, the generalized ${}_2F_1$-index transform of the Olevskii type is considered.

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1. Introduction

In this paper we deal with the so-called general index transform of the form

$$[Y_{i\tau}^{\varphi}f](\tau)=\tau\int_0^\infty Y_{i\tau}^{\varphi}(t)f(t)dt \quad (\tau>0), \tag{1.1}$$

where the kernel $Y_{i\tau}^{\varphi}(x)$ is the respective index kernel of the kind

$$Y_{i\tau}^{\varphi}(x)=\int_0^\infty K_{i\tau}(y)\varphi(xy)dy \quad (x>0), \tag{1.2}$$

involving the known Macdonald function $K_{i\tau}(x)$ [1] and an arbitrary characteristic function $\varphi(x)$ from some space $L_{\nu,p}(\mathbf{R}_+)$. The indicated Macdonald function with the imaginary index is the kernel of the known Kontorovich-Lebedev transform pair [2]

$$[\mathfrak{KL}f](\tau)\equiv g(\tau)=\tau\int_0^\infty K_{i\tau}(y)f(y)dy \quad (\tau>0), \tag{1.3}$$

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$$xf(x) = \frac{2}{\pi^2}\int_0^\infty \sinh(\pi\tau)K_{i\tau}(x)g(\tau)d\tau \quad (x>0). \tag{1.4}$$

As it is known, the Macdonald function has the expression [1]

$$K_{i\tau}(x) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-x\cosh\beta}e^{i\tau\beta}d\beta \quad (x>0). \tag{1.5}$$

By the analytic property of the integrand in (1.5) and by its asymptotic behavior at the contour we can shift its integral line to the horizontal open infinite strip $(i\delta-\infty, i\delta+\infty)$ with $\delta \in [0,\pi/2)$ as

$$K_{i\tau}(x) = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-x\cosh\beta}e^{i\tau\beta}d\beta \quad (x>0). \tag{1.6}$$

We note here the useful uniform estimate for the Macdonald function [9]

$$|K_{i\tau}(x)| \le C_\delta \frac{\tau+1}{\tau}e^{-\delta\tau - x\cos\delta} \quad (\tau, x>0), \tag{1.7}$$

where $0 \le \delta < \pi/2$ and C_δ is a positive constant depending only on δ. In [7] and later in [8] the generalization of the Kontorovich-Lebedev index transform (1.3)-(1.4) was constructed for the case of Meijer's G-function [1] as the kernel. As it was shown (see also [9]), this general index transform comprises enough wide class of integral transforms such as the Mehler-Fock transform [10], [12], the Olevskii transform [9], the Lebedev-Skalskaya transforms and its new generalizations [11]. Detailed information about index transforms and modern results in this field can be found in the book [10].

In this paper we continue our approach to construct and investigate index transforms in the weighted space $L_{\nu,p}(\mathbf{R}_+)$ with $1 \le p < \infty$ and $\nu \in \mathbf{R}_+ \equiv (0,\infty)$ of Lebesgue measurable complex valued functions f for which

$$||f||_{\nu,p} = \left(\int_0^\infty |t^\nu f(t)|^p \frac{dt}{t}\right)^{1/p} < \infty. \tag{1.8}$$

When $\nu = 1/p$, this space reduces to the usual Lebesgue space $L_p(\mathbf{R}_+)$. There are not so more papers dealt with the study of the mapping properties of the index transforms in L_p-spaces for arbitrary values of p. Most of the obtained results for index transforms were devoted to the cases of L_1 and L_2 spaces. The survey and historical notices can be seen in [9]. We note also that the L_p-theorems for the Mehler-Fock transform are given in [10], [12]. We will study here not only mapping properties of index transforms, but the hypergeometric approach [9] on their investigations, and the composition structure using the Mellin transform theory in $L_{\nu,p}$ and the theory of Mellin convolution type general transforms (see [6]). The approach allows us to obtain new interesting examples of index transforms with special functions of hypergeometric type as the kernel, and give inversion theorems in the spaces $L_{\nu,p}$.

For $f \in L_{\nu,p}(\mathbf{R}_+)$ with $1 < p \le 2$ the Mellin transform is defined by [6]

$$f^*(s) = \int_0^\infty f(t)t^{s-1}dt \quad (\mathrm{Re}(s) = \nu), \tag{1.9}$$

where the convergence of the integral (1.9) is understood in the norm of $L_q(\nu - i\infty, \nu + i\infty)$ $(q = p/(p-1))$. In particular, if $f \in L_{\nu,p}(\mathbf{R}_+) \cap L_{\nu,1}(\mathbf{R}_+)$, then the integral (1.9) is the usual improper absolutely convergent integral. In what follows the parameter p is taken as $1 \leq p < \infty$ when no otherwise is stated and the parameter q denotes $p/(p-1)$.

Let us give some useful results from [6].

Theorem 1. *If $f(x) \in L_{\nu,p}(\mathbf{R}_+)$ $(1 < p \leq 2, \nu \in \mathbf{R})$, then its Mellin transform $f^*(s) \equiv f^*(\nu + it)$ exists and belongs to the space $L_q(\mathbf{R})$, where $\mathbf{R} \equiv (-\infty, \infty)$.*

Theorem 2. *If $f^*(\nu + it) \in L_p(\mathbf{R})$ $(1 < p \leq 2, \nu \in \mathbf{R})$, then the inverse Mellin transform*

$$f(x) = \frac{1}{2\pi i}\int_{\nu - i\infty}^{\nu + i\infty} f^*(s) x^{-s} ds \quad (x > 0) \tag{1.10}$$

exists and $f(x) \in L_{\nu,q}(\mathbf{R}_+)$. Moreover, the equality

$$f(x) = \frac{1}{2\pi i}\frac{d}{dx}\int_{\nu - i\infty}^{\nu + i\infty} \frac{f^*(s)}{1-s} x^{1-s} ds \quad (x > 0). \tag{1.11}$$

is true for almost everywhere on $\mathbf{R}_+$.

Theorem 3. *If $f^*(\nu + it) \in L_p(\mathbf{R}), h(x) \in L_{1-\nu,p}(\mathbf{R}_+)$ $(1 < p \leq 2, \nu \in \mathbf{R})$, then the Mellin-Parseval equality*

$$\int_0^\infty f(xt)h(t)dt = \frac{1}{2\pi i}\int_{\nu - i\infty}^{\nu + i\infty} f^*(s) h^*(1-s) x^{-s} ds \tag{1.12}$$

takes place.

At the final of this section let us note the Hölder inequality for weighted spaces

$$\int_0^\infty |f(t)h(t)|dt \leq ||f||_{\nu,p}||h||_{1-\nu,q} \tag{1.13}$$

for $f(x) \in L_{\nu,p}(\mathbf{R}_+), h(x) \in L_{1-\nu,q}(\mathbf{R}_+)$, and the generalized Minkowski inequality

$$\left(\int_0^\infty dx \left|\int_0^\infty f(x,y)dy\right|^p\right)^{1/p} \leq \int_0^\infty dy \left(\int_0^\infty |f(x,y)|^p dx\right)^{1/p}. \tag{1.14}$$

2. General Results

In this section we will investigate the general index transform (1.1) in the space $L_{\nu,p}(\mathbf{R}_+)$ and will establish its inversion theorem.

First we need the following:

Lemma 1. *Let $\varphi(x) \in L_{\nu,p}(\mathbf{R}_+)$ with $\nu < 1$ and $p \geq 1$. Then for the index kernel $Y_{i\tau}^{\varphi}(x)$ the estimate*

$$||Y_{i\tau}^{\varphi}||_{\nu,p} \leq C_\delta \Gamma(1-\nu) \left(\cos^{\nu-1}\delta\right) \frac{\tau+1}{\tau} e^{-\delta\tau} ||\varphi||_{\nu,p} \quad (\tau > 0) \tag{2.1}$$

is true, where $0 \leq \delta < \pi/2$ and C_δ is a positive absolute constant depending only on δ.

Proof. In view of the generalized Minkowski inequality (1.14) and the estimate (1.7) of the Macdonald function $K_{it}(x)$ in (1.2), we obtain the following chain of expressions

$$\begin{aligned} ||Y_{i\tau}^{\varphi}||_{\nu,p} &= \left(\int_0^\infty x^{p\nu-1}dx \left|\int_0^\infty K_{i\tau}(y)\varphi(xy)dy\right|^p\right)^{1/p} \\ &\leq \int_0^\infty |K_{i\tau}(y)|\, dy \left(\int_0^\infty x^{p\nu-1}|\varphi(xy)|^p\, dx\right)^{1/p} = ||\varphi||_{\nu,p}\int_0^\infty y^{-\nu}|K_{i\tau}(y)|\, dy \\ &\leq C_\delta ||\varphi||_{\nu,p} \frac{\tau+1}{\tau} e^{-\delta\tau} \int_0^\infty e^{-y\cos\delta} y^{-\nu} dy \\ &= C_\delta \Gamma(1-\nu)\left(\cos^{\nu-1}\delta\right)\frac{\tau+1}{\tau}e^{-\delta\tau}||\varphi||_{\nu,p}. \end{aligned} \tag{2.2}$$

This completes the proof of Lemma 1.

Lemma 2. *Let $\varphi \in L_{1-\nu,q}(\mathbf{R}_+)$ and $f(x) \in L_{\nu,p}(\mathbf{R}_+)$ $(\nu > 0, p > 1)$. Then the index transform (1.1) has the estimate*

$$|[Y_{i\tau}^{\varphi} f](\tau)| \leq C_\delta \Gamma(\nu)\left(\cos^{-\nu}\delta\right)(\tau+1)e^{-\delta\tau}||\varphi||_{1-\nu,q}||f||_{\nu,p} \quad (\tau > 0), \tag{2.3}$$

where $0 \leq \delta < \pi/2$ and C_δ is a positive absolute constant depending only on δ.

Proof. Using Lemma 1 and the Hölder inequality (1.13), we have

$$\begin{aligned} |[Y_{i\tau}^{\varphi} f](\tau)| &\leq \tau \int_0^\infty |Y_{i\tau}^{\varphi}(t) f(t)|\, dt \\ &\leq C_\delta \Gamma(\nu)\left(\cos^{-\nu}\delta\right)(\tau+1)e^{-\delta\tau}||\varphi||_{1-\nu,q}||f||_{\nu,p}. \end{aligned} \tag{2.4}$$

Thus Lemma 2 is established.

Corollary 1. *The general index transform (1.1) is a bounded operator from the space $L_{\nu,p}(\mathbf{R}_+)$ $(\nu > 0, p > 1)$ into the space $L_r(\mathbf{R}_+)$ $(r \geq 1)$ and*

$$||\,[Y_{i\tau}^{\varphi} f]\,||_r \leq C_{\nu,\delta}||f||_{\nu,p} \tag{2.5}$$

for $0 < \delta < \pi/2$, where $C_{\nu,\delta}$ is a positive constant depending only on δ and ν.

Proof. By choosing some positive parameter $\delta \in (0, \delta/2)$, we obtain

$$||\,[Y_{i\tau}^{\varphi} f]\,||_r \leq C_\delta \Gamma(\nu)\left(\cos^{-\nu}\delta\right)||f||_{\nu,p}\left(\int_0^\infty (\tau+1)^r e^{-\delta r\tau}d\tau\right)^{1/r} = C_{\nu,\delta}||f||_{\nu,p} \tag{2.6}$$

from the estimate (2.4), which proves (2.5).

Let us consider the operator

$$(2.7)\qquad \left(I_\varepsilon^\psi g\right)(x)=\frac{2}{\pi^2}\int_0^\infty \sinh((\pi-\varepsilon)\tau)Y_{i\tau}^\psi(x)g(\tau)d\tau \quad (x>0),$$

where $\varepsilon\in(0,\pi)$ and $Y_{i\tau}^\psi(x)$ is the index kernel of the type (1.2) but with another characteristic function $\psi(x)$.

Theorem 4. *Let $\psi\in L_1(\mathbf{R}_+)\cap L_{\nu+1,1}(\mathbf{R}_+), \varphi\in L_{1-\nu,q}(\mathbf{R}_+)$ $(\nu>0, q>1)$. Then for the function $g(\tau)=\left[Y_{i\tau}^\varphi f\right](\tau)$ represented by the general index transform* (1.1) *with the density $f(y)\in L_{\nu,p}(\mathbf{R}_+)$, the operator* (2.7) *has the form*

$$(2.8)\qquad \left(I_\varepsilon^\psi g\right)(x)=\frac{\sin\varepsilon}{\pi}\int_0^\infty\int_0^\infty \frac{uvK_1\left(\sqrt{u^2+v^2-2uv\cos\varepsilon}\right)}{\sqrt{u^2+v^2-2uv\cos\varepsilon}}\psi(xu)(\Phi f)(v)dudv,$$

where $K_1(z)$ is the Macdonald function of index 1 *and $(\Phi f)(x)$ is the Mellin convolution type transform* [9]

$$(2.9)\qquad (\Phi f)(x)=\int_0^\infty \varphi(xy)f(y)dy \quad (x>0)$$

with the kernel $\varphi(x)$.

Proof. First we note that the composition

$$(2.10)\qquad \left[Y_{i\tau}^\varphi f\right](\tau)=\left[\mathfrak{KL}(\Phi f)\right](\tau)$$

holds for the index transform (1.1), from the estimate (2.4) and the Fubini theorem for the respective iterated integral in the right-hand side of (2.10), where $[\mathfrak{KL}f](\tau)$ is the Kontorovich-Lebedev transform (1.3) of an arbitrary function f. Substituting (2.7) instead of $g(\tau)$ the composition representation (2.10) and instead of the kernel $Y_{i\tau}^\psi(x)$ its definition (1.2), we obtain the iterated integral

$$(2.11)\qquad \left(I_\varepsilon^\psi g\right)(x)=\frac{2}{\pi^2}\int_0^\infty \sinh((\pi-\varepsilon)\tau)\int_0^\infty \psi(xv)K_{i\tau}(v)dv$$
$$\times\,\tau\int_0^\infty K_{i\tau}(u)\int_0^\infty f(y)\varphi(uy)dydud\tau \quad (x>0).$$

Using the inequality (1.7) and the condition on the function ψ,we deduce the estimate of the kernel $Y_{i\tau}^\psi(x)$

$$\left|Y_{i\tau}^\psi(x)\right|\le C_\delta\frac{\tau+1}{\tau}e^{-\delta\tau}\int_0^\infty|\psi(xv)|e^{-v\cos\delta}dv$$
$$\le C_1\frac{\tau+1}{\tau}e^{-\delta\tau}\int_0^1|\psi(xv)|dv+C_2\frac{\tau+1}{\tau}e^{-\delta\tau}\int_1^\infty|\psi(xv)|v^\nu dv$$
$$\le\left(C_1||\psi||_{L_1(\mathbf{R}_+)}+C_2x^{-\nu}||\psi||_{L_{\nu+1,1}(\mathbf{R}_+)}\right)\frac{\tau+1}{x\tau}e^{-\delta\tau}.$$

Hence invoking to (2.3) for each $x>0$, we can estimate the integral (2.11) for $\varepsilon\in(0,\pi)$

$$(2.12)\qquad \left|\left(I_\varepsilon^\psi g\right)(x)\right|\le\left(C_1x^{-1}||\psi||_{L_1(\mathbf{R}_+)}+C_2x^{-\nu-1}||\psi||_{L_{\nu+1,1}(\mathbf{R}_+)}\right)||\varphi||_{1-\nu,q}||f||_{\nu,p}$$
$$\times\int_0^\infty\frac{\sinh((\pi-\varepsilon)\tau)}{\tau}(\tau+1)^2e^{-(\delta_1+\delta_2)\tau}d\tau<\infty,$$

where $\nu > 0$ and we choose δ_i $(i = 1, 2)$ such that $\pi - \varepsilon < \delta_1 + \delta_2 < \pi$. Hence we observe that the integral by τ in the estimate (2.12) is convergent and we can apply the Fubini theorem to obtain (2.8) after using the formula [5, Vol.2, (2.16.52.6)]

$$\frac{2}{\pi}\int_0^\infty \tau \sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)K_{i\tau}(y)d\tau = \sin\varepsilon \frac{xyK_1\left(\sqrt{x^2+y^2-2xy\cos\varepsilon}\right)}{\sqrt{x^2+y^2-2xy\cos\varepsilon}}. \tag{2.13}$$

at the inside integral by τ . Thus Theorem 4 is proved.

The inversion of the general index transform (1.1) in $L_{\nu,p}$ is given by

Theorem 5. *Let $1 < p \le 2, 0 < \nu < 1$ and $g(\tau) = [Y_{i\tau}^{\varphi} f](\tau)$, where $f(x) \in L_{\nu,p}(\mathbf{R}_+) \cap L_{\nu,1}(\mathbf{R}_+)$ and $\varphi \in L_{1-\nu,q}(\mathbf{R}_+) \cap L_{1-\nu,1}(\mathbf{R}_+)$. Let the characteristic function $\psi(x)$ satisfying conditions in Theorem 4 be from the space $L_{1+\nu,p}(\mathbf{R}_+)$. Then the limit equality of the type*

$$\underset{\varepsilon\to 0+}{\text{l.i.m.}} \left(I_\varepsilon^\psi g\right)(x) = \frac{1}{x^2}\int_0^x f(y)dy \quad (x > 0) \tag{2.14}$$

in the $L_{1+\nu,p}$-norm is valid if and only if the equality

$$\psi^*(1+s)\varphi^*(1-s) = \frac{1}{1-s} \quad (\mathrm{Re}(s) = \nu) \tag{2.15}$$

is fulfilled, where ψ^* and φ^* denote the Mellin transform (1.9) of functions $\psi(x)$ and $\varphi(x)$, respectively. Further, the limit in (2.14) exists almost everywhere on $\mathbf{R}_+$.

Proof. The proof follows after the respective treatment of the integrals in (2.8). Indeed, by changing the variable $v = u(\cos\varepsilon + t\sin\varepsilon)$, we obtain the equality

$$\left(I_\varepsilon^\psi g\right)(x) = \frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty \frac{R(t,u,\varepsilon)}{t^2+1} u\psi(xu(\cos\varepsilon + t\sin\varepsilon)) \times (\cos\varepsilon + t\sin\varepsilon)(\Phi f)(u)dudt, \tag{2.16}$$

where $\varepsilon \in (0, \pi)$ and

$$R(t,u,\varepsilon) = \begin{cases} u\sin\varepsilon\sqrt{t^2+1}K_1\left(u\sin\varepsilon\sqrt{t^2+1}\right) & (t \ge -\cot\varepsilon), \\ 0 & (t < -\cot\varepsilon). \end{cases} \tag{2.17}$$

It is easily seen from the asymptotic behavior of the Macdonald function $K_1(z)$ [1]

$$|R(t,u,\varepsilon)| \le C$$

for any $t \in \mathbf{R}$, $u \in \mathbf{R}_+$ and $\varepsilon \in (0, \pi)$, where C is a positive constant, and

$$\lim_{\varepsilon\to 0+} R(t,u,\varepsilon) = 1.$$

Hence by estimating the norm (1.8) of the obtained operator (2.16) in the space $L_{\nu,p}(\mathbf{R}_+)$ by the aid of the generalized Minkowski inequality (1.14) and the conditions of this theorem, we find

$$(2.18)\quad ||\left(I_\varepsilon^\psi g\right)||_{\nu+1,p}$$
$$\leq \frac{C}{\pi}\int_0^\infty u|(\Phi f)(u)|\int_{-\cot\varepsilon}^\infty \frac{1}{t^2+1}||\psi(xu(\cos\varepsilon+t\sin\varepsilon))||_{\nu+1,p}(\cos\varepsilon+t\sin\varepsilon)dtdu$$
$$\leq \frac{C}{\pi}||\psi||_{\nu+1,p}\int_0^\infty u^{-\nu}\int_0^\infty |f(y)\varphi(uy)|dydu\int_{-\cot\varepsilon}^\infty \frac{(\cos\varepsilon+t\sin\varepsilon)^{-\nu}}{t^2+1}dt$$
$$\leq C_1||\psi||_{\nu+1,p}||\varphi||_{1-\nu,1}||f||_{\nu,1},$$

where C and C_1 are positive constants. When $0<\nu<1$, by virtue of the formula [5, Vol.1, (2.2.9.7)] we can estimate the integral

$$I_\varepsilon = \int_{-\cot\varepsilon}^\infty \frac{(\cos\varepsilon+t\sin\varepsilon)^{-\nu}}{t^2+1}dt$$

and obtain $I_\varepsilon \leq A\sin\varepsilon$ for A being a positive constant and $\varepsilon\in(0,\pi)$. Further, since $f(x)\in L_{\nu,p}(\mathbf{R}_+)$, then one can show that

$$\frac{1}{x^2}\int_0^x f(y)dy \in L_{\nu+1,p}(\mathbf{R}_+).$$

Indeed, using the inequality (1.14) we have that

$$(2.19)\quad \left\|\frac{1}{x^2}\int_0^x f(y)dy\right\|_{\nu+1,p} = \left\|\frac{1}{x}\int_0^1 f(xy)dy\right\|_{\nu+1,p}$$
$$\leq \int_0^1 dy\left(\int_0^\infty |\, x^\nu f(xy)\,|^p\,\frac{dx}{x}\right)^{1/p} = ||f||_{\nu,p}\int_0^1 y^{-\nu}dy<\infty.$$

So, by the known properties of the Poisson kernel $P(t)=1/\pi(t^2+1)$ we obtain from the equality (2.16) the estimate

$$(2.20)\quad \left\|\left(I_\varepsilon^\psi g\right)-\frac{1}{x^2}\int_0^x f(y)dy\right\|_{\nu+1,p}$$
$$\leq \frac{1}{\pi}\int_{-\infty}^\infty \frac{|\cos\varepsilon+t\sin\varepsilon\,|}{t^2+1}$$
$$\times\left\|\int_0^\infty u(\Phi f)(u)R(t,u,\varepsilon)\psi(xu(\cos\varepsilon+t\sin\varepsilon))du-\frac{1}{x^2}\int_0^x f(y)dy\right\|_{\nu+1,p}dt.$$

We must now establish that the right-hand side of inequality (2.20) tends to zero, as $\varepsilon\to 0+$, which will lead to the limit equality (2.14). In fact, using the above estimates and the Lebesgue theorem, we observe that the right-hand side of (2.20) tends to the expression

$$(2.21)\qquad \left\|\int_0^\infty u(\Phi f)(u)\psi(xu)du-\frac{1}{x^2}\int_0^x f(y)dy\right\|_{\nu+1,p},$$

which is equal to zero. Indeed, according to the Mellin-Parseval equality (1.12) and owing to Theorems 1-3, we can represent the Mellin convolution type transform (2.9) in the form

$$(\Phi f)(x) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)\varphi^*(1-s)x^{s-1}ds, \tag{2.22}$$

where $\varphi^*(s)$ is the Mellin transform (1.9) of the function $\varphi(x)$, and $\varphi^*(1-\nu-it) \in L_p(\mathbf{R})$. Since $f^*(\nu+it) \in L_q(\mathbf{R})$, then by the Hölder inequality the integral (2.22) is absolutely convergent and the estimate $|(\Phi f)(x)| \le Cx^{\nu-1}$ $(x > 0)$ is obtained. From $\psi \in L_{\nu+1,1}(\mathbf{R}_+)$ and by the Fubini theorem, we can change the order of integration in the iterated integral in (2.21) accounting (2.22) and have the equality

$$\begin{aligned}(2.23)\quad &\left\|\int_0^\infty u(\Phi f)(u)\psi(xu)du - \frac{1}{x^2}\int_0^x f(y)dy\right\|_{\nu+1,p}\\ &= \left\|\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)\psi^*(1+s)\varphi^*(1-s)x^{-s-1}ds - \frac{1}{x^2}\int_0^x f(y)dy\right\|_{\nu+1,p}\\ &= \left\|\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \frac{f^*(s)}{1-s}x^{-s-1}ds - \frac{1}{x^2}\int_0^x f(y)dy\right\|_{\nu+1,p} = 0,\end{aligned}$$

under the equality (1.11), if $\psi^*(1+s)\varphi^*(1-s) = (1-s)^{-1}$. Inversely, if

$$\left\|\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)\psi^*(1+s)\varphi^*(1-s)x^{-s-1}ds - \frac{1}{x^2}\int_0^x f(y)dy\right\|_{\nu+1,p} = 0, \tag{2.24}$$

then for almost all $x > 0$ it follows that

$$\int_0^x f(y)dy = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)\psi^*(1+s)\varphi^*(1-s)x^{1-s}ds$$

and from the equality (1.11) and Theorem 1, we have the relation (2.15). Thus from (2.23) we conclude the validity of the limit relation (2.14). The existence of the limit almost everywhere on $\mathbf{R}_+$ follows from the radial property of the Poisson kernel $P(t) = P(|t|) \in L_1(\mathbf{R}_+)$. Theorem 5 is proved.

3. ${}_2F_1$-Index Transform of the Olevskii Type

In this section we consider a very important example of the general index transform (1.1) being introduced by Olevskii [4], which contains the Gauss hypergeometric function [1] as the kernel and generalizes the famous Mehler-Fock index transform [10], [12] with the associated Legendre function of the first kind [1]. Indeed, putting into the formula (1.2) $\varphi(x) = x^{\alpha-1}J_\mu(x)$ $(x > 0, \mathrm{Re}(\alpha+\mu) > 0)$, where $J_\mu(x)$ is the Bessel function [1], we obtain the value of the respective index kernel $Y_{i\tau}^{\varphi}(x)$ using formula [5, Vol.2, (2.16.21.1)] as

$$\int_0^\infty y^{\alpha-1}J_\mu(xy)K_{i\tau}(y)dy \tag{3.1}$$

$$= 2^{\alpha-2}x^{\mu}\frac{\Gamma((\alpha+\mu+i\tau)/2)\Gamma((\alpha+\mu-i\tau)/2)}{\Gamma(\mu+1)}$$

$$\times {}_2F_1\left(\frac{\alpha+\mu+i\tau}{2}, \frac{\alpha+\mu-i\tau}{2}; \mu+1; -x^2\right) \quad (x>0).$$

Thus the formula (1.1) is reduced to the Olevskii transform

$$(3.2) \quad [{}_2F_1^{i\tau}f](\tau) = 2^{\alpha-2}\tau\frac{\Gamma((\alpha+\mu+i\tau)/2)\Gamma((\alpha+\mu-i\tau)/2)}{\Gamma(\mu+1)}$$

$$\times \int_0^\infty t^{\alpha+\mu-1}{}_2F_1\left(\frac{\alpha+\mu+i\tau}{2}, \frac{\alpha+\mu-i\tau}{2}; \mu+1; -t^2\right) f(t)dt.$$

Let us establish the inversion formula for the Olevskii transform (3.2) in the space $L_{\nu,p}$ following to Theorem 5. From the asymptotic behavior of the Bessel function [1] we find that of the characteristic function $\varphi(x) = x^{\alpha-1}J_\mu(x)$, as $x \to +0$ and $x \to \infty$. Precisely, we have

$$(3.3) \qquad x^{\alpha-1}J_\mu(x) = O(x^{\alpha+\mu-1}) \quad (x \to +0), \qquad x^{\alpha-1}J_\mu(x) = O(x^{\alpha-3/2}) \quad (x \to \infty).$$

Thus in order to be fulfilled the assumption of Theorem 5 $\varphi \in L_{1-\nu,q}(\mathbf{R}_+) \cap L_{1-\nu,1}(\mathbf{R}_+)$, we assume that the inequality

$$(3.4) \qquad -\frac{1}{2} < \nu - \mathrm{Re}(\alpha) < \mathrm{Re}(\mu)$$

holds as a corollary of the absolute convergency of the respective integral (1.8). Further, from the functional relation (2.15) we easily obtain the expression for the Mellin image of the characteristic function $\psi(x)$ at the point $1+s$ as

$$(3.5) \qquad \psi^*(1+s) = [\varphi^*(1-s)]^{-1}(1-s)^{-1} \quad (\mathrm{Re}(s) = \nu).$$

Invoking to the value of the integral [5, Vol.2, (2.12.2.2)]

$$(3.6) \qquad \varphi^*(s) = \int_0^\infty x^{\alpha+s-2}J_\mu(x)dx = 2^{\alpha+s-2}\frac{\Gamma((\alpha+s+\mu-1)/2)}{\Gamma((\mu-\alpha-s+3)/2)},$$

the equation (3.5) takes the form

$$(3.7) \qquad \psi^*(1+s) = 2^{-\alpha+s+1}\frac{1}{1-s}\frac{\Gamma(1+(-\alpha+s+\mu)/2)}{\Gamma((\mu+\alpha-s)/2)}.$$

Hence one can express the characteristic function $\psi(x)$ in terms of the Mellin-Barnes integrals using the Slater theorem [3], that is

$$(3.8) \qquad x\psi(x) = \frac{2^{-\alpha+1}}{2\pi i}\int_{\nu/2-i\infty}^{\nu/2+i\infty}\frac{\Gamma(1+(\mu-\alpha)/2+s)\Gamma(1/2-s)}{\Gamma((\mu+\alpha)/2-s)\Gamma(3/2-s)}(x/2)^{-2s}ds.$$

First let us discuss the convergence of the integral (3.8). By the asymptotic Stirling formula for the Euler gamma-function [1] we easily have that $\psi^*(1+\nu+it) = O(|t|^{\nu-\mathrm{Re}(\alpha)})$ $(|t| \to \infty)$. Therefore if we make to be precise the right inequality in (3.4) as $\nu - \mathrm{Re}(\alpha) <$

$\min[\mathrm{Re}(\mu), -1/q]$ $(q > 2)$, then we arrive at $\psi^*(1+\nu+it) \in L_q(\mathbf{R})$ for any $q > 2$ (see also Theorem 1) when $1 < p < 2$. So the integral (3.8) is convergent in the mean sense, but since $\nu - \mathrm{Re}(\alpha) < 0$ in the oscillation term of the Stirling formula, it allows us to conclude that the integral (3.8) is conditionally convergent, too. Moreover, from Theorem 2 it follows that $\psi(x) \in L_{1+\nu,p}(\mathbf{R}_+)$ $(1 < p < 2)$. Let us show that the value $p = 1$ is also possible in this case. For this, we will use the corollaries of the Slater theorem [3]. We note that it is not necessary to evaluate the integral (3.8), because the final purpose is to write the inversion operator like (2.7) for the introduced Olevskii transform and for this we need to evaluate the respective kernel $Y^{\psi}_{i\tau}(x)$. It is more easy to use the Mellin-Parseval formula (1.12) on this matter which is allowed to avoid the direct calculation of the function $\psi(x)$.

Now let us assume that the contour at the integral (3.8) separates the right and left poles of the integrand, for which let us set

$$-2 + \mathrm{Re}(\alpha - \mu) < \nu < 1. \tag{3.9}$$

Hence from the Slater theorem and its corollaries (see details [3]) it follows the asymptotic behavior of the function ψ

$$\psi(x) = O(x^{1+\mu-\alpha}) \quad (x \to 0+), \qquad \psi(x) = O(x^{-2}) \quad (x \to \infty). \tag{3.10}$$

Therefore, if $1 + \nu + \mathrm{Re}(\mu - \alpha) > 0$, then we achieve the property $\psi \in L_{\nu+1,1}(\mathbf{R}_+)$ taking into account the integral (1.8) and the asymptotic (3.10), because the function ψ has no other singularities as we concluded from the Slater theorem.

To write the inversion formula like (2.14) we need to evaluate only the integral (1.2) for the kernel $Y^{\psi}_{i\tau}(x)$. Making use of the Mellin-Barnes representation [3, §10, 9.3(1)] of the Macdonald function $K_{i\tau}(x)$

$$K_{i\tau}(x) = \frac{1}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} 2^{s-1}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)x^{-s}ds \quad (x > 0, \gamma > 0), \tag{3.11}$$

we substitute it into the formula (1.2) for the kernel $Y^{\psi}_{i\tau}(x)$. We perform to change the order of integration by the Fubini theorem in view of the Stirling formula, by noting the asymptotic (3.10) and by taking the positive parameter γ such that $2 + \mathrm{Re}(\mu - \alpha) - \gamma > 0$ to provide the property $x^{-\gamma}\psi(x) \in L_1(\mathbf{R}_+)$. Thus after evaluating the inside integral of the formula (3.7), we obtain the representation

$$Y^{\psi}_{i\tau}(x) = \frac{2^{-\alpha}}{4\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)\frac{\Gamma(1+(\mu-\alpha-s)/2)}{\Gamma((\mu+\alpha+s)/2)(1+s)}x^{s-1}ds. \tag{3.12}$$

Then from the differential properties of the last integrand and the uniform convergence of the integral (3.12), it can be written as

$$\begin{aligned} Y^{\psi}_{i\tau}(x) &= \frac{2^{-\alpha}x^{-2}}{4\pi i}\int_0^x dt\int_{\gamma-i\infty}^{\gamma+i\infty}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)\frac{\Gamma(1+(\mu-\alpha-s)/2)}{\Gamma((\mu+\alpha+s)/2)}t^s ds \\ &= 2^{-\alpha}x^{-2}\int_0^x F(t,i\tau)dt. \end{aligned} \tag{3.13}$$

Finally, calculating the Mellin-Barnes integral (3.13) by the Slater theorem, we arrive at relations depending on the values of variable t

$$(3.14)\quad F(t,i\tau)=t^{2+\mu-\alpha}\frac{\Gamma(1+(\mu-\alpha+i\tau)/2)\Gamma(1+(\mu-\alpha-i\tau)/2)}{\Gamma(\mu+1)}$$
$$\times\,{}_2F_1\left(1+\frac{\mu-\alpha-i\tau}{2},1+\frac{\mu-\alpha+i\tau}{2};\mu+1;-t^2\right)\quad(0<t\le 1),$$

$$(3.15)\quad F(t,i\tau)=t^{-i\tau}\frac{\Gamma(1+(\mu-\alpha-i\tau)/2)\Gamma(i\tau)}{\Gamma((\mu+\alpha+i\tau)/2)}$$
$$\times{}_2F_1\left(1+\frac{\mu-\alpha-i\tau}{2},1-\frac{\mu+\alpha+i\tau}{2};1-i\tau;-t^2\right)$$
$$+t^{i\tau}\frac{\Gamma(1+(\mu-\alpha+i\tau)/2)\Gamma(-i\tau)}{\Gamma((\mu+\alpha-i\tau)/2)}$$
$$\times\,{}_2F_1\left(1+\frac{\mu-\alpha+i\tau}{2},1-\frac{\mu+\alpha-i\tau}{2};1+i\tau;-t^2\right)\quad(t>1).$$

Thus as a conclusion we obtain:

Theorem 6. *Let $1<p<2, 0<\nu<1$ and $g(\tau)=[{}_2F_1^{i\tau}f]$ be the Olevskii type transform (3.2) of a function $f(x)\in L_{\nu,p}(\mathbf{R}_+)\cap L_{\nu,1}(\mathbf{R}_+)$. Then under condition (3.4) and inequality $1+\nu+\mathrm{Re}(\mu-\alpha)>0$, the inversion formula*

$$(3.16)\qquad \int_0^x f(y)dy=\underset{\varepsilon\to 0+}{\mathrm{l.i.m.}}\frac{2^{-\alpha+1}}{\pi^2}\int_0^\infty\int_0^x \sinh((\pi-\varepsilon)\tau)F(t,i\tau)g(\tau)dtd\tau\quad(x>0)$$

is true by the norm in the space $L_{\nu-1,p}(\mathbf{R}_+)$, and the limit in (3.16) exists almost everywhere on $\mathbf{R}_+$.

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References

[1] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vols. 1,2, McGraw-Hill, New York, 1953.

[2] N.N. Lebedev, Sur une formule d'inversion, *Doklady Akad. Nauk SSSR* (N.S.) **52**(1946), 655-658.

[3] O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions, Theory and Algorithmic Tables*, Ellis Horwood, Chichester, 1983.

[4] M.N. Olevskii, On the representation of an arbitrary function in the form of an integral with a kernel which is a hypergeometric function, *Doklady Akad. Nauk SSSR* (N.S.) **69**(1949), 11-14.

[5] A.P. Prudnikov, Yu.A. Brychkov and O.I. Marichev, *Integrals and Series*, Vols. 1-3, Gordon and Breach, New York *et alibi*, 1986, 1986, 1989.

[6] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford, 1937.

[7] J. Wimp, A class of integral transforms, *Proc. Edinburgh Math. Soc.* **14**(1964), 33-40.

[8] S.B. Yakubovich, Vu Kim Tuan, O.I. Marichev and S.L. Kalla, One class of index integral transforms, *Rev. Tec. Ing. Univ. Zulia, Edition, Especial* **10**(1987), 105-118.

[9] S.B. Yakubovich and Yu.F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*, (Ser. Math. Its Appl., 287), Kluwer Academic, Dordrecht, 1994.

[10] S.B. Yakubovich, On the Mehler-Fock integral transform in L_p-spaces, *Extracta Mathematicae* **8**(1993), 162-164.

[11] S.B. Yakubovich, M. Saigo and N.P. Lemeshevskaya, On the class of Lebedev-Skalskaya type index transforms, *Fukuoka Univ. Sci. Rep.* **24**(1994), 67-81.

[12] S.B. Yakubovich and M. Saigo, On the Mehler-Fock index transform in L_p-space, 京都大学数理解析研究所講究録 881,「Jack の補題とその周辺に関する研究」(1994), 130-144.