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Fuzzy Graph Rewritings

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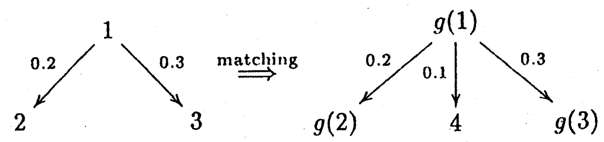
Abstract

This paper presents fuzzy graph rewriting systems with fuzzy relational calculus. In this paper fuzzy graph means crisp set of vetices and fuzzy set of edges. We provide fuzzy relational calculus with Heyting algebra. Formalizing rewriting system of fuzzy graphs it is important to choose how to match graphs. Therefore matching condition is argued. Moreover a variation of relatively pseudo-complement is studied for difference relation of edges. Two kind of matching conditions are introduced. One is rigorous matching and the other is ambiguous matching. Rigorous matching lead us to the theorem that resultant graph of rewriting and pushout construction of graphs are equivalent. Finally we study ambiguous matching.

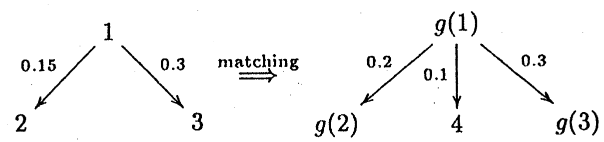
1 Introduction

Fuzzy theory has a notion "Fuzzy Graph" which graphically shows fuzzied relations of objects, for example fuzzy dynamic programming and fuzzied citation diagram of documents[KST+90]. In order to operate fuzzy graphs one may use representation with adjacent matrices. Though representation with adjacent matrices has an advantage of numeric calculation, it can neither deletes nor adds vetices. Adjacent matrices have no more than informations of relations and fuzziness on fixed vertices. On the other hand Ehrig et al[HMM91] presented an algebraic approach to graph transformation and Mizoguchi and Kawahara[MK95] generalized graph rewriting system with relational calculus. These researchers' works give a categorical aspects and one can view global observation of rewriting graphs. These theories intend to deal with "crisp" graphs.

This paper presents fuzzy graph rewriting systems with fuzzy relational calculus. In this paper fuzzy graph means crisp set of vetices and fuzzy set of edges. We operate fuzzy graph with fuzzy relational calculus which is originated from fuzzy relational algebra[KF95]. Fuzzy relational algebra is a fuzzy relation on single domain (called *homogeneous*). Though fuzzy relational calculus associates with the case of multi domain(called *heterogeneous*), we can make use of many results of fuzzy relational algebra. We provide set–theoretical operation (union, intersection, etc.) with Heyting algebra, and we give some consideration to the complements of relations because graph rewriting implies the difference of relations. In these consideration we are concerned about choosing how to match graphs. It is natural that graph rewriting system requires to match graphs as subgraphs. Working on "crisp" graphs one may define matching condition using morphisms. If we adopt only the inclusion of relation (which is the condition of morphisms) for matching condition of fuzzy graphs, then we have inappropriate examples for our intuition. In the following figures which present in the above situation, graphs in the left hand side match graphs in the right hand side and g is a morphism of matching:



Matching condition is given as morphism of graphs which implies inclusion with respect to relations of edges. Fuzzy theory defines set inclusion by order of membership function value, but adopting this set inclusion the following matching is possible:



We give some argument about matchings such as correspondence from $1 \overset{0.15}{\to} 2$ to $g(1) \overset{0.2}{\to} g(2)$. Intuitively we define matching condition which preserves subgraph and equality of membership value. We investigate two kind of matching. One matching described above is rigorous matching which requires equality of membership value of edges. The other is ambiguous matching which allows some ambiguity for membership value of edges.

As shown in Mizoguchi and Kawahara [MK95] the category of graphs and partial morphisms has pushouts, we show the same result in the category of fuzzy graphs and

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partial morphisms. Moreover confluency and critical pairs of fuzzy graphs are studied.

In section 2 we briefly review Heyting algebra and its properties. Set-theoretical arguments of fuzzy relation can be resolved into Heyting algebra. In section 3 we introduce fuzzy relational calculus. Relatively pseudo-complement and variation of complements are studied. In the last section we formalize fuzzy graph rewritings. As stated above matching condition is argued.

2 Heyting algebra

In this section we will review Heyting Algebra and its properties (ref. [Gol79]). Let $(L, \sqsubseteq)$ be a lattice. For $a, b \in L$ the **relatively pseudo-complement of a relative to b**, denoted by $a \Rightarrow b$, is the greatest element x such that $a \sqcap x \sqsubseteq b$. A lattice $(L, \sqsubseteq)$ is called **relatively pseudo-complemented lattice** if there exists the relatively pseudo-complement of a relative to b for any $a, b \in L$. In the case of $b = 0$, if 0 exists in $(L, \sqsubseteq)$, then it is called **pseudo-complement of a**, denoted by $\neg a$. Equivalently we can state as following; for any $x \in L$

$$a \sqcap x \sqsubseteq b \quad \text{if and only if} \quad x \sqsubseteq a \Rightarrow b$$

Heyting algebra is relatively pseudo-complemented lattice with the zero element. We review some properties without proof.

Proposition 2.1 *Let $(L, \sqsubseteq)$ be a Heyting algebra and let a, b, c be in L.*

1. *There exists the maximum element $1 \in L$, which is defined $1 = a \Rightarrow a$ for any a.*
2. *$a \sqsubseteq b$ if and only if $a \Rightarrow b = 1$.*
3. *$b \sqsubseteq a \Rightarrow b$.*
4. *$(a \Rightarrow b) \sqcap a = a \sqcap b$.*
5. *$(a \Rightarrow b) \sqcap b \sqsubseteq b$.*
6. *$(a \Rightarrow b) \sqcap (a \Rightarrow c) = a \Rightarrow b \sqcap c$.*
7. *$a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$, $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$.*
8. *$a \sqsubseteq \neg\neg a$.*

In a Heyting algebra double negation of an element is not equal to the original element, and the law of the extended middle does not hold. But both are equivalent.

Theorem 2.1 *In a Heyting algebra $(L, \sqsubseteq)$, for any $a \in L$*

$$a = \neg\neg a \quad \textit{if and only if} \quad a \sqcup (\neg a) = 1$$

3 Fuzzy relations

The fuzzy relation algebra in terms of homogeneous relations is presented by Kawahara and Furusawa[KF95] and they adapt scalar of relations in order to prove the representation theorem. We will show that heterogeneous fuzzy relations are a Heyting algebra in the sense of [KF95].

Let A, B be sets. A fuzzy relation (relation, for short) on A and B is a function α from the cartesian product $A \times B$ to the closed interval $[0, 1]$. We denote the set of all relation on A and B by $\mathbf{Rel}(A, B)$. The zero relation $0_{A,B}$ and the universal relation $\Theta_{A,B}$ are relations with $0_{A,B}(a, b) = 0$ and $\Theta_{A,B}(a, b) = 1$ for any $(a, b) \in A \times B$, respectively. Clearly $0_{A,B}, \Theta_{A,B} \in \mathbf{Rel}(A, B)$. We abbreviate $0_{A,B}$ and $\Theta_{A,B}$ to 0 and Θ if their domains are understood from the contexts. Throughout this paper we write $\alpha : A \rightharpoonup B$ for a relation α on A and B, and the uppercase letters, $A, B, C, \cdots$, and the 3 letters means sets and relations, respectively.

In the sense of [KF95] we define the order in $\mathbf{Rel}(A, B)$. The relation α is contained by β, denoted by $\alpha \sqsubseteq \beta$, if and only if $\alpha(a, b) \leq \beta(a, b)$ for any $(a, b) \in A \times B$. Obviously it holds that $0 \sqsubseteq \alpha \sqsubseteq \Theta$ for all $\alpha \in \mathbf{Rel}(A, B)$ and;

Proposition 3.1 *$\langle \mathbf{Rel}(A, B), \sqsubseteq \rangle$ is a partially ordered set.*

For a family $\{\alpha_\lambda\}_\lambda$ of relations we define fuzzy relations $\sqcap_\lambda \alpha_\lambda$ and $\sqcup_\lambda \alpha_\lambda$ as follows:

$$\begin{aligned} (\sqcap_\lambda \alpha_\lambda)(a, b) &= \bigwedge_\lambda [\alpha_\lambda(a, b)] \\ (\sqcup_\lambda \alpha_\lambda)(a, b) &= \bigvee_\lambda [\alpha_\lambda(a, b)] \end{aligned}$$

we call them the 2 and the infinimum of $\{\alpha_\lambda\}_\lambda$ respectively. For shorthand we write $\alpha \sqcap \beta$ and $\alpha \sqcup \beta$ for the 2 and the infinimum of $\{\alpha, \beta\}$. The operations $\sqcap$ and $\sqcup$ meets commutative law, associative law and absorption law. Moreover from proposition 3.1 we have:

Proposition 3.2 *$\langle \mathbf{Rel}(A, B), \sqsubseteq, \sqcap, \sqcup \rangle$ is a lattice with the zero element.*

Definition 3.1 *For α and β in $\mathrm{Rel}(A, B)$, the relation $\alpha \Rightarrow \beta$ is defined as*

$$(\alpha \Rightarrow \beta)(a, b) = \begin{cases} 1 & \text{if } \alpha(a, b) \leq \beta(a, b) \\ \beta(a, b) & \text{if } \alpha(a, b) > \beta(a, b) \end{cases}$$

Clearly $\alpha \Rightarrow \beta \in \mathbf{Rel}(A, B)$.

The binary operation $\Rightarrow$ determines the relatively pseudo-complement on $\mathbf{Rel}(A, B)$.

Proposition 3.3 *Let α and β be in $\mathrm{Rel}(A, B)$. For any relation $\gamma \in \mathbf{Rel}(A, B)$, it holds that*

$$\alpha \sqcap \gamma \sqsubseteq \beta \quad \textit{if and only if} \quad \gamma \sqsubseteq \alpha \Rightarrow \beta$$

Proof : Assume that $\gamma \sqsubseteq \alpha \Rightarrow \beta$, i.e. $\gamma(a,b) \leq (\alpha \Rightarrow \beta)(a,b)$ for any (a,b).

$$(\alpha \sqcap (\alpha \Rightarrow \beta))(a,b) = \begin{cases} \alpha(a,b) & \text{if } \alpha(a,b) \leq \beta(a,b) \\ (\alpha \sqcap \beta)(a,b) & \text{if } \alpha(a,b) > \beta(a,b) \end{cases}$$

Clearly we have $\alpha \sqcap (\alpha \Rightarrow \beta) = \alpha \sqcap \beta$. By assumption it holds that $\alpha \sqcap \gamma \sqsubseteq \alpha \sqcap (\alpha \Rightarrow \beta) \sqsubseteq = \alpha \sqcap \beta \sqsubseteq \beta$. Conversely assume that $\alpha \sqcap \gamma \sqsubseteq \beta$. If $\alpha(a,b) \leq \gamma(a,b)$, then $\alpha(a,b) = (\alpha \sqcap \gamma)(a,b) \leq \beta(a,b)$. So that we have $(\alpha \Rightarrow \beta)(a,b) = 1$. Therefore it holds that $\gamma(a,b) \leq (\alpha \Rightarrow \beta)(a,b)$. Otherwise, i.e. $\alpha(a,b) > \gamma(a,b)$, by assumption we have $\gamma(a,b) = (\alpha \sqcap \gamma)(a,b) \leq \beta(a,b) \leq (\alpha \Rightarrow \beta)(a,b)$. Hence $\gamma \sqsubseteq (\alpha \Rightarrow \beta)$. □

We denote the relatively pseudo-complements $\alpha \Rightarrow 0$ of a relative to 0 by $\neg a$. From the arguments so far, we have an important property of fuzzy relations.

Theorem 3.1 *For any sets A and B, $\langle \mathbf{Rel}(A,B), \sqsubseteq, \sqcap, \sqcup \rangle$ is a Heyting algebra.*

We call a relation $\alpha \in \mathbf{Rel}(A,B)$ **regular** if and only if $\alpha(a,b) = 0$ or $\alpha(a,b) = 1$ for any x and y.

Proposition 3.4 *If α and β are regular then $\neg\alpha \sqcup \beta = (\alpha \Rightarrow \beta)$.*

Proof : Since $\langle \mathbf{Rel}(A,B), \sqsubseteq, \sqcap, \sqcup \rangle$ is a Heyting algebra it holds that $\neg\alpha = \alpha \Rightarrow 0 \sqsubseteq \alpha \Rightarrow \beta$ and:

$$\alpha \sqcap \beta \sqsubseteq \beta \quad \text{if and only if} \quad \beta \sqsubseteq \alpha \Rightarrow \beta$$

Then we have $\neg\alpha \sqcup \beta \sqsubseteq (\alpha \Rightarrow \beta)$. Conversely, if $\alpha(a,b) = \beta(a,b)$ then $1 = (\alpha \Rightarrow \beta)(a,b) \leq (\neg\alpha \sqcup \beta)(a,b) = 1$. Else if $\alpha(a,b) < \beta(a,b)$, $\alpha(a,b) = 0$ and $\beta(a,b) = 1$ since α and β are regular. Hence we have $1 = (\alpha \Rightarrow \beta)(a,b) \leq (\neg\alpha \sqcup \beta)(a,b) = 1$. Otherwise, i.e. $\alpha(a,b) > \beta(a,b)$, similarly $\alpha(a,b) = 1$ and $\beta(a,b) = 0$. Therefore $0 = \beta(a,b) = (\alpha \Rightarrow \beta)(a,b) \leq (\neg\alpha \sqcup \beta)(a,b) = 1$. Hence we have $(\alpha \Rightarrow \beta) \sqsubseteq \neg\alpha \sqcup \beta$. □

The converse of proposition 3.4 does not necessarily holds. The following is a counterexample of it.

Example 3.1 *Set $0 < b < a < 1$. Let us define relations α and β from $A = [0,1]$ to $B = \{0\}$ (one-point set) as follows;*

$$\alpha(a,b) = \begin{cases} 0 & 0 \leq x < a \\ \frac{1}{2} & x = a \\ 1 & a < x \leq 1 \end{cases}$$

$$\beta(a,b) = \begin{cases} 0 & 0 \leq x < b \\ \frac{1}{2} & x = b \\ 1 & a < x \leq 1 \end{cases}$$

Though $(\neg\alpha \sqcup \beta)(a,b) = (\alpha \Rightarrow \beta)(a,b) = 1$ for any (a,b), that is, $\neg\alpha \sqcup \beta = (\alpha \Rightarrow \beta)$, α and β are not regular.

For relations $\alpha \in \mathbf{Rel}(A,B)$ and $\beta \in \mathbf{Rel}(B,C)$ we define the composition of α and β by

$$(\alpha\beta)(a,b) = \bigvee_{b \in B} [\alpha(a,b) \wedge \beta(b,c)]$$

Proposition 3.5 [KF95]

1. *For $\alpha \in \mathbf{Rel}(A,B)$, $\beta \in \mathbf{Rel}(B,C)$ and $\gamma \in \mathbf{Rel}(A,C)$, $\alpha\beta \sqcap \gamma \sqsubseteq \alpha(\beta \sqcap \alpha^\sharp\gamma)$.*
2. *A relation $\alpha \in \mathbf{Rel}(A,B)$ is regular if and only if there exists a relation $\beta \in \mathbf{Rel}(A,B)$ such that $\alpha \sqcup \beta = \Theta$ and $\alpha \sqcap \beta = 0$.*

Proof : The former is straightforward. We prove the latter. Suppose that α is not regular, i.e. there exists (a_0, b_0) such that $0 < \alpha(a_0, b_0) < 1$. This contradicts either $\alpha \sqcap \beta = 0$ or $\alpha \sqcup \beta = \Theta$. Hence α is regular. □

Proposition 3.6 *For $\alpha \in \mathbf{Rel}(A,B)$ and $\beta, \gamma \in \mathbf{Rel}(A,B)$, if $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}_B$, then it holds that $\alpha(\beta \Rightarrow \gamma) \sqsubseteq (\alpha\beta \Rightarrow \alpha\gamma)$. Especially $\alpha(\neg\beta) \sqsubseteq \neg(\alpha\beta)$. Moreover if $\mathrm{id}_A \sqsubseteq \alpha\alpha^\sharp$, then it holds that $\alpha(\beta \Rightarrow \gamma) = (\alpha\beta \Rightarrow \alpha\gamma)$.*

Proof : By Dedekind formula and assumption we have $\alpha(\beta \Rightarrow \gamma) \sqcap \alpha\beta \sqsubseteq \alpha[(\beta \Rightarrow \gamma) \sqcap \alpha^\sharp\alpha\beta] \sqsubseteq \alpha[(\beta \Rightarrow \gamma) \sqcap \beta] \sqsubseteq \alpha\gamma$. Hence $\alpha(\beta \Rightarrow \gamma) \sqsubseteq \alpha\beta \Rightarrow \alpha\gamma$. In addition to $\alpha^\sharp\alpha \sqsubseteq \mathrm{id}_B$ suppose that $\mathrm{id}_A \sqsubseteq \alpha\alpha^\sharp$. Let δ be arbitrary relation such that $\delta \sqsubseteq (\alpha\beta \Rightarrow \alpha\gamma)$. Equivalently it holds that $\alpha\beta \sqcap \delta \sqsubseteq \alpha\gamma$. By Dedekind formula we have the following: $\beta \sqcap \alpha^\sharp\delta \sqsubseteq \alpha^\sharp(\alpha\beta \sqcap \alpha\alpha^\sharp\delta) \sqsubseteq \alpha^\sharp(\alpha\beta \sqcap \delta) \sqsubseteq \alpha^\sharp\alpha\gamma \sqsubseteq \gamma$. Therefore it holds that $\alpha^\sharp\delta \sqsubseteq \beta \Rightarrow \gamma$, By the added assumption $\delta \sqsubseteq \alpha\alpha^\sharp\delta \sqsubseteq \alpha(\beta \Rightarrow \gamma)$. Hence $\alpha\beta \Rightarrow \alpha\gamma \sqsubseteq \alpha(\beta \Rightarrow \gamma)$. □

Proposition 3.7 *For relations $\alpha, \beta \in \mathbf{Rel}(A,B)$ it holds that $(\alpha \Rightarrow \beta)^\sharp = \alpha^\sharp \Rightarrow \beta^\sharp$.*

Proof : It is straightforward from definitions. □

If a relation $\alpha \in \mathbf{Rel}(A,B)$ satisfies univalency $\alpha^\sharp\alpha \sqsubseteq id_B$ it is called **partial function.** Moreover a partial function $\alpha \in \mathbf{Rel}(A,B)$ is called **(total) function** if it satisfies totality $id_A \sqsubset \alpha\alpha^\sharp$. We use lowercase letters, $f, g, h, \cdots$ for partial functions and functions. For simplicity we have arrow denotation for relations and functions. For $\alpha \in \mathbf{Rel}(A,B)$ we denote $\alpha : A \rightharpoonup B$ and for a partial function (resp. function) $f \in \mathbf{Rel}(A,B)$ we denote $f : A \to B$: pfn (resp. tfn) . We define subtraction of relations by: $\alpha - \beta = \alpha \sqcap (\neg\beta)$.

Proposition 3.8 *If $\alpha, \beta : A \rightharpoonup B$ are relations and $f : X \to A$ and $g : Y \to B$ are partial functions, then $f(\alpha \sqcap \beta)g^\sharp = f\alpha g^\sharp \sqcap f\beta g^\sharp$. Moreover if f and g are functions then $f(\alpha - \beta)g^\sharp = f\alpha g^\sharp - f\beta g^\sharp$.*

Proof : We need to show $f\alpha g^\sharp \sqcap f\beta g^\sharp \sqsubseteq f(\alpha \sqcap \beta)g^\sharp$. By Dedekind formula $f\alpha g^\sharp \sqcap f\beta g^\sharp \sqsubseteq f(\alpha g^\sharp g \sqcap f^\sharp f\beta)g \sqsubseteq f(\alpha \sqcap \beta)g^\sharp$. In addition let us suppose that f and g are functions. Then $f(\alpha - \beta)g^\sharp = f\alpha g^\sharp \sqcap f(\neg\beta)g^\sharp$. By proposition 3.6 and proposition 3.7 $f(\neg\beta)g^\sharp = \neg(f\beta g^\sharp)$. Hence $f(\alpha - \beta)g^\sharp = f\alpha g^\sharp - f\beta g^\sharp$. □

The regularity of relation is absolutely determined either related or not. We present the extended regularity relative to a given relation.

Definition 3.2 *Let $\alpha, \beta \in$ **Rel**(A, B) and $\alpha \sqsubseteq \beta$. The relation α is regular with respect to β if and only if $\alpha \sqsubseteq \beta$ and $\alpha(a, b) \neq 0$ implies $\alpha(a, b) = \beta(a, b)$.*

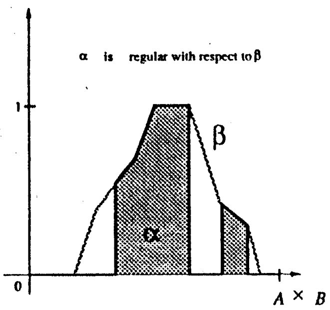


Figure 1: Regularity with respect to some relation

Proposition 3.9 *If $\alpha, \alpha' \in$ **Rel**(A, B) are regular with respect to $\beta \in$ **Rel**(A, B) then $\alpha \sqcap \alpha'$ and $\alpha \sqcup \alpha'$ are regular with respect to β.*

Proposition 3.10 *If α is regular with respect to β then there exists a relation γ such that $\alpha \sqcup \gamma = \beta$ and $\alpha \sqcap \gamma = 0$.*

Proof : Choose γ as

$$\gamma(a, b) = \begin{cases} \beta(a, b) & \text{if} \quad \alpha(a, b) = 0 \\ 0 & \text{if} \quad \alpha(a, b) = \beta(a, b). \end{cases}$$

It can be proved that γ holds the condition. □

We call such relation γ **quasi–complement of α with respect to β** , denoted by $\overset{\beta}{\neg}\alpha$. The next lemmas show that regularity relative to some relation is extension of regularity and quasi–complement is weaker negation than pseudo–complement.

Theorem 3.2 *Let $\alpha, \beta \in$ **Rel**(A, B) and $\alpha \sqsubseteq \beta$. Then the following statements are equivalent.*

1. *the relation α is regular with respect to β.*
2. *For every $k \in [0, 1]$, $\alpha(a, b) \wedge k\beta(a, b) = k\alpha(a, b)$.*
3. $x \sqsubseteq \beta \wedge \alpha \sqcap x = 0 \;\leftrightarrow\; x \sqsubseteq \overset{\beta}{\neg}\alpha$.

Proof : (1.→ 3.) Suppose that $x \sqsubseteq \gamma$ and $\alpha \sqcap x = 0$. For any a, b it holds that $x(a, b) \leq \gamma(a, b)$ and $(a, b) \wedge \beta(a, b) = 0$. If $\alpha(a, b) = 0$ then $(\overset{\gamma}{\neg}\alpha)(a, b) = \gamma(a, b)$. Else if $\alpha(a, b) \neq 0$ then $(\overset{\gamma}{\neg}\alpha)(a, b) = 0$. So that $x(a, b) \leq (\overset{\gamma}{\neg}\alpha)(a, b)$. Hence $x \sqsubseteq \overset{\gamma}{\neg}\alpha$. Conversely suppose that $x \sqsubseteq \overset{\gamma}{\neg}\alpha$. For any a, b it holds that $x(a, b) \leq (\overset{\gamma}{\neg}\alpha)(a, b)$. From the definition of quasi–complement we have $x(a, b) \leq (\overset{\gamma}{\neg}\alpha)(a, b) \leq \gamma(a, b)$. Moreover from proposition 3.10,

$$\begin{aligned} \alpha(a, b) \wedge x(a, b) &\leq \alpha(a, b) \wedge (\overset{\gamma}{\neg}\alpha)(a, b) \\ &= (\alpha \sqcap (\overset{\gamma}{\neg}\alpha))(a, b) \\ &= 0 \end{aligned}$$

Hence we have that $x \sqsubseteq \gamma$ and $\alpha \sqcap x = 0$.

(3.→ 1.) Suppose that $\alpha(a, b) \neq \gamma(a, b)$ for any a, b. Then $(\overset{\gamma}{\neg}\alpha)(a, b) = \gamma(a, b)$ from the definition. By assumption take the relation $\overset{\gamma}{\neg}\alpha$ as x then, $(\overset{\gamma}{\neg}\alpha)(a, b) \leq \gamma(a, b)$ and $\alpha(a, b) \wedge (\overset{\gamma}{\neg}\alpha)(a, b) = \alpha(a, b) \wedge \gamma(a, b) = 0$. As $\alpha \sqsubseteq \gamma$ it holds that $0 = \alpha(a, b) \wedge \gamma(a, b) = \alpha(a, b)$. Hence, if $\alpha(a, b) \neq 0$ then $\alpha(a, b) = \gamma(a, b)$. The proof completes. □

Lemma 3.1 *The relation $\alpha \in$ **Rel**(A, B) is regular if and only if it is regular with respect to $\Theta_{A,B}$.*

Proof : For only–if part Obviously $\alpha \sqsubseteq \Theta$. If $\alpha(a, b) \neq 0$, then $\alpha(a, b) = 1 = \Theta(a, b)$ by the regularity. Conversely suppose that $\alpha(a, b) \neq 0$. From assumption $\alpha(a, b) = \Theta(a, b) = 1$. □

Lemma 3.2 *Let $\alpha, \beta, \gamma \in$ **Rel**(A, B). If α and β are regular with respect to γ, then*

1. *$\overset{\gamma}{\neg}\alpha$ is regular with respect to γ,*
2. $\overset{\gamma}{\neg}\overset{\gamma}{\neg}\alpha = \alpha$,
3. *if $\alpha \sqsubseteq \beta$ then $\overset{\gamma}{\neg}\beta \sqsubseteq \overset{\gamma}{\neg}\alpha$,*
4. $\overset{\gamma}{\neg}(\alpha \sqcup \beta) = (\overset{\gamma}{\neg}\alpha) \sqcap (\overset{\gamma}{\neg}\beta)$,
5. $(\overset{\gamma}{\neg}\alpha) \sqcup (\overset{\gamma}{\neg}\beta) \sqsubseteq \overset{\gamma}{\neg}(\alpha \sqcap \beta)$,
6. *$\overset{\gamma}{\neg}\alpha \sqsubseteq \neg\alpha$, and*
7. $\beta \sqcap (\neg\alpha) = \beta \sqcap (\overset{\gamma}{\neg}\alpha)$.

Proof :

1. Trivial by the definition of quasi–complement.
2. Easy from 1. and proposition 3.10.
3. Let $x \sqsubseteq \overset{\gamma}{\neg}\beta$. Then $x \sqsubseteq \gamma$ and $x \sqcap \beta = 0$. Therefore $x \sqcap \alpha \sqsubseteq x \sqcap \beta = 0$.
4. Let $x \sqsubseteq \overset{\gamma}{\neg}(\alpha \sqcup \beta)$. Equivalently we have $x \sqsubseteq \gamma$ and $x \sqcap \alpha = 0$ and $x \sqcap \beta = 0$. Hence $x \sqsubseteq (\overset{\gamma}{\neg}\alpha) \sqcap (\overset{\gamma}{\neg}\beta)$.

5. If $x \sqsubseteq (\overset{\gamma}{\neg}\alpha) \sqcup (\overset{\gamma}{\neg}\beta)$ then we have $x \sqsubseteq \gamma$ and $(x \sqcap \alpha) \sqcap (x \sqcap \beta) = 0$, that is, $x \sqsubseteq \gamma$ and $x \sqcap \alpha = 0$ and $x \sqcap \beta = 0$. Hence it holds that $x \sqsubseteq \overset{\gamma}{\neg}(\alpha \sqcap \beta)$. Converse does not necessarily holds.

6. We need to show that $\alpha \sqcap \overset{\gamma}{\neg}\alpha = 0$.

$$(\alpha \sqcap (\overset{\gamma}{\neg}\alpha))(a,b) = \begin{cases} 0 \wedge \gamma(a,b) & \text{if } \alpha(a,b) = 0 \\ \gamma(a,b) \wedge 0 & \text{if } \alpha(a,b) = \gamma(a,b) \end{cases}$$

is equal to 0. Hence $\overset{\gamma}{\neg}\alpha \sqsubseteq \neg\alpha$.

7. From the above clearly $\beta \sqcap (\overset{\gamma}{\neg}\alpha) \sqsubseteq \beta \sqcap (\neg\alpha)$. Conversely $(\gamma \sqcap (\neg\alpha)) \sqsubseteq \gamma$ and $(\gamma \sqcap (\neg\alpha)) \sqcap \alpha = \gamma \sqcap (\neg\alpha) \sqcap \alpha = 0$. Hence we have $(\gamma \sqcap (\neg\alpha)) \sqsubseteq (\overset{\gamma}{\neg}\alpha) = \gamma \sqcap (\overset{\gamma}{\neg}\alpha)$.

The proof completes. □

Definition 3.3 *Given $0 < \epsilon < 1$. A relation α is ϵ-regular with respect to γ if for any x, y,*

$$\alpha(a,b) \neq 0 \rightarrow \gamma(a,b) \neq 0 \wedge |\alpha(a,b) - \gamma(a,b)| \leq \epsilon$$

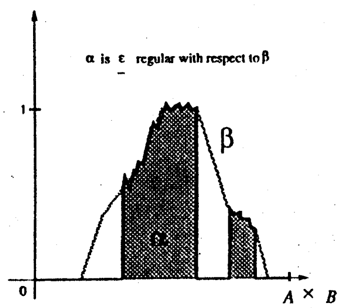


Figure 2: ϵ-regularity

Needless to say, if a relation is regular with respect to some relation then it is ϵ-regular. We say that two relation are *ϵ-equal* if each of them is ϵ-regular with respect to the other.

Proposition 3.11 *If relations α and β are ϵ-regular with respect to γ then $\alpha \sqcup \beta$ is ϵ-regular with respect to γ.*

Proof : Suppose that $(\alpha \sqcup \beta)(a,b) \neq 0$. Then we have $\alpha(a,b) \neq 0$ and $\beta(a,b) \neq 0$. By assumption, it holds that $\gamma(a,b) \neq 0$ and

$$|\gamma(a,b) - \alpha(a,b)| \leq \epsilon, \; |\gamma(a,b) - \beta(a,b)| \leq \epsilon$$

So that

$$|\gamma(a,b) - (\alpha \sqcup \beta)(a,b)| \leq \epsilon$$

holds. Hence $\alpha \sqcup \beta$ is ϵ-regular with respect to γ. □

Lemma 3.3 *Let $\alpha : A \rightharpoonup A$ be ϵ-regular with respect to $\gamma : A \rightharpoonup A$, and let f be a function from B to A. If f is regular and injective then $f^\sharp \alpha f$ is ϵ-regular with respect to $f^\sharp \gamma f$.*

$$\begin{array}{ccc} A & \xrightarrow[\gamma]{\alpha} & A \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[f^\sharp \gamma f]{f^\sharp \alpha f} & B \end{array}$$

Proof : For any a and b suppose that $(f^\sharp \alpha f)(a,b) \neq 0$. As f is regular and injective there uniquely exist c_1 and c_2 such that

$$f^\sharp(a,c_1) \wedge \alpha(c_1,c_2) \wedge f(c_2,b) \neq 0.$$

By regularity of f it holds that $f^\sharp(a,c_1) = f(c_2,b) = 1$ and $\alpha(c_1,c_2) \neq 0$. Since α is ϵ-regular with respect to γ we have $\gamma(c_1,c_2) \neq 0$ and

$$|\gamma(c_1,c_2) - \alpha(c_1,c_2)| \leq \epsilon$$

Note that $f^\sharp(a,c_1) \wedge \gamma(c_1,c_2) \wedge f(c_2,b) = \gamma(c_1,c_2)$ and $f^\sharp(a,c_1) \wedge \alpha(c_1,c_2) \wedge f(c_2,b) = \alpha(c_1,c_2)$, then

$$|(f^\sharp \sqcup \gamma \sqcup f)(a,b) - (f^\sharp \sqcup \alpha \sqcup f)(a,b)| \leq \epsilon.$$

Hence $f^\sharp \alpha f$ is ϵ-regular with respect to $f^\sharp \gamma f$. □

Lemma 3.4 *If a relation α is ϵ-regular with respect to γ then there exists $\hat{\alpha}$ such that it is regular with respect to γ and $\neg\alpha = \neg\hat{\alpha}$.*

Proof : If $((\neg\alpha) \sqcap \gamma)(a,b) \neq 0$ then $((\neg\alpha) \sqcap \gamma)(a,b) = \gamma(a,b)$ by the regularity of $\neg\alpha$. Therefore $((\neg\alpha) \sqcap \gamma)$ is regular with respect to γ. Put $\hat{\alpha} = \overset{\gamma}{\neg}((\neg\alpha) \sqcap \gamma)$. Take $x \sqsubseteq \neg\alpha \sqcap \hat{\alpha}$. Equivalently $x \sqsubseteq \neg\alpha$ and $x \sqsubseteq \gamma$ and $x \sqcap (\neg\alpha) \sqcap \gamma = 0$. Hence $x = 0$, that is, $\neg\alpha \sqcap \hat{\alpha} = 0$. Conversely observe that

$$\begin{aligned} \neg \overset{\gamma}{\neg}((\neg\alpha) \sqcap \gamma) \sqcap \alpha &\sqsubseteq \neg \overset{\gamma}{\neg}((\neg\alpha) \sqcap \gamma) \sqcap \neg\neg\alpha \\ &= \neg(\overset{\gamma}{\neg}((\neg\alpha) \sqcap \gamma) \sqcup \neg\alpha) \end{aligned}$$

If $(\neg\alpha)(a,b) = 0$, then $\alpha(a,b) \neq 0$. By ϵ-regularity $\gamma(a,b) \neq 0$. Therefore $(\overset{\gamma}{\neg}((\neg\alpha) \sqcap \gamma) \sqcup \neg\alpha)(a,b) = \gamma(a,b) \neq 0$. Else if $(\neg\alpha)(a,b) = 1$ then obviously $(\overset{\gamma}{\neg}((\neg\alpha) \sqcap \gamma) \sqcup \neg\alpha)(a,b) \neq 0$. So that $\neg(\overset{\gamma}{\neg}((\neg\alpha) \sqcap \gamma) \sqcup \neg\alpha) = 0$. Hence $\neg \overset{\gamma}{\neg}((\neg\alpha) \sqcap \gamma) \sqcap \alpha = \neg\hat{\alpha} \sqcap \alpha = 0$. Then we can conclude $\neg\alpha = \neg\hat{\alpha}$. □

Domains of relations can be represented by relations. For relation $\alpha : A \rightharpoonup B$ the **domain** of α is a relation $d(\alpha) = \alpha\alpha^\sharp \sqcap id_A$. We review properties domain of relations from [MK95].

Proposition 3.12 [MK95] *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$ be relations and $f : A \rightarrow B$ be a partial function.*

1. $d(\alpha\beta)d(\alpha) = d(\alpha\beta)$.

2. $d(f\beta)f = fd(\beta)$.

Proof : By Dedekind formula $d(\alpha\beta) = \alpha\beta\beta^\sharp\alpha^\sharp \sqcap id_A \sqsubseteq \alpha(\beta\beta^\sharp\alpha^\sharp \sqcap \alpha^\sharp) \sqsubseteq \alpha\beta\beta^\sharp\alpha^\sharp \sqcap \alpha\alpha^\sharp$. Therefore $d(\alpha\beta) = (\alpha\beta\beta^\sharp\alpha^\sharp \sqcap id_A) \sqcap (\alpha\beta\beta^\sharp\alpha^\sharp \sqcap \alpha\alpha^\sharp) = \alpha\beta\beta^\sharp\alpha^\sharp \sqcap \alpha\alpha^\sharp \sqcap id_A \sqsubseteq d(\alpha)$. By Dedekind formula and univalency of f we have: $d(f\beta)f = (f\beta\beta^\sharp f^\sharp \sqcap id_A)f \sqsubseteq f\beta\beta^\sharp f^\sharp f \sqcap f \sqsubseteq f\beta\beta^\sharp \sqcap f \sqsubseteq f(\beta\beta^\sharp \sqcap f^\sharp f) \sqsubseteq f(\beta\beta^\sharp \sqcap id_B) = fd(\beta)$. □

Proposition 3.13 *Let $\alpha : A \rightarrow A$, $\theta : B \rightarrow B$ and $f : A \rightarrow B$: pfn . If $\theta \sqsubseteq f^\sharp\alpha f$ then $\theta = f^\sharp f\theta f^\sharp f$.*

Proof : Obviously we have $f^\sharp f\theta f^\sharp f \sqsubseteq \theta$ since f is a partial function. Choose $\Theta_{A,A}$ for α. By assumption $\theta \sqsubseteq f^\sharp\Theta_{A,A}f$. Thus $\theta = \theta \sqcap f^\sharp\Theta_{A,A}f \sqsubseteq f^\sharp(f\theta f^\sharp \sqcap \Theta_{A,A})f = f^\sharp f\theta f^\sharp f$ by Dedekind formula. □

4 Rewriting for graphs

Mizoguchi and Kawahara[MK95] presented graph rewriting with regular relations. This section presents the formalization of fuzzy graph rewriting system in the same manner as [MK95].

A **graph** $\langle A, \alpha\rangle$ is a pair of set A and a relation $\alpha : A \rightarrow A$. A **partial morphism** of graph $\langle A, \alpha\rangle$ into $\langle B, \beta\rangle$ is a regular–partial function $f : A \rightarrow B$ satisfying $d(f)\alpha f \sqsubseteq f\beta$. If f is a total function then we merely say **morphism**. Note that we will deal with "regular(crisp)" sets of nodes and "fuzzy" relation of edges and morphisms are regular relation.

It is easy to show that all of graphs and their partial morphisms form a category, which we denote by **Pfn($\mathcal{F}$-Graph)**. In the sense of regular relations it is proved that the category of sets and partial functions (denoted by **Pfn**) has pushouts(ref. [Kaw90]).

Theorem 4.1 *The category* **Pfn** *has pushouts.*

This theorem lead us to the same fact as [MK95] in the sense of **Pfn($\mathcal{F}$-Graph)**.

Corollary 4.1 *The category* **Pfn($\mathcal{F}$-Graph)** *has pushouts.*

Let us consider the following diagrams, which are in **Pfn($\mathcal{F}$-Graph)** and **Pfn**. In the left hand side f and g are partial morphisms. By the theorem 4.1 one can construct pushout in **Pfn**, which is in the middle.

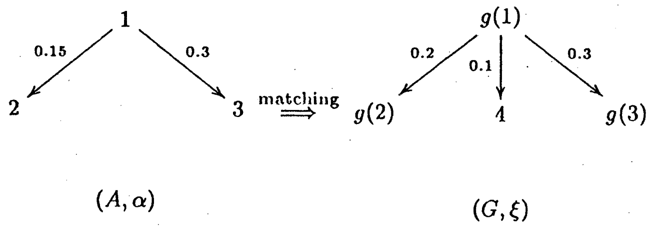
$$\begin{array}{ccc} \langle A,\alpha\rangle & \xrightarrow{f} & \langle B,\beta\rangle \\ {\scriptstyle g}\downarrow & & \\ \langle C,\gamma\rangle & & \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \vdots\,{\scriptstyle h} \\ C & \xrightarrow[k]{\cdots} & D \end{array} \qquad \begin{array}{ccc} \langle A,\alpha\rangle & \xrightarrow{f} & \langle B,\beta\rangle \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \langle C,\gamma\rangle & \xrightarrow[k]{} & \langle D,\delta\rangle \end{array}$$

Define $\delta = k^\sharp\gamma k \sqcup h^\sharp\beta h$. Then we obtain the pushout square in the right hand side.

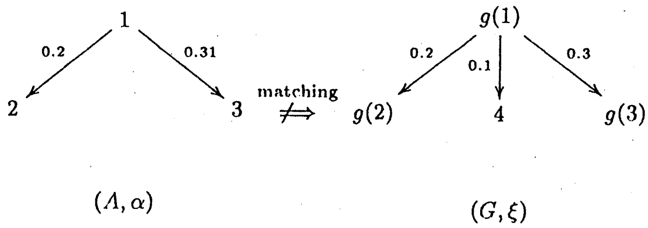
Rewriting consists of two notion. One is "rewriting rules" which is correspondence between nodes and can be formalized as partial functions on sets of nodes. The other is "matchings" into which rewriting rules are applied. Matchings must indicate appropriate subgraphs in objective graphs.

Definition 4.1 **A rewriting rule** *is a triple $p = (\langle A, \alpha\rangle, \langle B, \beta\rangle, f : A \rightarrow B)$ where f is a partial function (It is not necessarily a partial morphism).*

Now we will consider matching conition. Working on "crisp" graphs one may define matching condition using partial morphisms. If we adopt only the inclusion of relation (which is the condition of partial morphisms) for matching condition, i.e., $\alpha g \sqsubseteq g\beta$, then we have inappropriate examples for our intuition.



In this example $\langle A, \alpha\rangle$ is matched into $\langle G, \xi\rangle$, but in the next example $\langle A, \alpha\rangle$ is not matched into $\langle G, \xi\rangle$.



In order to except such examles we should preserve equality of membership value.

Definition 4.2 *A morphism g from $\langle A, \alpha\rangle$ into $\langle G.\xi\rangle$ is a* **matching** *from $\langle A, \alpha\rangle$ into $\langle G.\xi\rangle$ if $g^\sharp\alpha g$ is regular relative to ξ.*

By theorem 4.1 we can construct the pushout from rewriting rule and matching. For partial functions in definition 4.1 we have the following pushout square in **Pfn**.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\downarrow & & \vdots\,{\scriptstyle h} \\ G & \xrightarrow[k]{\cdots} & H \end{array}$$

The graph $\langle G, \xi\rangle$ is said to be **rewritten into** $\langle H, \eta\rangle$ **by applying a rewriting rule** p **along a matching** g if the relation η is defined as $\eta = k^\sharp\beta k \sqcup h^\sharp(\xi - g^\sharp\alpha g)k$. We denote by $\langle G, \xi \rangle \underset{p/g}{\Longrightarrow} \langle H, \eta \rangle$. Applying rewriting rule is viewed as a **rewriting square**:

$$
\begin{tikzcd}
\langle A, \alpha \rangle \arrow[r, "f"] \arrow[d, "g"'] & \langle B, \beta \rangle \arrow[d, "h"] \\
\langle G, \xi \rangle \arrow[r, "k"'] & \langle H, \eta \rangle
\end{tikzcd}
$$

Mizoguchi and Kawahara [MK95] show that the rewriting square is a pushout square in **Pfn(Graph)** if the function in rewriting rule is a partial morphism of graph. In our case the similar results can be obtained. Moreover the construction of pushouts in the category of fuzzy graph is the same as [MK95].

Theorem 4.2 *Let p be a rewriting rule $(\langle A, \alpha \rangle, \langle B, \beta \rangle, f : A \to B)$ and $g : \langle A, \alpha \rangle \to \langle B, \beta \rangle$ be a matching. If $f : \langle A, \alpha \rangle \to \langle B, \beta \rangle$ is a partial morphism, then rewriting square $\langle G, \xi \rangle \underset{p/g}{\Leftrightarrow} \langle H, \eta \rangle$ is a pushout square in* **Pfn(Graph)**.

Proof: As f is a partial homomorphism we have $f^\sharp \alpha f = (f \sqcap f^\sharp f f^\sharp) \alpha f \sqsubseteq f^\sharp (id_A \sqcap f f^\sharp) \alpha f \sqsubseteq f^\sharp d(f) \alpha f \sqsubseteq f^\sharp f \beta \sqsubseteq \beta$ by Dedekind formula. We need to show that $\eta = h^\sharp \beta h \sqcup k^\sharp \alpha k$. From the construction of η and by lemma 3.2,

$$
\begin{aligned}
\eta &= h^\sharp \beta h \sqcup k^\sharp (\xi - g^\sharp \alpha g) k \\
&= h^\sharp \beta h \sqcup k^\sharp (\xi \sqcap (\overset{\xi}{\neg} g^\sharp \alpha g)) k \\
&\sqsupseteq h^\sharp f^\sharp \alpha f h \sqcup k^\sharp (\xi \sqcap (\overset{\xi}{\neg} g^\sharp \alpha g)) k \\
&= k^\sharp g^\sharp \alpha g k \sqcup k^\sharp (\xi \sqcap (\overset{\xi}{\neg} g^\sharp \alpha g)) k \\
&= k^\sharp (g^\sharp \alpha g \sqcup (\xi \sqcap (\overset{\xi}{\neg} g^\sharp \alpha g))) k \\
&= k^\sharp (\xi \sqcap (g^\sharp \alpha g \sqcup (\overset{\xi}{\neg} g^\sharp \alpha g))) k \\
&= k^\sharp \xi k
\end{aligned}
$$

Hence it holds that $\eta \sqsubseteq h^\sharp \beta h \sqcup k^\sharp (\xi - g^\sharp \alpha g) k \sqcup k^\sharp \xi k \sqsubseteq h^\sharp \beta h \sqcup k^\sharp \xi k$. The proof completes. □

Next we define the ambiguous matching condition. Resultant graph applied along an ambiguous matching can be equivalent to pushout construction in the sense of that ambiguity.

Definition 4.3 *Fix $\epsilon \in [0, 1]$. $g : A \to G$ is a ϵ–* **matching** *from $\langle A, \alpha \rangle$ to $\langle G, \xi \rangle$ if and only if the relation $g^\sharp \alpha g$ is ϵ–regular with respect to ξ.*

Definition 4.4 *Two graphs $\langle A, \alpha \rangle$ and $\langle B, \beta \rangle$ are ϵ–equal if and only if each of α and β is ϵ–regular with respect to the other.*

By lemma3.3 we can state that;

Theorem 4.3 *If rewriting function f is an injective partial morphism and matching is ϵ–matching, then the resultant graph of rewriting and resultant graph of pushout square are ϵ–equivalent.*

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