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DEMOCRATIC COMPACTIFICATION OF CONFIGURATION SPACES OF POINT SETS ON THE REAL PROJECTIVE LINE

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To the memory of late cat Oshin (? - 02-01-1995)

Introduction

The configuration space of distinct colored $n(\geq 3)$ points on the real projective line P^1 is defined by

$$X(n) = PGL(2)\backslash\{(P^1)^n - \Delta\},$$

where $\Delta = \{(x_1,\dots,x_n) \mid x_i = x_j \text{ for some } i \neq j\}$, and the group $PGL(2)$ of projective transformations acts diagonally and freely. The space $X(n)$ admits a natural action of the symmetric group S_n through the permutations of n points.

If one destroys the symmetry and let $x_1 = 0, x_2 = 1, x_n = \infty$, then the quotient space $X(n)$ can be identified to the complement of hyperplanes in $\mathbb{R}^{n-3}$:

$$\{(x_3,\dots,x_{n-1}) \in \mathbb{R}^{n-3} \mid x_j \neq 0, 1, x_k (k \neq j)\}.$$

Let us have a look at one of its connected components bounded by hyperplanes, say,

$$x_3 = 1, \quad x_3 = x_4, \ \dots \ , x_{n-2} = x_{n-1};$$

see Figure 0. Each point in the domain represents an arrangement of n points $x_1,\dots,x_n$ such that

$$x_1(=0) < x_2(=1) < x_3 < \cdots < x_{n-1} < x_n(=\infty) \ \in P^1.$$

Thus it is natural to label this domain by the sequence $12\cdots n$; according to the fact that P^1 is a circle and to the action of $PGL(2)$, we should think that the following sequences

$$\begin{array}{l} 23\cdots n1, \ 3\cdots n12, \ \ \cdots \ \ , n12\cdots , \\ n\cdots 321, \ 1n\cdots 32, \cdots \end{array}$$

are equivalent to $12\cdots n$; the equivalence class is called an n-juzu (see Figure 1), and will be denoted by one of the sequences above.

Thus the space $X(n)$ consists of $(n-1)!/2$ chambers (connected components) coded by n-juzus.

We add to $X(n)$ various degenerate arrangements as well as extra varieties to get a smooth variety $\bar{X}(n)$ so that $\bar{X}(n) - X(n)$ is purely 1-codimensional with normal crossings only, and that the group S_n acts bi-regularly on it. We describe the shape of chambers in $\bar{X}(n)$, and clarify what kind of degenerate arrangements and extra varieties are added.

The chamber above can also be obtained in the following way. Let Δ be an $(n-3)$-dimensional simplex. Make the barycentric subdivision of Δ, then $(n-2)!$ small simplices are cut out by the hyperplanes; call one of these C. Truncate C along the $(n-5)$-dimensional faces along which the hyperplanes are not normally crossing, and along their intersections. The truncated one is just the prescribed chamber.

Apart from its own interest of the construction, it relates to the following subjects.

(1) The complexification of the space $X(n)$ is the natural domain of definition of the Appell-Lauricella hypergeometric differential equation F_D in $n-3$ variables, which is also called the one of type $(2, n)$; cf. [K], [Sek], [ST], [Ter], [Oda].

(2) If you are interested in an integral of a Selberg type:

$$\int \prod_{i=1}^{n} t_i^{\alpha_i} \prod_{j=1}^{n} (1-t_j)^{\beta_j} \prod_{i<j} (t_i - t_j)^{\gamma_{ij}} f(t, x) dt,$$

then you know that the intersection numbers of loaded cycles concerning to the integral play an important role; cf. [KY] and [CMY]. To compute them, one blows up along the non-normally crossing part of the divisor

$$\prod_{i=1}^{n} t_i \prod_{j=1}^{n} (1-t_j) \prod_{i<j} (t_i - t_j) = 0;$$

the consequent manifold would just be $\bar{X}(n+3)$. Notice that real blow-ups can be described by truncations.

I shall explain the shape of the chamber $12\cdots n$, since all the chambers will have the same shape (they are transitive under S_n), by describing its boundary in $\bar{X}(n)$. In order to help the reader understand what is going on, we describe it for $n = 3, \cdots, 7$, and then give the general results.

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1. Geometric observations

For an integer $n \geq 4$, let C be the $(n-3)$-dimensional chamber of $X(n)$ coded by the n-juzu $12\cdots n$. Degenrate arrangements such that

$$x_1 < \cdots x_{j-1} < x_j = x_{j+1} < x_{j+1} < \cdots < x_n$$

form an $(n-4)$-dimensional space $C(j)$, which is isomorphic to a chamber of $X(n-1)$. The $C(j)$'s $(j = 1, \ldots, n)$ can be considered as part of the *boundary* of C. Suppose, by induction, we know the boundary of each $C(j)$, and patch them along its boundary; we shall find (when $n \geq 6$) that they do not bound C, i.e. there are many holes. We shall find that each hole is a direct product of chambers with lower dimension. In this section, we give an intuitive description of the situation when $3 \leq n \leq 7$. A complete description will be made in the next section by introducing an algebra.

1.1. $X(3)$

Since any three distinct points on the real projective line P^1 can be transformed to any other such, $X(3)$ consists of a point, which will be denoted by 123.

1.2. $X(4)$

The space $X(4)$ is isomorphic to

$$P^1 - \{\text{three distinct points}\},$$

(it is well-known that a cross-ratio gives an isomorphism), consisting of three open intervals coded by

$$1243, \quad 1234, \quad 1324.$$

Since the space $X(4)$ is 1-dimensional, it can be uniquely compactified by adding three points, which represent arrangements that two points coincide. The arrangement of three points $x_1 = x_2 < x_3 < x_4$ shall be denoted by (12)34, and so on. By the uniqueness above, the two arrangements

$$(12)34 \quad \text{and} \quad 12(34)$$

must be identified. The compactification $\bar{X}(4)$ can be illustrated as follows:

$$\begin{array}{ccccccc} & \circ & & \circ & & \circ & \\ 1243 & (12)34 & 1234 & 1(23)4 & 1324 & (13)24 & 1342 \\ & \| & & \| & & \| & \\ & 12(34) & & 23(41) & & 13(42) & \end{array}$$

Thus the boundary $\partial\{1234\}$ of the chamber 1234 is given as follows

$$\partial\{1234\} = \{12(34) = (12)34\} \cup \{23(41) = (23)41\}.$$

Notice that the chamber 1243, which is adjacent to 1234 through the point $(12)34 = 12(34)$ is obtained from 1234 by performing a permutation (12) or equivalently by (34).

1.3. $X(5)$

Denote the set of arrangements

$$x_1 = x_2 < x_3 < x_4 < x_5,$$

by the juzu (12)345, which can be considered as a chamber of $X(4)$. Juzus with parentheses are defined accordingly. The chamber 12345 is bounded by five segments

$$(12)345, \quad (34)512, \quad (51)234, \quad (23)451, \quad (45)123.$$

This can be shown as follows: the boundary of each segment is given by the formula above:

$$\begin{aligned} \partial\{(12)345\} &= (12)3(45) \cup (12)(34)5, \\ \partial\{(23)451\} &= (23)4(51) \cup (23)(45)1, \\ \partial\{(34)512\} &= (34)5(12) \cup (34)(51)2, \\ \partial\{(45)123\} &= (45)1(23) \cup (45)(12)3, \\ \partial\{(51)234\} &= (51)2(34) \cup (51)(23)4. \end{aligned}$$

See Figure 2. Thus one sees that the five segments form a circle bounding a polygon 12345:

$$\partial\{12345\} = (12)345 \cup (34)512 \cup (51)234 \cup (23)451 \cup (45)123.$$

Recall that we are forced, by the argument in the previous section, to identify for instance the following three arrangements:

$$(12)3(45), \quad (123)45 \quad 12(345).$$

The chamber 12354, obtained from 12345 by performing the permutation (45), is adjacent to 12345 through the segment (45)123; the chamber 13254, obtained from 12345 by performing the permutations (23) and (45), touches to 12345 at a point (45)1(23). Figure 3 will give you a global idea, in which (12)345 and (12)3(45) are replaced by (12) and (12)(45).

Remark. Let Δ_2 be a 2-simplex, a triangle bounded by three (= 5 − 2) lines. Make the barycentric subdivision by three (= (5 − 2)(5 − 3)/2) lines, and one gets six (=(5 − 2)!) small triangles; call one of these D. The triangle D has two vertices where three lines meet. Truncate D at the two vertices and one gets a pentagon. See Figure 4.

1.4. $X(6)$

The 2-dimensional chambers (pentagons)

$$(12)3456, \dots, (61)2345$$

are part of the boundary of the chamber 123456. Figure 5 illustrates the pentagon 1234(56) with its boundary consisting of five segments:

$$(12)34(56),\ 1(23)4(56),\ 12(34)(56), \quad 123(4(56)),\ 234((56)1).$$

Glue the six pentagons along their possible boundaries. Figure 6 shows a stereographic image of this object, where (12)34(56) and 123(4(56)) are replaced by (12)(56) and (4(56)). In this figure we find three rectangular holes, which may be coded by

$$(123) \times (456), \quad (234) \times (561), \quad (345) \times (612).$$

The first one represents the product of the two segments (123)456 and 123(456). In this way the six pentagons and the three rectangles are glued along their boundaries to form a 2-sphere bounding the chamber 123456.

The (3-dimensional) chamber 213456 obtained from 123456 by the permutation (12) is adjacent to the chamber 123456 through the pentagon (12)3456.

The chamber 321456 = 123654 obtained from 123456 by reversing the sequence 123, or by reversing 456, is adjacent to the chamber 123456 through the rectangle $(123) \times (456)$.

The chamber 214356 obtained from 123456 by the permutations (12) and (34) is adjacent to the chamber 123456 along the segment (12)(34)56, around which are four chambers corresponding to the permutations $(), (12), (34), (12)(34)$.

The chamber 231456 obtained from 123456 by permuting (12) and then reversing the sequence 213, or by reversing the sequence 123 and then permuting (12) is adjacent to the chamber 123456 along the segment ((12)3)456. Around the segment there are four chambers $123456, 213456, 321456$ and 312456.

The chamber 214365 obtained from 123456 by the permutations (12), (34) and (56) touches the chamber 123456 at a point, around which are eight chambers corresponding to the permutations

$$(),\ (12),\ (34),\ (56),\ (12)(34),\ (34)(56),\ (56)(12),\ (12)(34)(56).$$

The chamber 312546 touches the chamber 123456 at a point, around which are eight chambers:

$$123456,\ 213456,\ 123546,\ 123654,\ 312456,\ 213546,\ 123645,\ 312546.$$

Remark. Let Δ_3 be a 3-simplex, a tetrahedron bounded by four $(= 6 - 2)$ planes. Make the barycentric subdivision by six $(= (6 - 2)(6 - 3)/2)$ planes, and one gets twenty four $(= (6 - 2)!)$ small tetrahedra; call one of these D. The tetrahedron D has three edges along which three planes meet; the three edges do not form a triangle but form the letter Z; they meet at the two vertices which are the two cusps of Z. See Figure 7. Truncate D at the two vertices and along the three edges. Let us describe this polytope: after the truncation, the four triangular faces and the two vertices become pentagons, and the three edges become rectangles. You see this is just the shape of the chamber 123456 described above. You can make a paper model by six regular pentagons and three scquares, according to the pachwork shown in Figure 6.

1.5. $X(7)$

The seven 3-dimensional chambers (the polytope described above)

$$(12)34567, \dots, (71)23456$$

are part of the boundary of the chamber 1234567. Figure 8 illustrates the polytope 12345(67) with its boundary consisting of six pentagons and three rectangles:

$$(12)345(67),\ 1(23)45(67),\ 12(34)5(67),\ 123(45)(67),\quad 1234(5(67)),\ 2345((67)1),$$
$$(123) \times (45(67)),\ (234) \times (5(67)1),\ (345) \times ((67)12)$$

Glue the seven polytopes along their possible pentagonal faces, and imagine the sterographic image in the 3-dimensional space; you shall find seven holes with the shape of pentagonal prism (the product of a pentagon and a segment):

$$(123)4567 \times 123(4567)$$

and their cyclic permutations; see Figure 9.

In this way the seven polytopes and the seven pentagonal prisms are glued along their boundaries to form a 3-sphere bounding the chamber 1234567.

1.6. $X(8), \cdots$

Now you can guess what will be going on with the $(n-3)$-dimensional chamber $1 \cdots n$. It would be bounded by the $(n-4)$-chambers

$$(12)3 \cdots n,\ (23)4 \cdots n1,\ \cdots\ , (n1)2 \cdots (n-1),$$

and the direct products of $(n-k-2)$-dimensional and $(k-2)$-dimensional chambers:

$$(12\cdots k)\cdots n \times 12\cdots k(k+1\cdots n), \quad k=3,\cdots,n-3;$$

they would be patched along their boundaries to form an $(n-4)$-dimensional sphere bounding the chamber $12\cdots n$. But since we are doing mathematics, guess is not enough. In the next section, we are going to fix an orientation on each chamber, and to define the boundary operator ∂, and to show $\partial\partial = 0$, which proves the above guess.

Our idea is as follows:

(1) introduce dummy chambers and think, for example,

$$\begin{aligned}
12345(67) &\quad \text{as} \quad 12345(67) \times (12345)67,\\
(12)345(67) &\quad \text{as} \quad (12)345(67) \times 12(34567) \times (12345)67,\\
1234(5(67)) &\quad \text{as} \quad 1234(567) \times (1234)5(67) \times (12345)67,\\
(123) \times (45(67)) &\quad \text{as} \quad (123)4567 \times 123(4567) \times (12345)67,
\end{aligned}$$

(2) fix an orientation for each chamber,

(3) think the product $\times$ above as the exterior product $\wedge$,

(4) define the boundary operator ∂ in the exterior algebra.

2. An algebraic formulation

Consider for an integer $n \geq 3$ a free $\mathbb{Z}$ module F_n generated by the following elements

$$\begin{aligned}
&(\overbrace{1\cdots a_1}^{a_1})(\overbrace{a_1+1\cdots a_1+a_2+1}^{a_2})\cdots(\overbrace{\cdots n}^{a_p})\\
&= (-)^{p+1}(\overbrace{a_1+1\cdots}^{a_2})\cdots(\overbrace{\cdots n}^{a_p})(\overbrace{1\cdots a_1}^{a_1})\\
&= (-)^{2(p+1)}(\overbrace{\cdots}^{a_3})\cdots(\overbrace{\cdots n}^{a_p})(\overbrace{1\cdots a_1}^{a_1})(\overbrace{\cdots}^{a_2})\\
&= \cdots\\
&= (-)^{p+1}(\overbrace{\cdots n}^{a_p})(\overbrace{1\cdots a_1}^{a_1})\cdots(\overbrace{\cdots}^{a_{p-1}})
\end{aligned}$$

and their cyclic permutations of $1,2,\cdots,n$, where

$$a_j \geq 1, \quad p \geq 3.$$

The element above will be called an (oriented) p-juzu, and each part sandwiched by parenthesis a bead. When $a_i = 1$ we often write

$$j \;\text{ in place of }\; (\overbrace{\;j\;}^{a_i}).$$

Under this convention, F_3 is generated by one element $123 = 231 = 312$.

F_4 is generated by $1234 = -2341 = 3412 = -4123$ and

$$12(34) = 2(34)1 = (34)12,\ 23(41),\ 34(12),\ 41(23).$$

F_5 is generated by $12345 = 23451 = \cdots$ and the cyclic permutations of $1,2,3,4,5$ of

$$123(45), \quad 1(23)(45), \quad \text{and} \quad 12(345);$$

notice that $123(45) = -23(45)1 = 3(45)12 = -(45)123$.

F_6 is generated by $123456 = -234561 = \cdots$ and the cyclic permutations of

$$1234(56),\ 123(456),\ 12(34)(56),\ 1(23)4(56),\ (12)(34)(56),\ 12(3456),\ 1(23)(456).$$

Let Λ_n be the algebra generated by F_n equipped with the product $\wedge$ such that

$$J_p \wedge J_q = (-)^{(p+1)(q+1)} J_q \wedge J_p,$$

where J_p is a p-juzu, and

$$J_p \wedge J_q = 0 \quad \text{if } J_p \text{ and } J_q \text{ have a common bead.}$$

On the algebra Λ_n we are going to define a boundary operator $\partial = \partial_n$. We define it for the n-juzu $12\cdots n$, and extend it by $\mathbb{Z}$-linearlity and the chain rule

$$\partial\{J_p \wedge J_q\} = \partial_p\{J_p\} \wedge J_q + (-)^{p+1} J_p \wedge \partial_q\{J_q\}.$$

When describing a juzu in F_n, we often omit redundant informations, for instance, in place of

$$12\cdots(n-a)(\overbrace{n-a+1\cdots n}^{a}),$$

we would write

$$\cdots(\overbrace{\cdots n}^{a}), \quad \cdots(n-a)(\overbrace{\cdots}^{a}) \quad \text{or} \quad 1\cdots(\overbrace{\cdots}^{a}).$$

Definition.

$$\partial\{1\cdots n\} = \sum_{\text{cyclic}} \sum_{a=2}^{(n-1)/2} \cdots(\overbrace{\cdots n}^{a}) \wedge \cdots n(\overbrace{\cdots}^{n-a}), \quad n\text{: odd}$$

$$\partial\{1\cdots n\} = \sum_{\pm\text{cyclic}} \sum_{a=2}^{n/2-1} (-)^a \cdots(\overbrace{\cdots n}^{a}) \wedge \cdots n(\overbrace{\cdots}^{n-a})$$

$$+ (-)^{n/2} \sum_{\pm\text{cyclic}/2} \cdots(\overbrace{\cdots n}^{n/2}) \wedge \cdots n(\overbrace{\cdots}^{n/2}), \quad n\text{: even,}$$

where the sum over $\pm$cyclic means the alternating sum for $1,2,\ldots,n$, and the sum over $\pm$cyclic/2 means the alternating sum for $1,2,\ldots,n/2$. Please check that this definition is consistent under cyclic permutations of $1,2,\ldots,n$.

Let us show some examples:

$$
\begin{aligned}
\partial\{123\} &= 0, \\
\partial\{1234\} &= 12(34) \wedge 34(12) - 23(41) \wedge 41(23), \\
\partial\{12345\} &= 123(45) \wedge 45(123) + 234(51) \wedge 51(234) + \cdots + 512(34) \wedge 34(512), \\
\partial\{123456\} &= 1234(56) \wedge 56(1234) - 2345(61) \wedge 61(2345) + \cdots - 6123(45) \wedge 45(6123) \\
&\quad - \{123(456) \wedge 456(123) - 234(561) \wedge 561(234) + 345(612) \wedge 612(345)\}, \\
\partial\{1234567\} &= 12345(67) \wedge 67(12345) + \cdots + 71234(56) \wedge 56(71234) \\
&\quad + 1234(567) \wedge 567(1234) + \cdots + 7123(456) \wedge 456(7123), \\
\partial\{12345678\} &= 123456(78) \wedge 78(123456) - \cdots + \cdots - 812345(67) \wedge 67(812345) \\
&\quad - \{12345(678) \wedge 678(12345) - \cdots + \cdots - 81234(567) \wedge 567(81234)\} \\
&\quad + 1234(5678) \wedge 5678(1234) - \cdots + \cdots - 4567(8123) \wedge 8123(4567);
\end{aligned}
$$

now you can do it by yourself. The rest of this section is devoted to a proof of the following

Theorem.

$$\partial\partial = 0.$$

Lemma 1. *For even n,*

$$\partial\left[\sum_{\pm\text{cyclic}/2} \cdots(\overbrace{\cdots n}^{n/2}) \wedge \cdots n(\overbrace{\cdots}^{n/2})\right] = \sum_{\pm\text{cyclic}} \partial\{\cdots(\overbrace{\cdots n}^{n/2})\} \wedge \cdots n(\overbrace{\cdots}^{n/2}).$$

Proof. We have

$$
\begin{aligned}
\partial\left[\sum\cdots\right] = &\sum_{\pm\text{cyclic}/2} \partial\{\cdots(\overbrace{\cdots n}^{n/2})\} \wedge \cdots n(\overbrace{\cdots}^{n/2}) \\
&+ (-)^{n/2} \sum_{\pm\text{cyclic}/2} \cdots(\overbrace{\cdots n}^{n/2}) \wedge \partial\{\cdots n(\overbrace{\cdots}^{n/2})\};
\end{aligned}
$$

the last term is a product of an $(n/2+1)$-juzu and an $n/2$-juzu, so they are commutative; thus by the definition of $\pm$cyclic/2, the two sums can be unified to the sum over $\pm$cyclic. □

Thus we have

Lemma 2.

$$\partial\partial\{1\cdots n\} \quad n\text{: even}$$

$$= \sum_{\pm\text{cyclic}} \sum_{a=2}^{n/2-1} \left[(-)^a \partial\{\cdots(\overbrace{\cdots n}^{a})\} \wedge \cdots n(\overbrace{\cdots}^{n-a}) + 1\cdots(\overbrace{\cdots n}^{a}) \wedge \partial\{\cdots n(\overbrace{\cdots}^{n-a})\}\right]$$

$$+(-)^{n/2}\sum_{\pm\text{cyclic}} \partial\{\cdots(\overbrace{\cdots n}^{n/2})\}\wedge\cdots n(\overbrace{\cdots}^{n/2}),$$

$$\partial\partial\{1\cdots n\} \quad n\text{: odd}$$

$$= \sum_{\text{cyclic}} \sum_{a=2}^{(n-1)/2} \left[\partial\{\cdots(\overbrace{\cdots n}^{a})\} \wedge \cdots n(\overbrace{\cdots}^{n-a}) + (-)^{a+1}\cdots(\overbrace{\cdots n}^{a}) \wedge \partial\{\cdots n(\overbrace{\cdots}^{n-a})\}\right].$$

Lemma 3. *For even n,*

$$\partial\{1\cdots(\overbrace{\cdots n}^{a})\}$$

$$= \sum_{b=2}^{[(n-a)/2]} (-)^{ab}\left[\sum_{k=0}^{b-1}(-)^{ka}\cdots(\overbrace{\cdots k}^{a+b-1})\wedge\cdots(\overbrace{\cdots n}^{a})\cdots k(\overbrace{\cdots}^{n-a-b+1})\right.$$

$$\left.+\sum_{k=b}^{n-a}(-)^{ka}\cdots(\overbrace{\cdots n}^{a})\cdots(\overbrace{\cdots k}^{b})\wedge\cdots k(\overbrace{\cdots}^{n-b})\right]$$

$$+\,\delta^{\text{even}}_{a+1}(-)^{(n-a+1)/2}\sum_{k=0}^{(n-a+1)/2-1}(-)^k\cdots(\overbrace{\cdots k}^{(n-a+1)/2})\wedge\cdots(\overbrace{\cdots n}^{a})\cdots k(\overbrace{\cdots}^{(n-a+1)/2}),$$

where $\delta^{even}_a = 0$ or 1 if a is odd or even, respectively.

Proof. Writing the cyclic sums explicitly, in the definition of the boundary operator, we have

$$\partial\{1\cdots n\} = \sum_{a=2}^{(n-1)/2}\sum_{k=0}^{n-1}\cdots(\overbrace{\cdots k}^{a})\wedge\cdots k(\overbrace{\cdots}^{n-a}), \quad n\text{: odd}$$

$$\partial\{1\cdots n\} = \sum_{a=2}^{n/2-1}(-)^a\sum_{k=0}^{n-1}(-)^k\cdots(\overbrace{\cdots k}^{a})\wedge\cdots k(\overbrace{\cdots}^{n-a})$$

$$+(-)^{n/2}\sum_{k=0}^{n/2-1}(-)^k\cdots(\overbrace{\cdots k}^{n/2})\wedge\cdots k(\overbrace{\cdots}^{n/2}), \quad n\text{: even}$$

where $k=0$ means $k=n$. We can make them together, i.e. for any $n\geq 3$, we have

$$\partial\{1\cdots n\} = \sum_{a=2}^{[(n-1)/2]}(-)^{a(n+1)}\sum_{k=0}^{n-1}(-)^{k(n+1)}\cdots(\overbrace{\cdots k}^{a})\wedge\cdots k(\overbrace{\cdots}^{n-a})$$

$$+\,\delta^{\text{even}}_n(-)^{n/2}\sum_{k=0}^{n/2-1}(-)^k\cdots(\overbrace{\cdots k}^{n/2})\wedge\cdots k(\overbrace{\cdots}^{n/2}).$$

Let us add in this formula an extra information the place of n:

$$
\begin{aligned}
\partial\{1\cdots n\} = \sum_{a=2}^{[(n-1)/2]} (-)^{a(n+1)} \Bigg[& \sum_{k=0}^{a-1}(-)^{k(n+1)} \cdots(\overbrace{\cdots n\cdots k}^{a}) \wedge \cdots n \cdots k(\overbrace{\cdots}^{n-a}) \\
& + \sum_{k=a}^{n-1}(-)^{k(n+1)} \cdots n \cdots (\overbrace{\cdots k}^{a}) \wedge \cdots k(\overbrace{\cdots n \cdots}^{n-a}) \Bigg] \\
& + \delta_n^{\mathrm{even}}(-)^{n/2} \sum_{k=0}^{n/2-1} (-)^k \cdots (\overbrace{\cdots n \cdots k}^{n/2}) \wedge \cdots n \cdots k(\overbrace{\cdots}^{n/2}).
\end{aligned}
$$

Put $n = m$ in this formula, and subtitute m in the juzus by $(\overbrace{\cdots n}^{b})$, and the index m by $n - b + 1$; then we get

$$
\begin{aligned}
& \partial\{1\cdots(\overbrace{\cdots n}^{b})\} \\
& = \sum_{a=2}^{[(n-b)/2]} (-)^{a(n-b)} \Bigg[\sum_{k=0}^{a-1}(-)^{k(n-b)} \cdots(\overbrace{\cdots k}^{a+b-1}) \wedge \cdots (\overbrace{\cdots n}^{b}) \cdots k(\overbrace{\cdots}^{n-a-b+1}) \\
& \qquad + \sum_{k=a}^{n-b}(-)^{k(n-b)} \cdots (\overbrace{\cdots n}^{b}) \cdots (\overbrace{\cdots k}^{a}) \wedge \cdots k(\overbrace{\cdots}^{n-a}) \Bigg] \\
& + \delta_{n-b+1}^{\mathrm{even}}(-)^{(n-b+1)/2} \sum_{k=0}^{(n-b+1)/2-1} (-)^k \cdots (\overbrace{\cdots k}^{(n-b+1)/2}) \wedge \cdots (\overbrace{\cdots n}^{b}) \cdots k(\overbrace{\cdots}^{(n-b+1)/2}).
\end{aligned}
$$

One has only to exchange the indices a and b, and assume n even. □

We shall prove the theorem only when n is even, because the odd case can be proved quite analogously. By Lemma 3, one knows that every term of the expression of $\partial\partial\{1\cdots n\}$ in Lemma 2 is the product of three juzus in the form

$$
\cdots(\overbrace{\cdots}^{a})\cdots(\overbrace{\cdots}^{b})\cdots, \quad \cdots(\overbrace{\cdots}^{n-a}) \quad \text{and} \quad \cdots(\overbrace{\cdots}^{n-b}),
$$

where

$$
a, b \geq 2, \quad a + b \leq n - 3,
$$

which shall be called a term of type $(a)(b) = (b)(a)$. See Figure 10. Without loss of generaity we assume $a \leq n/2$. When $b \leq [(n-a)/2]$, the sum of the terms of type $(a)(b)$ in the expression is $(-)^{ab}$ times

$$
\begin{aligned}
& \sum_{\pm\text{cyclic}} (-)^a \sum_{k=b}^{n-a} (-)^{ka} \cdots (\overbrace{\cdots n}^{a}) \cdots (\overbrace{\cdots k}^{b}) \wedge \cdots k(\overbrace{\cdots}^{n-b}) \wedge \cdots n(\overbrace{\cdots}^{n-a}) \\
& + \sum_{\pm\text{cyclic}} (-)^b \sum_{k=a}^{n-b} (-)^{kb} \cdots (\overbrace{\cdots n}^{b}) \cdots (\overbrace{\cdots k}^{a}) \wedge \cdots k(\overbrace{\cdots}^{n-a}) \wedge \cdots n(\overbrace{\cdots}^{n-b}),
\end{aligned}
$$

where both terms comes from the second term in the expression in Lemma 3. We deform the expression of the second term by the formulae

$$\cdots k(\overbrace{\cdots}^{n-a}) \wedge \cdots n(\overbrace{\cdots}^{n-b}) = (-)^{ab} \cdots n(\overbrace{\cdots}^{n-b}) \wedge \cdots k(\overbrace{\cdots}^{n-a}),$$

$$\cdots (\overbrace{\cdots n}^{b}) \cdots (\overbrace{\cdots k}^{a}) = (-)^{(n-a-b+1)(k-a+1)} \cdots (\overbrace{\cdots k}^{a}) \cdots (\overbrace{\cdots n}^{b}),$$

and put $k = n - l$; then the second term equals

$$\sum_{\pm\text{cyclic}} \sum_{l=b}^{n-a} (-)^K \cdots (\overbrace{\cdots n-l}^{a}) \cdots (\overbrace{\cdots n}^{b}) \wedge \cdots n(\overbrace{\cdots}^{n-b}) \wedge \cdots (n-l)(\overbrace{\cdots}^{n-a}),$$

where

$$K = b + kb + ab + (n - a - b + 1)(k - a + 1).$$

Perform the $\pm$cyclic permutation to change n to l (accordingly $n - l$ changes to n), which causes the multiplication by $(-)^l$, and change the index l into k; then the consequent expression is exactly the same to the first term times $(-)^{K+l}$. Since

$$K + l \equiv a + ak + 1 \mod 2,$$

the sum of the first and the second terms is zero.

When $[(n-a)/2] < b < n/2$, the terms of type $(a)(b)$ appear in the first and the second lines in Lemma 3; when $b > n/2$, such terms appear in the first term in Lemma 3 and the second term of the sum in Lemma 2. One can prove the vanishing of the sum in the same way as in the case above. When a or b equals $n/2$, situation is simpler. In this way we complete the proof of the Theorem.

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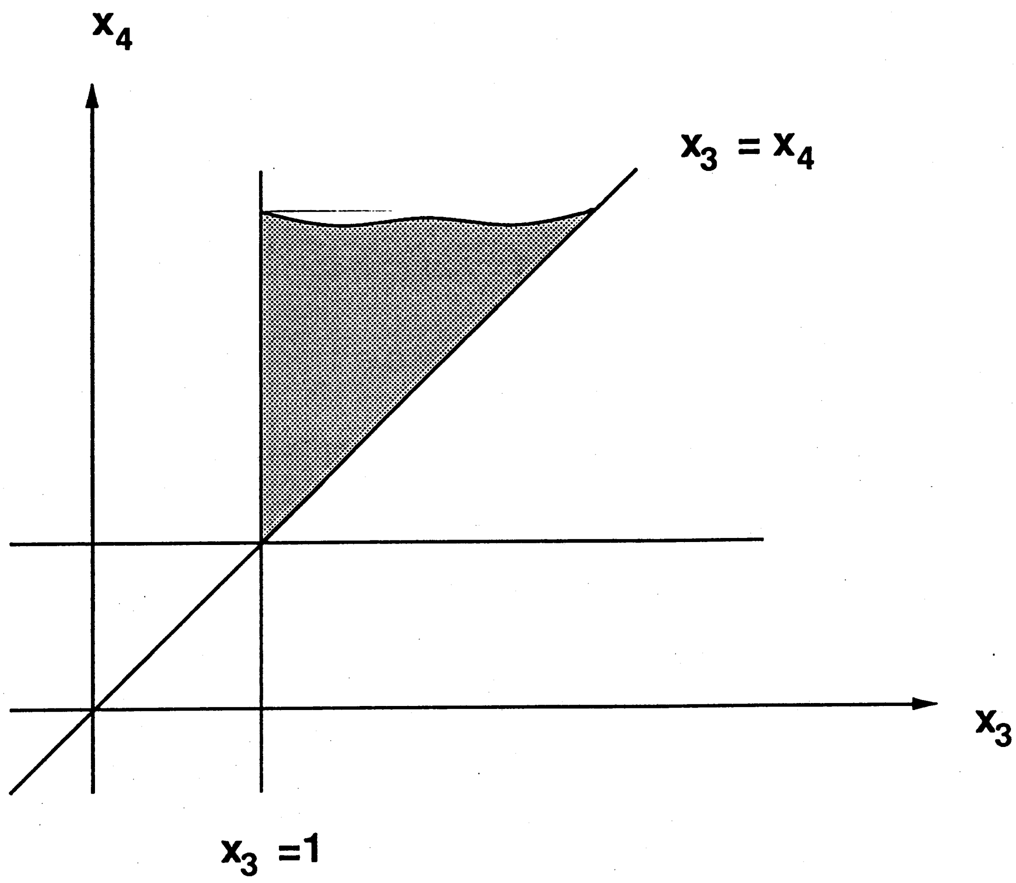

Figure 0

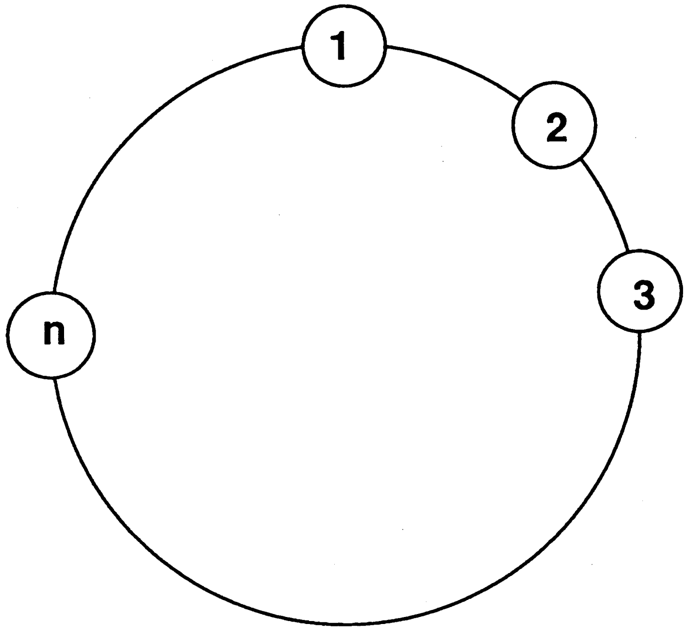

Figure 1. Juzu 12...n

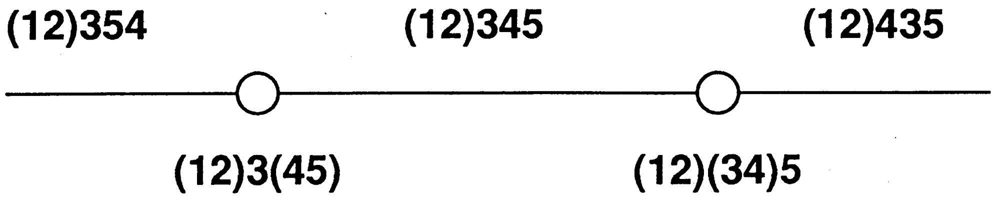

Figure 2. Segment (12)345

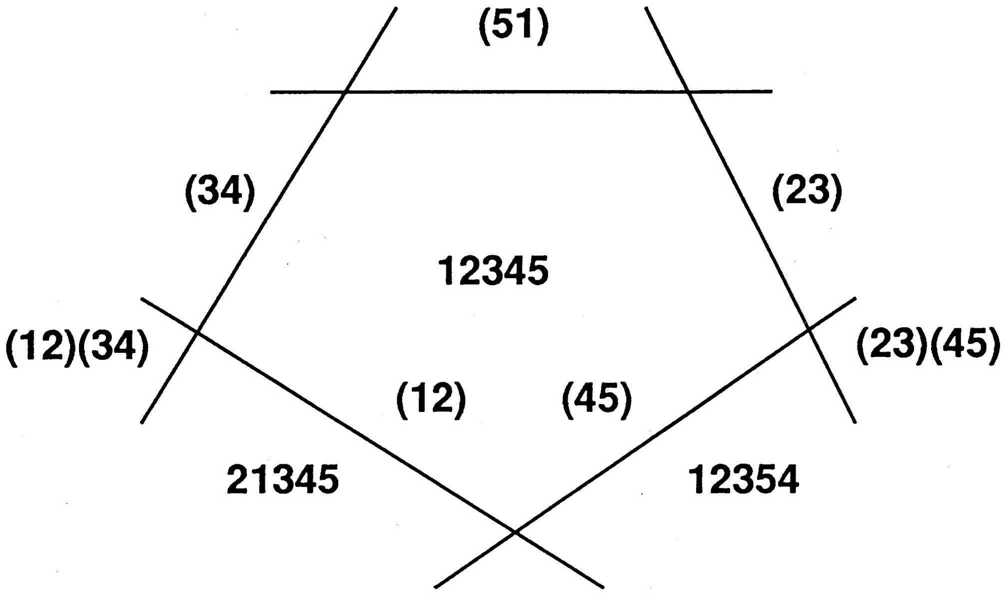

Figure 3. Boundary of 12345

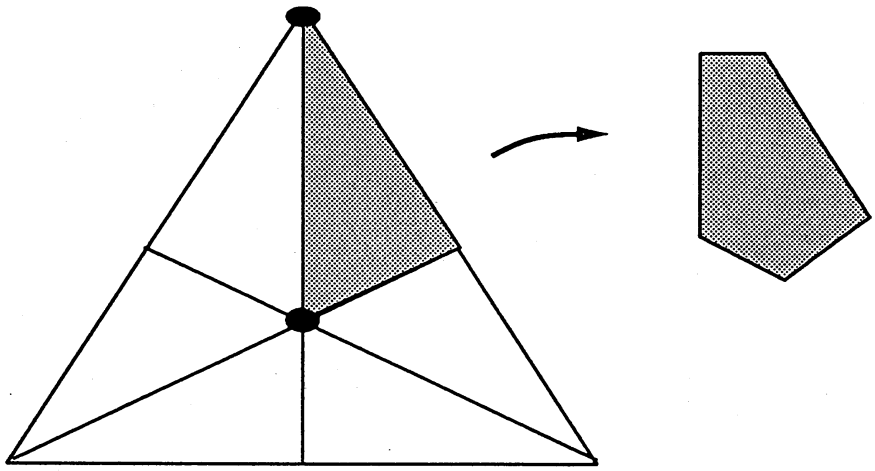

Figure 4. Truncation of a 2-simplex

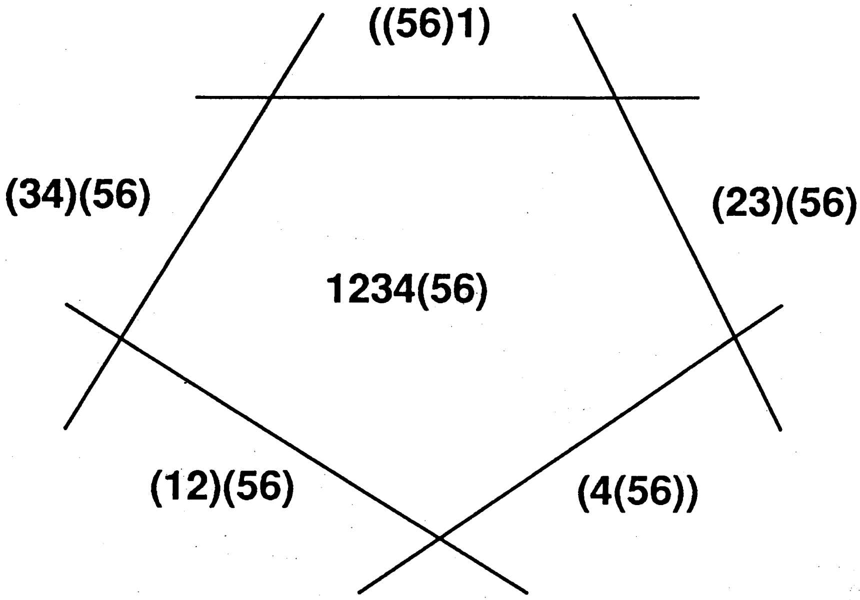

Figure 5. Boundary of 1234(56)

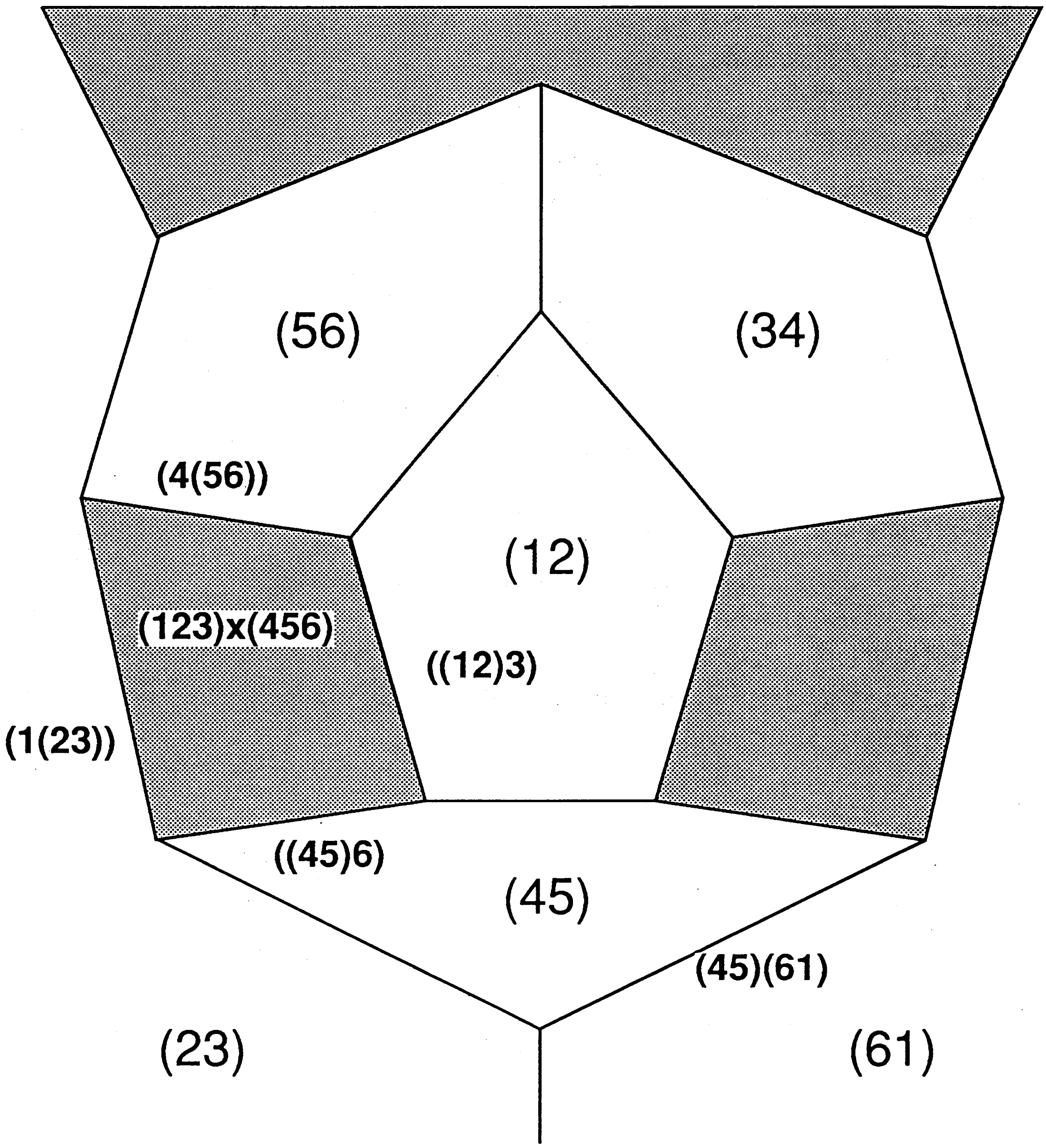

Figure 6. Boundary of 123456

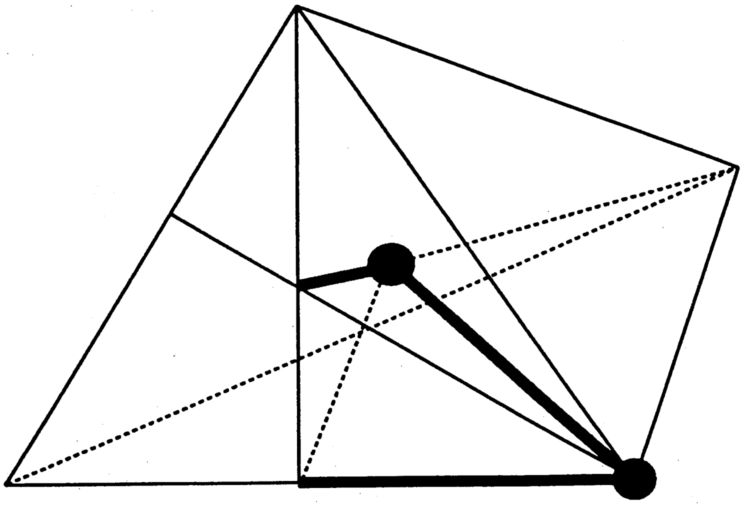

Figure 7. Barycentric subdivision

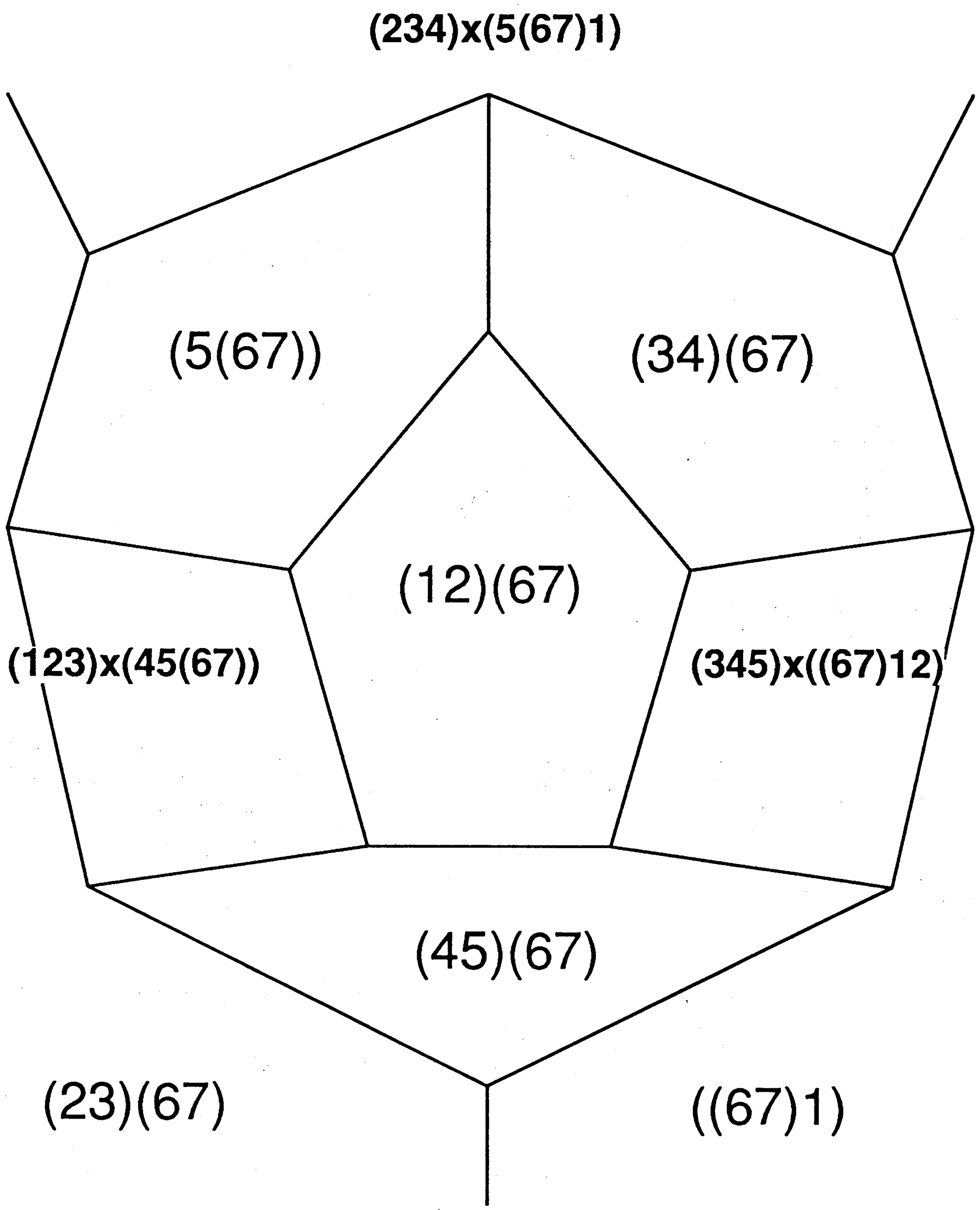

Figure 8. Boundary of 12345(67)

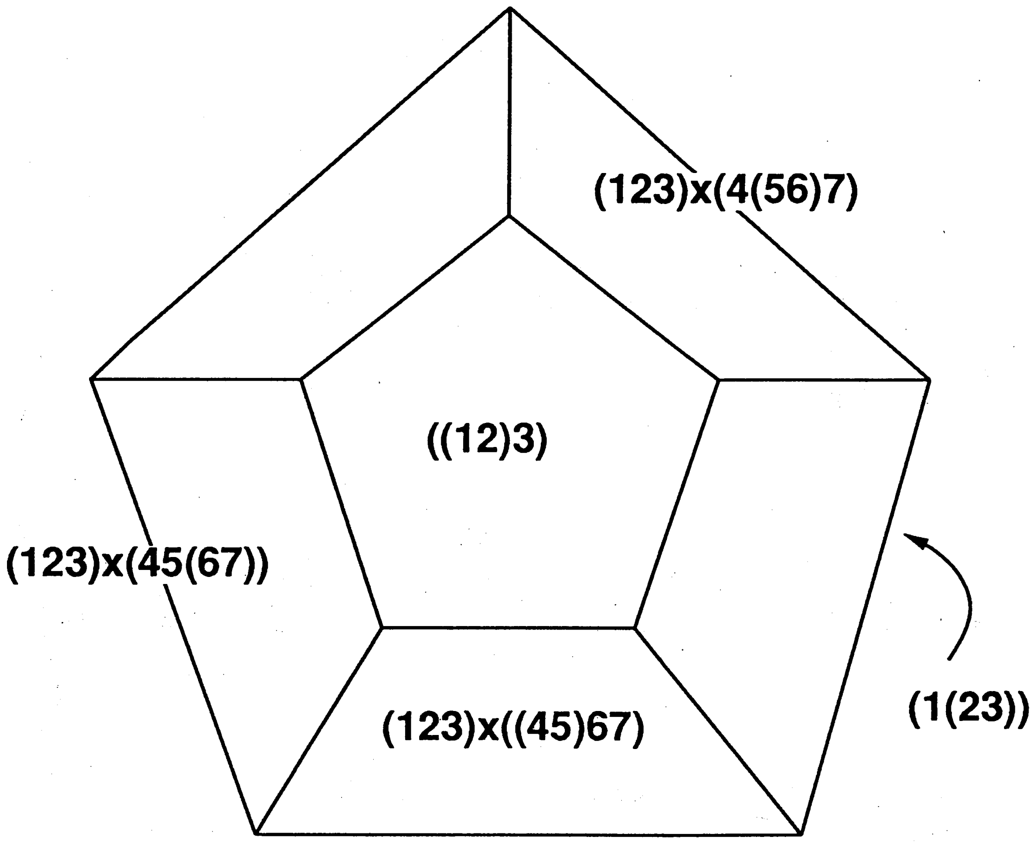

Figure 9. Prism (123)x(4567)

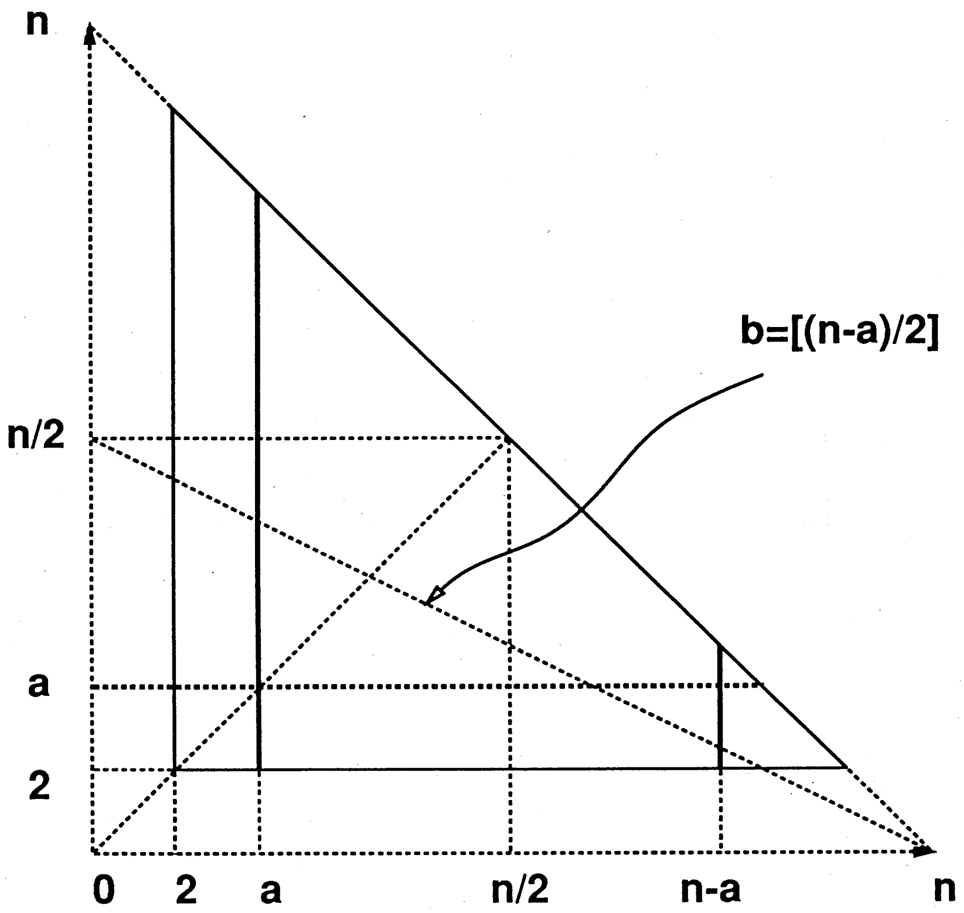

Figure 10. Terms of type (a)(b)