数理解析研究所講究録
918巻 1995年 206-223

Extracting a reduction system from a conjunction calculus

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July 1995

1 Introduction

The purpose of this paper is to show that we can obtain a strongly normalizing and confluent abstract reduction system from a variant of Lawvere style deductive system [Lambek 94] for a propositional calculus with conjunctions. The deductive system is reminiscent of a sequent calculus, and consists of Lawvere style deductions, namely each of which has exactly one input and one output, and inference rules that includes, initial deduction, composition rule, and also left and right rules for conjunctions. It enjoys the *composition rule elimination theorem*, which is thought of as a kind of cut elimination theorem. In order to analyze the computational aspects, in particular the operational semantics, we introduce a Σ-term algebra whose sorts are the deductions and operation symbols correspond to the inference rules. Then each Σ-term corresponds to the unique derivation of a deduction, and vice versa. First we show that we can eliminate operation symbols corresponding to composition rule. This result, the weak normalization theorem, amounts to the composition rule elimination theorem of the deductive system. Next, from the proof of the theorem, we extract a binary relation on the Σ-terms so that the Σ-terms and the relation form an abstract reduction system, which is not a term rewriting system by some reason. Finally we show that it is strongly normalizing and confluent. As an application the word problem for the equivalence relation generated by it is thus decidable as expected.

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The underlying motivation of this study is to investigate the connection between reduction systems and deductive systems. For a given deductive system D, we would like to find an abstract reduction system $\langle T, R\rangle$, with T being a set of terms in which each term interprets a proof of D, and R being a binary relation on T. As a computational model, R is desired to be strongly normalizing and confluent. An application of such $\langle T, R\rangle$ is to show that the word problem for the equivalence relation generated by R is decidable by means of confluence method. The reduction system $\langle T, R\rangle$ might also be used as a basis of constructive programming.

As a special case, we can think of the following correspondence between Σ-term algebras and sequent calculi, which is called *sequents-as-sorts interpretation*. Let S be a given sequent calculus that enjoys cut-elimination. Let Σ be a signature whose sorts are the sequents and operations correspond to the inference rules. Then T_Σ, the set of Σ-terms, is a sound and complete interpretation of derivations of S. Since S enjoys cut-elimination, T_Σ is weakly normalizing. Let R be a binary relation on T_Σ extracted from the weak normalization. Then the reduction system $\langle T_\Sigma, R\rangle$ may have further properties like strong normalization and confluence.

2 A conjunction calculus

In this section we introduce a variant of Lawvere style deductive system for a propositional calculus with conjunctions. Then we introduce a Σ-term algebra whose sorts are the deductions and operation symbols correspond to inference rules. For each deduction, there is a bijection between Σ-terms of the deduction and derivation of the deduction.

Definition 1 *Let $\mathcal{PS}$ be some set of propositional symbols. The set of* **propositional conjunction formulae**, *notation $\mathcal{F}$, is defined inductively as follows*

$$\mathcal{F} \ ::= \ \mathcal{PS} \mid (\mathcal{F} \wedge \mathcal{F}).$$

Definition 2 *The set of* **Lawvere style deductions** *over $\mathcal{F}$ is defined by*

$$\mathcal{D} \ ::= \ \mathcal{F} \longrightarrow \mathcal{F}.$$

Definition 3 *A deduction $A \longrightarrow B \in \mathcal{D}$ is* **derivable**, *notation $\vdash A \longrightarrow B$, if there is a* **derivation** *of the deduction, in other words if it can be produced using the following rules.*

• *The identity axiom and the composition rule*

$$A \longrightarrow A \;\; \text{ID} \qquad \frac{A \longrightarrow C \quad C \longrightarrow B}{A \longrightarrow B}\,\text{COMP}$$

We call C the composition formula.
• *The rules for conjunctions*

$$\frac{A \longrightarrow C}{A \wedge B \longrightarrow C}\,\wedge\text{-L} \qquad \frac{B \longrightarrow C}{A \wedge B \longrightarrow C}\,\wedge\text{-L'} \qquad \frac{C \longrightarrow A \quad C \longrightarrow B}{C \longrightarrow A \wedge B}\,\wedge\text{-R}$$

Definition 4 (Signature of Proof Terms) *The signature Σ over $\mathcal{D}$ consists of $\mathcal{D}$ and a $\mathcal{D}^* \times \mathcal{D}$-indexed family*

$$\langle \Sigma_{w,s} \mid w \in \mathcal{D}^*,\ s \in \mathcal{D} \rangle$$

of sets, where for any $A, B, C \in \mathcal{F}$,

$$\begin{array}{rcll}
1_A & \in & \Sigma_{\lambda,\ A \longrightarrow A} \quad (\lambda \textit{ is the empty word}) & \text{(identity)}, \\
\sigma^C_{A,B} & \in & \Sigma_{A \longrightarrow C\ C \longrightarrow B,\ A \longrightarrow B} & \text{(composition)}, \\
\pi_{A,B,C} & \in & \Sigma_{A \longrightarrow C,\ A \wedge B \longrightarrow C} & \text{(projection)}, \\
\pi'_{A,B,C} & \in & \Sigma_{B \longrightarrow C,\ A \wedge B \longrightarrow C} & \text{(projection)}, \\
\Pi_{C,A,B} & \in & \Sigma_{C \longrightarrow A\ C \longrightarrow B,\ C \longrightarrow A \wedge B} & \text{(product)}, \\
\Sigma_{w,s} & = & \emptyset \quad \textit{otherwise}. &
\end{array}$$

Definition 5 (Proof Terms over $\mathcal{D}$) *The set of* **proof terms** *over $\mathcal{D}$, notation $\mathcal{T}$, is defined as the set of ground Σ-terms that is $\mathcal{T} = T_\Sigma$.*

Below the notation $f : A \longrightarrow B$ stands for $f \in \mathcal{T}_{A \longrightarrow B}$.
The following proposition obviously holds.

Proposition 6 (Subformula Property of Operation Symbols)

1. $1_A : A \longrightarrow A$,
2. *if* $\sigma^C_{A,B}(f,g) : A \longrightarrow B$ *then* $f : A \longrightarrow C$ *and* $g : C \longrightarrow B$,
3. *if* $\pi_{A,B,C}(f) : A \wedge B \longrightarrow C$ *then* $f : A \longrightarrow C$,
4. *if* $\pi'_{A,B,C}(f) : A \wedge B \longrightarrow C$ *then* $f : B \longrightarrow C$,

5. *if* $\Pi_{C,A,B}(f,g) : C \longrightarrow A \wedge B$ *then* $f : C \longrightarrow A$ *and* $g : C \longrightarrow B$.

This property permits us to omit subscripts of operation symbols of a term except composition formula of composition symbols provided that the sort of the term is known.
It is clear that the next proposition holds, which says that the derivations of deductions can be interpreted soundly and completely by the Σ-term algebra. We call this interpretation the deductions-as-sorts interpretation, which is a kind of sequents-as-sorts interpretation.

Proposition 7 (Deductions-as-Sorts Interpretation) *For each deduction* $A \longrightarrow B$ *there exists a bijection between the derivations of* $A \longrightarrow B$ *and* $\mathcal{T}_{A \longrightarrow B}$.

3 A reduction system

In the process of proving the weak normalization theorem in the Σ-term algebra, which amounts to the composition rule elimination theorem in the deductive system, we extract a reduction relation, which turns out to be strongly normalizing and confluent.

Definition 8 *The set of* **normal forms** *of sort* $A \longrightarrow B$, *notation* $\mathcal{N}_{A \longrightarrow B}$, *is the set of ground* Σ*-terms of sort* $A \longrightarrow B$ *which contains no occurrences of composition symbols.*

Definition 9 *The* **degree** *of a* **formula** A, *notation* $\partial(A)$, *is defined as follows*

- $\partial(P) = 1, \qquad$ *where* $P \in \mathcal{PS}$;
- $\partial(A \wedge B) = \max(\partial(A), \partial(B)) + 1, \qquad$ *where* $A, B \in \mathcal{F}$.

The **degree of a composition symbol** σ^C, *notation* $\partial(\sigma^C)$, *is defined to be the* **degree of the composition formula** C *that is* $\partial(\sigma^C) = \partial(C)$.
The **degree of a proof term** f, *notation* $\partial(f)$, *is the sup of the degrees of its composition symbols, so* $\partial(f) = 0$ *iff* f *is a normal form.*
The **height of a proof term** f, *notation* $h(f)$, *is that of its associated tree.*

Definition 10 (Redex) *Let* $A, B, C \in \mathcal{F}$. *Let* $f : A \longrightarrow C$ *and* $g : C \longrightarrow B$ *such that* $\partial(f), \partial(g) < \partial(C)$. *Then a proof term being of the form*

$$\sigma^C(f,g)$$

is called a **redex**.

Note that the degree of a redex $\sigma^C(f,g)$ is the degree of the composition formula C. Then next lemma and proposition can be shown in a usual constructive manner as in [Girard 89] for example.

Lemma 11 (Principal lemma) *Let C be a formula of degree d. Suppose that*

$$\sigma^C(f,g) : A \longrightarrow B$$

is a redex. Then we can make a proof term

$$h : A \longrightarrow B$$

such that $\partial(h) < d$.

See appendix A for the proof of the principal lemma. Precise trace of the proof of this lemma suggests a one-step reduction relation as follows. The resulting relation is somewhat cumbersome, since we do not ignore the degrees of terms. But the degree information ensures that the degree of a term decreases as reduction proceeds. The relation and the terms form an ARS (abstract reduction system) but not a TRS.

Definition 12 (Principal Reduction $\rightarrow_p$) *We define a relation $\rightarrow_p$ by*

$$\rightarrow_p = \rightarrow_1 \cup \rightarrow_2 \cup \rightarrow_3 \cup \rightarrow_4 \cup \rightarrow_6 \cup \rightarrow_7 \cup \rightarrow_8 \cup \rightarrow_9 \cup \rightarrow_{10},$$

where $\rightarrow_i$'s, which are extracted from the proof of the principal lemma, are defined as follows. Below the notation $\sigma^C(f,g) \rightarrow_i h$ denotes that $\rightarrow_i$ is the $\mathcal{D}$-indexed family of relations

$$\{(\sigma^C(f,g),h) \in \mathcal{T}_{A\longrightarrow B} \times \mathcal{T}_{A\longrightarrow B} \mid \partial(f), \partial(g) < \partial(C)\}.$$

$$\begin{aligned}
&\sigma^A(1,f) \rightarrow_1 f,\\
&\sigma^B(f,1) \rightarrow_2 f,\\
&\sigma^C(\sigma^D(f,g),h) \rightarrow_3 \sigma^D(f,\sigma^C(g,h)),\\
&\sigma^C(f,\sigma^D(g,h)) \rightarrow_4 \sigma^D(\sigma^C(f,g),h),\\
&\sigma^C(\pi(f),g) \rightarrow_6 \pi(\sigma^C(f,g)),\\
&\sigma^C(\pi'(f),g) \rightarrow_7 \pi'(\sigma^C(f,g)),\\
&\sigma^C(f,\Pi(g_1,g_2)) \rightarrow_8 \Pi(\sigma^C(f,g_1),\sigma^C(f,g_2)),\\
&\sigma^{C_1\wedge C_2}(\Pi(f_1,f_2),\pi(g)) \rightarrow_9 \sigma^{C_1}(f_1,g),\\
&\sigma^{C_1\wedge C_2}(\Pi(f_1,f_2),\pi'(g)) \rightarrow_{10} \sigma^{C_2}(f_2,g).
\end{aligned}$$

Definition 13 (One-step Reduction $\rightarrow$) *The relation $\rightarrow$ is the compatible relation generated by $\rightarrow_p$, i.e.*

$$\begin{array}{rcll} f & \rightarrow & g & \text{if } f \rightarrow_p g, \\ \sigma^C(f_1, f_2) & \rightarrow & \sigma^C(g_1, f_2) & \text{if } f_1 \rightarrow g_1, \\ \sigma^C(f_1, f_2) & \rightarrow & \sigma^C(f_1, g_2) & \text{if } f_2 \rightarrow g_2, \\ \pi(f) & \rightarrow & \pi(g) & \text{if } f \rightarrow g, \\ \pi'(f) & \rightarrow & \pi'(g) & \text{if } f \rightarrow g, \\ \Pi(f_1, f_2) & \rightarrow & \Pi(g_1, f_2) & \text{if } f_1 \rightarrow g_1, \\ \Pi(f_1, f_2) & \rightarrow & \Pi(f_1, g_2) & \text{if } f_2 \rightarrow g_2. \end{array}$$

Definition 14 *The ARS $\mathcal{C}$ is defined by $\mathcal{C} = \langle \mathcal{T}, \rightarrow \rangle$.*

By using the principal lemma we have the following proposition. See appendix B for the proof of this proposition.

Proposition 15 *Let $f : A \longrightarrow B$ such that $\partial(f) > 0$. Then we can construct a proof term $g : A \longrightarrow B$ such that $\partial(f) > \partial(g)$ and $f \rightarrow^* g$.*

By iterating the above proposition we have the next result.

Theorem 16 (Weak Normalization) *For every proof term $f : A \longrightarrow B$, there is a normal form $\mathrm{nf}(f) \in \mathcal{N}_{A \longrightarrow B}$ such that $f \rightarrow^* \mathrm{nf}(f)$.*

We have the following results. See appendix C for the proof of this theorem.

Theorem 17 (Strong Normalization) *Every term is strongly normalizing.*

To show that $\rightarrow$ is locally confluent we need to introduce the next congruence relation. This relation cannot be extracted from the principal lemma since it does not involve any composition symbols.

Definition 18 (Congruence relation $=_\Pi$) *We define the relation $=_\Pi$ on proof terms as the $\mathcal{D}$-indexed family of the congruence relations generated by the union of the relations*

$$\begin{array}{l} \{(l, r) \in \mathcal{T}_{A_1 \wedge A_2 \longrightarrow B_1 \wedge B_2} \times \mathcal{T}_{A_1 \wedge A_2 \longrightarrow B_1 \wedge B_2} \mid \\ l = \Pi(\pi(f), \pi(g)), r = \pi(\Pi(f, g)) \\ \textit{for some } f : A_1 \longrightarrow B_1,\ g : A_1 \longrightarrow B_2\} \end{array}$$

and

$$\{(l,r) \in \mathcal{T}_{A_1 \wedge A_2 \longrightarrow B_1 \wedge B_2} \times \mathcal{T}_{A_1 \wedge A_2 \longrightarrow B_1 \wedge B_2} \mid \\ l = \Pi(\pi'(f), \pi'(g)), r = \pi'(\Pi(f,g)) \\ \textit{for some } f : A_2 \longrightarrow B_1,\ g : A_2 \longrightarrow B_2\}.$$

By checking all the critical situations we get the following result. See appendix D for the proof of this proposition.

Proposition 19 (Local Confluence) *The relation $\rightarrow$ is locally confluent modulo the congruence $=_\Pi$.*

As a corollary we have the following theorem by the Newman's lemma.

Theorem 20 *The ARS C is strongly normalizing and confluent modulo the congruence $=_\Pi$.*

As an application the word problem for the equivalence relation generated by the ARS C is thus decidable as expected.

4 Related Work

There have been some attempts to investigate equational meaning of cut elimination from the categorical point of view. Among such attempts, [Lambek 68] is an early investigation of categorical semantics of the Syntactic calculus, which is now called Lambek calculus, and a recent paper by the same author [Lambek 93] dealt with the same subject in a more sophisticated manner. A categorical semantics of propositional fragment of Gentzen's LJ was explored in [Szabo 74]. Categorical semantics of sequent calculus of ~~propositional~~ intuitionistic first-order logic is systematically described in [Szabo 78]. These investigations has brought us a basis of the denotational semantics of reduction systems extracted from cut-elimination theorems.

On the other hand, the intention of this study is to understand the connection between sequent calculi and reduction systems by means of the sequents-as-sorts interpretation. This kind of investigation will bring us a basis of the operational semantics of reduction systems extracted from cut-elimination.

5 Conclusion

We have attempted a method to investigate operational semantics of deductive systems although the target deductive system was quite simple. Firstly, we introduced a Σ-term algebra corresponding to a variant of conjunction calculus by means of sequents-as-sorts interpretation. This interpretation is clearly sound and complete. Secondly, after having showed the weak normalization theorem, we extracted a reduction relation from the principal lemma. Finally the reduction system was shown to be strongly normalizing and confluent modulo a congruence relation.

What we have shown in this paper indicates that the sequents-as-sorts interpretation is an effective way to investigate the operational semantics of sequent calculi. There do not seem any significant obstacles to obtain Σ-term algebra interpretations of other sequent calculi, including substructural logics [Ono 90], or even Gentzen's LJ. To extract strongly normalizing and confluent reduction relations, we may, however, need to invent some technique to handle structural rules. This issue will be a future work.

References

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A Proof of Principal lemma

Let C be a formula of degree d. Suppose that $\sigma^C(f,g) : A \longrightarrow B$ is a redex. We construct a proof term h by induction on $h(f) + h(g)$.

1. Suppose that $C = A$ and that $f = 1 : A \longrightarrow A$ and $g : A \longrightarrow B$. Then h is g.

2. Suppose that $C = B$ and that $f : A \longrightarrow B$ and $g = 1 : B \longrightarrow B$. Then h is f.

3. Suppose that
$$f = \sigma^D(f_1, f_2) : A \longrightarrow C$$
where $f_1 : A \longrightarrow D$ and $f_2 : D \longrightarrow C$ for some $D \in \mathcal{F}$ such that $\partial(D) < d$. Then by the induction hypothesis for f_2 and g, we have a proof term $h_1 : D \longrightarrow B$ such that $\partial(h_1) < d$. And so we obtain a proof term h as follows
$$h = \sigma^D(f_1, h_1) : A \longrightarrow B.$$

4. Suppose that
$$g = \sigma^D(g_1, g_2) : C \longrightarrow B$$
where $g_1 : C \longrightarrow D$ and $g_2 : D \longrightarrow B$ for some $D \in \mathcal{F}$ such that $\partial(D) < d$. Then by the induction hypothesis for f and g_1, we have a proof term $h_1 : A \longrightarrow D$ such that $\partial(h_1) < d$. And so we obtain a proof term h as follows
$$h = \sigma^D(h_1, g_2) : A \longrightarrow B.$$

6 ~~5~~. Suppose that $A = A_1 \wedge A_2$ and that
$$f = \pi(f_1) : A_1 \wedge A_2 \longrightarrow C,$$
where $f_1 : A_1 \longrightarrow C$. Then by the induction hypothesis for f_1 and g, we have a proof term $h_1 : A_1 \longrightarrow B$ such that $\partial(h_1) < d$. And so we obtain a proof term h as follows
$$h = \pi(h_1) : A_1 \wedge A_2 \longrightarrow B.$$

7 ~~6~~. Suppose that $A = A_1 \wedge A_2$ and that
$$f = \pi'(f_1) : A_1 \wedge A_2 \longrightarrow C,$$
where $f_1 : A_2 \longrightarrow C$. Then by the induction hypothesis for f_1 and g, we have a proof term $h_1 : A_2 \longrightarrow B$ such that $\partial(h_1) < d$. And so we obtain a proof term h as follows
$$h = \pi'(h_1) : A_1 \wedge A_2 \longrightarrow B.$$

8 ~~7~~. Suppose that $B = B_1 \wedge B_2$ and that

$$g = \Pi(g_1, g_2) : C \longrightarrow B_1 \wedge B_2,$$

where $g_1 : C \longrightarrow B_1$ and $g_2 : C \longrightarrow B_2$. Then by the induction hypotheses for f and g_1 and for f and g_2, we have a proof terms $h_1 : A \longrightarrow B_1$ and $h_2 : A \longrightarrow B_2$ such that $\partial(h_1), \partial(h_2) < d$. And so we obtain a proof term h as follows

$$h = \Pi(h_1, h_2) : A \longrightarrow B_1 \wedge B_2.$$

9 ~~8~~. Suppose that $C = C_1 \wedge C_2$ and that

$$f = \Pi(f_1, f_2) : A \longrightarrow C_1 \wedge C_2 \quad \text{and} \quad g = \pi(g_1) : C_1 \wedge C_2 \longrightarrow B,$$

where $f_1 : A \longrightarrow C_1$, $f_2 : A \longrightarrow C_2$, and $g_1 : C_1 \longrightarrow B$. Since $\partial(C_1) < \partial(C)$ we obtain a proof term h as follows

$$h = \sigma^{C}(f_1, g_1) : A \longrightarrow B.$$

10 ~~9~~. Suppose that $C = C_1 \wedge C_2$ and that

$$f = \Pi(f_1, f_2) : A \longrightarrow C_1 \wedge C_2,$$

where $f_1 : A \longrightarrow C_1$ and $f_2 : A \longrightarrow C_2$, and also that

$$g = \pi'(g_1) : C_1 \wedge C_2 \longrightarrow B,$$

where $g_1 : C_2 \longrightarrow B$. Since $\partial(C_2) < \partial(C)$ we obtain a proof term h as follows

$$h = \sigma^{C_2}(f_2, g_1) : A \longrightarrow B.$$

□

B Proof of Proposition 15

Let $f : A \longrightarrow B$ such that $\partial(f) > 0$. By induction on $h(f)$ we show that there is a proof term $g : A \longrightarrow B$ such that $\partial(f) > \partial(g)$ and $f \rightarrow^* g$.

1. Suppose that $A = B$ and $f = 1 : A \longrightarrow A$. Then $\partial(f) = 0$, and so the claim holds vacuously.

2. Suppose that
$$f = \sigma^C(f_1, f_2) : A \longrightarrow B,$$
for some $f_1 : A \longrightarrow C$ and $f_2 : C \longrightarrow B$.

There are several cases.

(a) Suppose that $\partial(f_1), \partial(f_2) \leq \partial(C) = d$. Then by the induction hypothesis for f_1 and f_2, there exist $f_1' : A \longrightarrow C$ and $f_2' : C \longrightarrow B$ such that $\partial(f_1'), \partial(f_2') < \partial(C)$, $f_1 \rightarrow^* f_1'$ and $f_2 \rightarrow^* f_2'$. Then $\sigma^C(f_1', f_2')$ $\cancel{\partial(f_1'), \partial(f_2') \leq \partial(C) = d}$. is a redex, and so we obtain a proof term g by the above lemma.

(b) Suppose that $\partial(f_1), \partial(f_2) > \partial(C)$. Then by the induction hypotheses for f_1 and f_2 we have proof terms $g_1 : A \longrightarrow C$ and $g_2 : C \longrightarrow B$ such that $\partial(g_1), \partial(g_2) < d$, $f_1 \rightarrow^* g_1$ and $f_2 \rightarrow^* g_2$. And so we obtain a proof term g as follows
$$g = \sigma^C(g_1, g_2) : A \longrightarrow B.$$

3. Suppose that $A = A_1 \wedge A_2$ and that
$$f = \pi(f_1) : A_1 \wedge A_2 \longrightarrow B,$$
where $f_1 : A_1 \longrightarrow B$ such that $\partial(f_1) = d$. Then by the induction hypothesis for f_1, we have a proof term $g_1 : A_1 \longrightarrow B$ such that $\partial(g_1) < d$ and $f_1 \rightarrow^* g_1$. And so we obtain a proof term g as follows
$$g = \pi(g_1) : A_1 \wedge A_2 \longrightarrow B.$$

4. Suppose that $A = A_1 \wedge A_2$ and that $f = \pi'(f_1) : A_1 \wedge A_2 \longrightarrow B$, where $f_1 : A_2 \longrightarrow B$ such that $\partial(f_1) = d$. Same as the above case.

5. Suppose that $B = B_1 \wedge B_2$ and that $f = \Pi(f_1, f_2) : A \longrightarrow B_1 \wedge B_2$, where $f_1 : A \longrightarrow B_1$ and $f_2 : A \longrightarrow B_2$ such that $\partial(f_1) = d$ or $\partial(f_2) = d$. Same as the above case.

□

C Proof of Strong Normalization

It is clear that if f, f_1 and f_2 are strongly normalizing, then $\pi(f)$, $\pi'(f)$ and $\Pi(f_1, f_2)$ are. Below we implicitly use this fact.

Proposition 21 *Let $f : A \longrightarrow C$ and $g : C \longrightarrow B$. If f and g are strongly normalizing, then $\sigma^C(f, g)$ is strongly normalizing.*

Proof Let $f : A \longrightarrow C$ and $g : C \longrightarrow B$. Suppose that f and g are strongly normalizing. To show that $\sigma^C(f,g)$ is strongly normalizing it is sufficient to show that whenever $\sigma^C(f,g) \to h$, h is strongly normalizing. Induction on $\nu(f)+\nu(g)$, where $\nu(f)$ is a upper bound of length of every normalization sequence beginning with f.

1. $\partial(f), \partial(g) < \partial(C)$. Then $\sigma^C(f,g)$ is a redex. Suppose that h is the result of the reduction. By induction on the degree of the composition formula C we show that $h : A \longrightarrow B$ is strongly normalizing. Note that $\partial(f), \partial(g) < \partial(C)$ since $\sigma^C(f,g)$ is a redex.

   (a) Suppose that $C \in \mathcal{PS}$. By induction on the sum of the lengths of f and g, we show that h is strongly normalizing. Note that f cannot be of the form $\Pi(f_1, f_2)$ for some f_1 and f_2, and also both of f and g are normal forms since $\partial(f) = \partial(g) = 0$. And so there are next five cases.

      i. $C = A$ and $f = 1$. Then $\sigma^C(f,g) = \sigma^A(1,g)$, and so $h = g$. By assumption g is strongly normalizing.

      ii. $C = B$ and $g = 1$. Same as the above case.

      iii. $A = A_1 \wedge A_2$ for some A_1 and A_2, and $f = \pi(f_1)$. Then
      $$\sigma^C(f,g) = \sigma^C(\pi(f_1), g),$$
      and so
      $$h = \pi(\sigma^C(f_1, g)).$$
      Since f is strongly normalizing, f_1 is, and so by the induction hypothesis for f_1 and g, $\sigma^C(f_1,g)$ is strongly normalizing. Therefore h is.

      iv. $A = A_1 \wedge A_2$ for some A_1 and A_2, and $f = \pi'(f_1)$. This case is same as the above case.

      v. $B = B_1 \wedge B_2$ for some B_1 and B_2, and $g = \Pi(g_1, g_2)$. Then
      $$\sigma^C(f,g) = \sigma^C(f, \Pi(g_1, g_2)),$$
      and so
      $$h = \Pi(\sigma^C(f, g_1), \sigma^C(f, g_2)).$$
      Since g is strongly normalizing, g_1 and g_2 are, and so, by the induction hypotheses for f and g_1 and for f and g_2, $\sigma^C(f,g_1)$ and $\sigma^C(f,g_2)$ are strongly normalizing. Therefore h is.

(b) $C = C_1 \wedge C_2$ for some C_1 and C_2. Note that $\partial(f), \partial(g) < \partial(C)$. By induction on the sum of the lengths of f and g, we show that h is strongly normalizing. We consider the next four cases since other cases are same as the base step.

i. $f = \sigma^D(f_1, f_2)$ for some D, f_1 and f_2. Then

$$\sigma^C(f,g) = \sigma^C(\sigma^D(f_1, f_2), g),$$

and so

$$h = \sigma^D(f_1, \sigma^C(f_2, g)).$$

Since f is strongly normalizing, f_2 is, and so, by the induction hypothesis for f_2 and g, $\sigma^C(f_2, g)$ is strongly normalizing. Note that f_1 is strongly normalizing since f is, and $\partial(D) < \partial(C)$ since $\partial(f) < \partial(C)$. Therefore, by the induction hypothesis for D, h is.

ii. $g = \sigma^D(g_1, g_2)$ for some D, g_1 and g_2. Same as the above case.

iii. $f = \Pi(f_1, f_2)$ for some f_1 and f_2, and $g = \pi(g_1)$ for some g_1. Then

$$\sigma^C(f,g) = \sigma^{C_1 \wedge C_2}(\Pi(f_1, f_2), \pi(g_1)),$$

and so

$$h = \sigma^{C_1}(f_1, g_1).$$

Note that $\partial(C_1) < \partial(C)$, and that f_1 and g_1 are strongly normalizing since f and g are. And so, by the induction hypothesis for C_1, h is strongly normalizing.

iv. $f = \Pi(f_1, f_2)$, for some f_1 and f_2, and $g = \pi'(g_1)$ for some g_1. Same as the above case.

2. $f \to f'$. Then $\sigma^C(f,g) \to \sigma^C(f', g)$. Note that f' is strongly normalizing since f is. And so, by the induction hypothesis for f' and g, $\sigma^C(f', g)$ is strongly normalizing.

3. $g \to g'$. This case is same as the above.

□

Proof of Theorem 17 Let $f : A \longrightarrow B$ be an arbitrary term. By the induction on f, we show that f is strongly normalizing.

1. $A = B$ and $f = 1$. Then clearly f is strongly normalizing.
2. $f = \sigma^C(f_1, f_2)$ for some C and f_1 and f_2. Then by the induction hypotheses for f_1 and f_2, they are strongly normalizing. Hence f is by the above proposition.
3. $f = \pi(f_1)$ for some f_1. Then, by the induction hypothesis for f_1, it is strongly normalizing, and so f is.
4. $f = \pi'(f_1)$ for some f_1. Same as above case.
5. $f = \Pi(f_1, f_2)$ for some f_1 and f_2. Then, by the induction hypotheses for f_1 and f_2, they are strongly normalizing, and so f is.

□

D Proof of Local Confluence

For $f : A \longrightarrow C$ and $g : C \longrightarrow B$, we introduce a notation $f \triangleright^C g$ which stands for $\sigma^C(f, g)$. Below we implicitly use Theorem 17.

Lemma 22 *Let $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$, where $\partial(B) = \partial(C)$, be normal forms. And let m and n be normal forms of $f \triangleright^B g$ and $g \triangleright^C h$ respectively. Then there exists a term $k : A \longrightarrow D$ such that $m \triangleright^C h \rightarrow^* k$ and $f \triangleright^B n \rightarrow^* k$.*

We can prove this lemma by induction on the sum of lengths f, g and h.

Proof of Proposition 19 Let $f, h, k \in \mathcal{T}_{A \longrightarrow B}$. Assume that $f \rightarrow h$ and $f \rightarrow k$. Then we need to show that there exists $g \in \mathcal{T}_{A \longrightarrow B}$ such that $h \rightarrow^* g$ and $k \rightarrow^* g$. We check all the critical situations.

1. Suppose that $B = A$ and $f = 1 \triangleright^A 1$. Since $\partial(1) = 0 < \partial(A)$, we have $f \rightarrow_1 h$ and $f \rightarrow_2 k$, where $h = k = 1$. We set $g = 1$.
2. Suppose that
$$f = 1 \triangleright^A (f_1 \triangleright^D f_2)$$

for some $f_1 : A \longrightarrow D$ and $f_2 : D \longrightarrow B$ such that $\partial(f_1 \triangleright^D f_2) < \partial(A)$. Then $f \rightarrow_1 h$ and $f \rightarrow_4 k$, where

$$h = f_1 \triangleright^D f_2 \quad \text{and} \quad k = (1 \triangleright^A f_1) \triangleright^D f_2.$$

But since $\partial(f_1) < \partial(A)$ we have $1 \triangleright^A f_1 \rightarrow_1 f_1$, and so $k \rightarrow h$. We set $g = h$.

3. Suppose that $B = B_1 \wedge B_2$ and

$$f = 1 \triangleright^A \Pi(f_1, f_2)$$

for some $f_1 : A \longrightarrow B_1$ and $f_2 : A \longrightarrow B_2$ such that $\partial(\Pi(f_1, f_2)) < \partial(A)$. Then $f \rightarrow_1 h$ and $f \rightarrow_8 k$ where

$$h = \Pi(f_1, f_2) \quad \text{and} \quad k = \Pi(1 \triangleright^A f_1, 1 \triangleright^A f_2).$$

But since $\partial(f_1), \partial(f_2) < \partial(A)$ we have $1 \triangleright^A f_i \rightarrow_1 f_i$ for $i = 1, 2$, and so $k \rightarrow^+ h$. We set $g = h$.

4. Suppose that

$$f = (f_1 \triangleright^D f_2) \triangleright^B 1.$$

for some $f_1 : A \longrightarrow D$ and $f_2 : D \longrightarrow B$ such that $\partial(f_1 \triangleright^D f_2) < \partial(B)$. Then $f \rightarrow_2 h$ and $f \rightarrow_3 k$ where

$$h = f_1 \triangleright^D f_2 \quad \text{and} \quad k = f_1 \triangleright^D (f_2 \triangleright^B 1).$$

But since $\partial(f_2) < \partial(B)$ we have $f_2 \triangleright^B 1 \rightarrow_2 f_2$, and so $k \rightarrow h$. We set $g = h$.

5. Suppose that $A = A_1 \wedge A_2$ and

$$f = \pi(f_1) \triangleright^B 1$$

for some $f_1 : A_1 \longrightarrow B$ such that $\partial(\pi(f_1)) < \partial(B)$. Then $f \rightarrow_2 h$ and $f \rightarrow_6 k$ where

$$h = \pi(f_1) \quad \text{and} \quad k = \pi(f_1 \triangleright^B 1).$$

But since $\partial(f_1) < \partial(B)$, we have $f_1 \triangleright^B 1 \rightarrow_2 f_1$ and so $k \rightarrow h$. We set $g = h$.

6. Suppose that $A = A_1 \wedge A_2$ and $f = \pi'(f_1) \triangleright^B 1$ for some $f_1 : A_2 \longrightarrow B$ such that $\partial(\pi'(f_1)) < \partial(B)$. Same as the above case.

7. Suppose that
$$f = (f_1 \triangleright^D f_2) \triangleright^C (f_3 \triangleright^E f_4),$$
for some $f_1 : A \longrightarrow D$, $f_2 : D \longrightarrow C$, $f_3 : C \longrightarrow E$, and $f_4 : E \longrightarrow B$ such that $\partial(f_1 \triangleright^D f_2), \partial(f_3 \triangleright^E f_4) < \partial(B)$. Then $f \rightarrow_3 h$ and $f \rightarrow_4 k$ where
$$h = f_1 \triangleright^D (f_2 \triangleright^C (f_3 \triangleright^E f_4)),$$
$$k = ((f_1 \triangleright^D f_2) \triangleright^C f_3) \triangleright^E f_4.$$
But since $\partial(f_i), \partial(D), \partial(E) < \partial(C)$ we have
$$\begin{array}{llll} h & \rightarrow & f_1 \triangleright^D ((f_2 \triangleright^C f_3) \triangleright^E f_4) & \rightarrow^* \; n_1 \triangleright^D (n_2 \triangleright^E n_3) = h', \\ \text{and} & & & \\ k & \rightarrow & (f_1 \triangleright^D (f_2 \triangleright^C f_3)) \triangleright^E f_4 & \rightarrow^* \; (n_1 \triangleright^D n_2) \triangleright^E n_3 = k', \end{array}$$
where n_1, n_2 and n_3 are normal forms of f_1, $f_2 \triangleright^C f_3$ and f_4 respectively. There are three cases.

   - $\partial(D) < \partial(E)$. Then we have $k' \rightarrow h'$, thereby we set $g = k'$.
   - $\partial(E) < \partial(D)$. Then we have $h' \rightarrow k'$, thereby we set $g = h'$.
   - $\partial(D) = \partial(E)$. Let p and q be normal forms of $n_2 \triangleright^E n_3$ and $n_1 \triangleright^D n_2$ respectively. Then, by the above lemma, there exists r such that $n_1 \triangleright^D p \rightarrow^* r$ and $q \triangleright^E n_3 \rightarrow^* r$. And so $h \rightarrow^* r$ and $k \rightarrow^* r$. Thereby we set $g = r$.

8. Suppose that $B = B_1 \wedge B_2$ and
$$f = (f_1 \triangleright^D f_2) \triangleright^C \Pi(f_3, f_4),$$
for some $f_1 : A \longrightarrow D$, $f_2 : D \longrightarrow C$, $f_3 : C \longrightarrow B_1$, and $f_4 : C \longrightarrow B_2$ such that $\partial(f_1 \triangleright^D f_2), \partial(\Pi(f_3, f_4)) < \partial(B)$. Then $f \rightarrow_3 h$ and $f \rightarrow_8 k$ where
$$\begin{array}{lll} h & = & f_1 \triangleright^D (f_2 \triangleright^C \Pi(f_3, f_4)), \\ k & = & \Pi((f_1 \triangleright^D f_2) \triangleright^C f_3, (f_1 \triangleright^D f_2) \triangleright^C f_4) \end{array}$$

But since $\partial(f_i), \partial(D) < \partial(C)$ we have

$$\begin{array}{rcl} h & \to & f_1 \rhd^D \Pi(f_2 \rhd^C f_3, f_2 \rhd^C f_4) \to^* f_1 \rhd^D \Pi(n_1, n_2) \\ & \to & \Pi(f_1 \rhd^D n_1, f_1 \rhd^D n_2), \\ k & \to & \Pi(f_1 \rhd^{B_1} (f_2 \rhd^C f_3), f_1 \rhd^{B_2} (f_2 \rhd^C f_4)) \to^* \Pi(f_1 \rhd^D n_1, f_1 \rhd^D n_2), \end{array}$$

where n_1 and n_2 are normal forms of $f_2 \rhd^C f_3$ and $f_2 \rhd^C f_4$ respectively. Thereby we set $g = \Pi(f_1 \rhd^D n_1, f_1 \rhd^D n_2)$.

9. Suppose that $A = A_1 \wedge A_2$ and

$$f = \pi(f_1) \rhd^C (f_2 \rhd^D f_3)$$

for some $f_1 : A_1 \longrightarrow C$, $f_2 : C \longrightarrow D$ and $f_3 : D \longrightarrow B$ such that $\partial(\pi(f_1)), \partial(f_2 \rhd^D f_3) < \partial(B)$. Then $f \to_4 h$ and $f \to_6 k$, where

$$\begin{array}{rcl} h & = & (\pi(f_1) \rhd^C f_2) \rhd^D f_3, \\ k & = & \pi(f_1 \rhd^C (f_2 \rhd^D f_3)). \end{array}$$

But since $\partial(f_1), \partial(D) < \partial(B)$ we have

$$\begin{array}{rcl} h & \to & \pi(f_1 \rhd^C f_2) \rhd^D f_3 \to \pi((f_1 \rhd^C f_2) \rhd^D f_3), \\ k & \to & \pi((f_1 \rhd^C f_2) \rhd^D f_3). \end{array}$$

Thereby we set $g = \pi((f_1 \rhd^C f_2) \rhd^D f_3)$.

10. Suppose that $A = A_1 \wedge A_2$ and $f = \pi'(f_1) \rhd^C (f_2 \rhd^D f_3)$ for some $f_1 : A_2 \longrightarrow C$, $f_2 : C \longrightarrow D$ and $f_3 : D \longrightarrow B$ such that $\partial(\pi(f_1)), \partial(f_2 \rhd^D f_3) < \partial(B)$. Same as the above case.

11. Suppose that $A = A_1 \wedge A_2$, $B = B_1 \wedge B_2$ and

$$f = \pi(f_1) \rhd^C (\Pi(f_2, f_3))$$

for some $f_1 : A_1 \longrightarrow C$, $f_2 : C \longrightarrow B_1$ and $f_3 : C \longrightarrow B_2$ such that $\partial(\pi(f_1)), \partial(\Pi(f_2, f_3)) < \partial(B)$.

Then $f \to_6 h$ and $f \to_8 k$, where

$$\begin{array}{rcl} h & = & \pi(f_1 \rhd^C \Pi(f_2, f_3)), \\ k & = & \Pi(\pi(f_1) \rhd^C f_2, \pi(f_1) \rhd^C f_3). \end{array}$$

But since $\partial(f_1), \partial(f_2), \partial(f_3) < \partial(B)$ we have

$$\begin{array}{rcl} h & \to & \pi(\Pi(f_1 \triangleright^C f_2, f_1 \triangleright^C f_3)) = h', \\ k & \to & \Pi(\pi(f_1 \triangleright^C f_2), \pi(f_1 \triangleright^C f_3)) = k'. \end{array}$$

Thereby we set $g = h' =_\Pi k'$.

12. Suppose that $A = A_1 \wedge A_2$, $B = B_1 \wedge B_2$ and

$$f = \pi'(f_1) \triangleright^C \Pi(f_2, f_3)$$

for some $f_1 : A_2 \longrightarrow C$, $f_2 : C \longrightarrow B_1$ and $f_3 : C \longrightarrow B_2$ such that $\partial(\pi'(f_1)), \partial(\Pi(f_2, f_3)) < \partial(B)$. Then $f \to_7 h$ and $f \to_8 k$, where

$$\begin{array}{rcl} h & = & \pi'(f_1 \triangleright^C \Pi(f_2, f_3)), \\ k & = & \Pi(\pi'(f_1) \triangleright^C f_2, \pi'(f_1) \triangleright^C f_3). \end{array}$$

This case is same as the above case.

□