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楕円曲線上の共形場理論に付随した微分方程式系について

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ABSTRACT. We study the $SU(2)$ WZNW model over a family of elliptic curves. Starting from the formulation developed in [TUY], we derive a system of differential equations which contains the Knizhnik-Zamolodchikov-Bernard equations[Be1][FW]. Our system completely determines the N-point functions and is regarded as a natural elliptic analogue of the system obtained in [TK] for the projective line. We also calculate the system for the 1-point functions explicitly. This gives a generalization of the results in [EO2] for $\widehat{sl}(2,\mathbb{C})$-characters.

§0. Introduction.

We consider the Wess-Zumino-Novikov-Witten (WZNW) model. A mathematical formulation of this model on general algebraic curves is given in [TUY], where the correlation functions are defined as flat sections of a certain vector bundle over the moduli space of curves. On the projective line $\mathbb{P}^1$, the correlation functions are realized more explicitly in [TK] as functions which take their values in a certain finite-dimensional vector space, and characterized by the system of equations containing the Knizhnik-Zamolodchikov (KZ) equations[KZ]. One aim in the present paper is to have a parallel description on elliptic curves. Namely, we characterize the N-point functions as vector-valued functions by a system of differential equations containing an elliptic analogue of the KZ equations by Bernard[Be1]. Furthermore we write down this system explicitly in the 1-pointed case.

To explain more precisely, first let us review the formulation in [TUY] roughly. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ and $\widehat{\mathfrak{g}}$ the corresponding affine Lie algebra. We fix a positive integer ℓ (called the level) and consider the integrable highest weight modules of $\widehat{\mathfrak{g}}$ of level ℓ. Such modules are parameterized by the set of highest weight P_ℓ and we denote by $\mathcal{H}_\lambda$ the left module corresponding to $\lambda \in P_\ell$. By $M_{g,N}$ we denote the moduli space of N-pointed curves of genus g. For $\mathfrak{X} \in M_{g,N}$ and $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$, we associate the space of conformal blocks $\mathcal{V}_g^\dagger(\mathfrak{X};\vec{\lambda})$. The space $\mathcal{V}_g^\dagger(\mathfrak{X};\vec{\lambda})$ is the finite dimensional subspace of $\mathcal{H}_{\vec{\lambda}}^\dagger := \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_N}, \mathbb{C})$ defined by "the gauge conditions". Consider the vector bundle $\tilde{\mathcal{V}}_g^\dagger(\vec{\lambda}) = \cup_{\mathfrak{X}\in M_{g,N}} \mathcal{V}_g^\dagger(\mathfrak{X};\vec{\lambda})$ over $M_{g,N}$. On this vector bundle, projectively flat connections are defined through the Kodaira-Spencer theory, and flat sections of $\tilde{\mathcal{V}}_g^\dagger(\vec{\lambda})$ with respect to these connections are called the N-point correlation functions (or N-point functions). In the rest of this paper we set $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}E \oplus \mathbb{C}F \oplus \mathbb{C}H$ for simplicity, where E, F and H are the basis of $\mathfrak{g}$ satisfying $[H,E] = 2E$, $[H,F] = -2F$, $[E,F] = H$. We identify P_ℓ with the set $\{\ 0, \frac{1}{2}, \dots, \frac{\ell}{2}\ \}$ by the mapping $\lambda \mapsto \frac{\lambda(H)}{2}$.

In the case of genus 0, the space of conformal blocks is injectively mapped into $V_{\vec{\lambda}}^\dagger := \mathrm{Hom}_{\mathbb{C}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}, \mathbb{C})$ by the restriction map, where $V_\lambda \subset \mathcal{H}_\lambda$ denotes the finite dimensional irreducible highest weight left $\mathfrak{g}$-module with highest weight λ. This injectivity makes it possible to treat this model in a more explicit way as above, and the N-point functions are described by the vacuum expectation values of vertex operators.

On the other hand, in the case of genus 1 this injectivity does not hold, and in order to recover it we twist the space of conformal blocks by introducing a new parameter following [Be1,2][EO1][FW]. Because of the twisting, any N-point function in genus 1 can be calculated from its restriction to $V_{\bar{\lambda}}$ (Proposition 3.3.2). It is natural to ask how the restrictions of the N-point functions are characterized as $V_{\bar{\lambda}}^{\dagger}$-valued functions. It turns out that the restricted N-point functions satisfy the equations (E1)–(E3) in Proposition 3.3.3. These equations are essentially derived by Bernard[Be1] for traces of vertex operators

$$\mathrm{Tr}_{\mathcal{H}_\mu}(\varphi_1(z_1)\cdots\varphi_N(z_N)q^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}}) \in V_{\bar{\lambda}}^{\dagger},$$

where $z_1,\ldots,z_N,q,\xi$ are the variables in $\mathbb{C}^*$ with $|q|<1$, $\varphi_j : V_{\lambda_j}\otimes\mathcal{H}_{\mu_j}\to\hat{\mathcal{H}}_{\mu_{j-1}}$ $(j=1,\ldots,N)$ are the vertex operators for some μ_i $(i=0,\ldots,N)$ with $\mu_0=\mu_N=\mu$, L_0 is defined by (1.2.1) and $c_v = 3\ell/(\ell-2)$ (for the details, see §§3.4). It is proved that the space of restricted N-point functions is spanned by traces of vertex operators (Theorem 3.4.3) and hence Bernard's approach is equivalent to ours. However, the system (E1)–(E3) is not complete since it has infinite-dimensional solution space.

We will show that the integrability condition

$$\left(E\otimes t^{-1}\right)^{\ell-2\lambda+1}|\bar{v}(\lambda)\rangle = 0$$

for the highest weight vector $|\bar{v}(\lambda)\rangle \in \mathcal{H}_\lambda$ implies the differential equations (E4), which determine the N-point functions completely combining with (E1)–(E3).

For 1-point functions, the equation (E4) can be written down explicitly, and the system (E1)–(E4) reduces to the two equations (F1)(F2) in Theorem 4.2.4. In the simplest case, the 1-point functions are given by the characters

$$\mathrm{Tr}_{\mathcal{H}_\mu} q^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}} \quad \left(\mu = 0,\frac{1}{2},\ldots,\frac{\ell}{2}\right),$$

and our system coincide with the one obtained in [EO2].

Recently Felder and Wieczerkowski give a conjecture on the characterization of the restricted N-point functions in genus 1 by using the modular properties and certain additive conditions[FW]. They confirm their conjecture in some cases by explicit calculations. We recover this result in $\mathfrak{sl}(2,\mathbb{C})$ case by solving the equation (F2) (Proposition 4.2.5). The equation (F1) can be also integrated when the dimension of the solution space is small, and we can calculate the 1-point functions explicitly.

§1. Representation theory for $\widehat{\mathfrak{sl}}(2,\mathbb{C})$.

For the details of the contents in this section, we refer the reader to [Kac].

1.1 Integrable highest weight modules.

By $\mathbb{C}[[x]]$ and $\mathbb{C}((x))$, we mean the ring of formal power series in x and the field of formal Laurent series in x, respectively. We put $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. Let $\mathfrak{h} = \mathbb{C}H$ be a Cartan subalgebra of $\mathfrak{g}$ and $(\ ,\ ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the Cartan-Killing form normalized by the condition $(H, H) = 2$. We identify the set P_+ of dominant integral weights with $\frac{1}{2}\mathbb{Z}_{\geq 0}$. For $\lambda \in P_+$, we denote by V_λ the irreducible highest weight left $\mathfrak{g}$-module with highest weight λ and by $|v(\lambda)\rangle$ its highest weight vector.

The affine Lie algebra $\widehat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined by

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((x)) \oplus \mathbb{C}c,$$

where c is an central element of $\widehat{\mathfrak{g}}$ and the Lie algebra structure is given by

$$[X \otimes f(x), Y \otimes g(x)] = [X, Y] \otimes f(x)g(x) + c \cdot (X, Y) \operatorname*{Res}_{x=0} (g(x) \cdot df(x)),$$

for $X, Y \in \mathfrak{g},\ f(x), g(x) \in \mathbb{C}((\xi))$. We use the following notations:

$$X_n = X \otimes x^n,\ X = X_0\,,$$
$$\widehat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[[x]]x,\ \widehat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbb{C}[x^{-1}]x^{-1},$$
$$\widehat{\mathfrak{p}}_\pm = \widehat{\mathfrak{g}}_\pm \oplus \mathfrak{g} \oplus \mathbb{C}c.$$

Fix a positive integer ℓ (called the level) and put $P_\ell = \{\ 0, \frac{1}{2}, \dots, \frac{\ell}{2}\ \} \subset P_+$. For $\lambda \in P_\ell$, we define the action of $\widehat{\mathfrak{p}}_+$ on V_λ by $c = \ell \times id$ and $a = 0$ for all $\mathfrak{a} \in \widehat{\mathfrak{g}}_+$, and put

$$\mathcal{M}_\lambda = U(\widehat{\mathfrak{g}}) \otimes_{\widehat{\mathfrak{p}}_+} V_\lambda.$$

Then $\mathcal{M}_\lambda$ is a highest weight left $\widehat{\mathfrak{g}}$-module and it has the maximal proper submodule $\mathcal{J}_\lambda$, which is generated by the singular vector $E_{-1}^{\ell-2\lambda+1}|v(\lambda)\rangle$:

$$\mathcal{J}_\lambda = U(\widehat{\mathfrak{p}}_-)E_{-1}^{\ell-2\lambda+1}|v(\lambda)\rangle.$$

The integrable highest weight left $\widehat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ with highest weight λ is defined as the quotient module $\mathcal{M}_\lambda/\mathcal{J}_\lambda$. We denote by $|\bar{v}(\lambda)\rangle$ the highest weight vector in $\mathcal{H}_\lambda$. We introduce the lowest weight right $\widehat{\mathfrak{g}}$-module structure on

$$\mathcal{H}_\lambda^\dagger = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_\lambda, \mathbb{C})$$

in the usual way, and denote its lowest weight vector by $\langle \bar{v}(\lambda)|$.

1.2. Segal-Sugawara construction and the filtration on $\mathcal{H}_\lambda$.

Fix a weight $\lambda \in P_\ell$. On $\mathcal{H}_\lambda$, elements L_n $(n \in \mathbb{Z})$ of the Virasoro algebra act with the central charge $c_v = 3\ell/(\ell+2)$ through the Segal-Sugawara construction

$$L_n = \frac{1}{2(\ell+2)} \sum_{m \in \mathbb{Z}} \left\{ {}^\circ_\circ \frac{1}{2} H_m H_{n-m} {}^\circ_\circ + {}^\circ_\circ E_m F_{n-m} {}^\circ_\circ + {}^\circ_\circ F_m E_{n-m} {}^\circ_\circ \right\}, \tag{1.2.1}$$

where $\circ\atop\circ$ $\circ\atop\circ$ denotes the standard normal ordering. Put

$$X(z) = \sum_{n\in\mathbb{Z}} X_n z^{-n-1} \ \ (X \in \mathfrak{g}),\ T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}.$$

The module $\mathcal{H}_\lambda$ has the decomposition $\mathcal{H}_\lambda = \oplus_{d\geq 0}\mathcal{H}_\lambda(d)$, where

$$\mathcal{H}_\lambda(d) = \{\ |u\rangle \in \mathcal{H}_\lambda\ ;\ L_0|u\rangle = (\Delta_\lambda + d)|u\rangle\ \},$$
$$\Delta_\lambda = \frac{\lambda(\lambda+1)}{\ell+2}.$$

We define the filtration $\{\mathcal{F}_\bullet\}$ on $\mathcal{H}_\lambda$ by

$$\mathcal{F}_p\mathcal{H}_\lambda = \sum_{d\leq p}\mathcal{H}_\lambda(d)$$

and put $\hat{\mathcal{H}}_\lambda = \prod_{d\geq 0}\mathcal{H}_\lambda(d)$.

1.3. The Lie algebra $\widehat{\mathfrak{g}}_N$.

Put $L\mathfrak{g} = \mathfrak{g}\otimes\mathbb{C}((x))$. For a positive integer N, we define a Lie algebra $\widehat{\mathfrak{g}}_N$ by

$$\widehat{\mathfrak{g}}_N = \oplus_{j=1}^N L\mathfrak{g}_{(i)} \oplus \mathbb{C}c,$$

where $L\mathfrak{g}_{(i)}$ denotes a copy of $L\mathfrak{g}$ and c is a center. The commutation relations are given by

$$[\oplus_{j=1}^N X_j\otimes f_j, \oplus_{j=1}^N Y_j\otimes g_j] =$$
$$\oplus_{j=1}^N [X_j, Y_j]\otimes f_j g_j \ + \sum_{j=1}^N (X_j, Y_j)\operatorname*{Res}_{\xi_j=0}(g_j\cdot df_j)\cdot c.$$

For each $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \in (P_\ell)^N$ a left $\widehat{\mathfrak{g}}_N$-module $\mathcal{H}_{\vec{\lambda}}$ is defined by

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1}\otimes\cdots\otimes\mathcal{H}_{\lambda_N}.$$

Similarly a right $\widehat{\mathfrak{g}}_N$-module $\mathcal{H}^\dagger_{\vec{\lambda}}$ is defied by

$$\mathcal{H}^\dagger_{\vec{\lambda}} = \mathcal{H}^\dagger_{\lambda_1}\hat{\otimes}\cdots\hat{\otimes}\mathcal{H}^\dagger_{\lambda_N} \cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}, \mathbb{C}).$$

The $\widehat{\mathfrak{g}}_N$-action on $\mathcal{H}_{\vec{\lambda}}$ is given by

$$c = \ell\cdot id$$
$$(\oplus_{j=1}^N a_j)|u_1\otimes\cdots\otimes u_N\rangle = \sum_{j=1}^N \rho_j(a_j)|u_1\otimes\cdots\otimes u_N\rangle$$

for $a_j \in L\mathfrak{g}_{(j)} (j = 1, \dots, N)$, where we used the notations

$$|u_1 \otimes \cdots \otimes u_N\rangle = |u_1\rangle \otimes \cdots \otimes |u_N\rangle,$$
$$\rho_j(a)|u_1 \otimes \cdots \otimes u_N\rangle = |u_1 \otimes \cdots \otimes a \cdot u_j \otimes \cdots \otimes u_N\rangle$$

for $|u_i\rangle \in \mathcal{H}_{\lambda_i}$ $(i = 1, \dots, N)$ and $a \in L\mathfrak{g}$. The right action on $\mathcal{H}^\dagger_{\vec{\lambda}}$ is defined similarly. The module $\mathcal{H}_{\vec{\lambda}}$ has the filtration induced from those of $\mathcal{H}_{\lambda_j}$ $(j = 1, \dots, N)$:

$$\mathcal{F}_p\mathcal{H}_{\vec{\lambda}} = \sum_{d \le p} \mathcal{H}_{\vec{\lambda}}(d),$$

where

$$\mathcal{H}_{\vec{\lambda}}(d) = \sum_{d_1 + \cdots + d_N = d} \mathcal{H}_{\lambda_1}(d_1) \otimes \cdots \otimes \mathcal{H}_{\lambda_N}(d_N).$$

We put

$$V_{\vec{\lambda}} = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N} \cong \mathcal{H}_{\vec{\lambda}}(0), \ V^\dagger_{\vec{\lambda}} = \mathrm{Hom}_{\mathbb{C}}(V_{\vec{\lambda}}, \mathbb{C}).$$

§2 The WZNW model in genus 0.

In this section we review the SU(2) WZNW model on the projective line $\mathbb{P}^1$.

2.1. The space of conformal blocks.

In this subsection we define the N-point functions on $\mathbb{P}^1$ following [TUY] as sections of a vector bundle on the manifold

$$R_N = \{ (z_1, \dots, z_N) \in (\mathbb{C}^*)^N \ ; \ z_i \ne z_j \text{ if } i \ne j \}.$$

For a meromorphic function $f(t)$ on $\mathbb{P}^1$ and $w \in \mathbb{C}$, put

$$X[f(t)]_w = \operatorname*{Res}_{t=w} f(t)X(t-w)dt,$$
$$T[f(t)\frac{d}{dt}]_w = \operatorname*{Res}_{t=w} f(t)T(t-w)dt.$$

If $f(t)$ has an Laurent expansion $f(t) = \sum_{n \ge M} a_n(t-w)^n$ then $X[f(t)]_w$ is an element of $\widehat{\mathfrak{g}}$ given by

$$X[f(t)]_w = \sum_{n \ge M} a_n X_n.$$

For $z = (z_1, \dots, z_N) \in R_N$, we set

$$\widehat{\mathfrak{g}}(z) = H^0(\mathbb{P}^1, \mathfrak{g} \otimes \mathcal{O}_{\mathbb{P}^1}(* \sum_{j=1}^N z_j)).$$

Then we have the following injection:

$$\begin{aligned} \widehat{\mathfrak{g}}(z) &\to \widehat{\mathfrak{g}}_N, \\ X \otimes f(z) &\mapsto X[f] := \oplus_{j=1}^N X[f]_{z_j}. \end{aligned}$$

Through this map we regard $\widehat{\mathfrak{g}}(z)$ as a subspace of $\widehat{\mathfrak{g}}_N$ and the residue theorem implies that $\widehat{\mathfrak{g}}(z)$ is a Lie subalgebra of $\widehat{\mathfrak{g}}_N$. We also use the following notation

$$T[g] = \oplus_{j=1}^N T[g]_{z_j}$$

for $g \in H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1}(*\sum_{j=1}^N z_j))$, where $\Theta_{\mathbb{P}^1}$ denotes the sheaf of vector fields on $\mathbb{P}^1$.

Definition 2.1.1. For $z = (z_1, \ldots, z_N) \in R_N$ and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \in (P_\ell)^N$ we put

$$\begin{aligned} \mathcal{V}_0(z\,;\vec{\lambda}) &= \mathcal{H}_{\vec{\lambda}} / \widehat{\mathfrak{g}}(z)\mathcal{H}_{\vec{\lambda}}, \\ \mathcal{V}_0^\dagger(z\,;\vec{\lambda}) &= \{\, \langle\Psi| \in \mathcal{H}_{\vec{\lambda}}^\dagger \,;\, \langle\Psi|\widehat{\mathfrak{g}}(z) = 0 \,\} \\ &\cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_0(z\,;\vec{\lambda}), \mathbb{C}). \end{aligned}$$

We call $\mathcal{V}_0^\dagger(z\,;\vec{\lambda})$ the space of conformal blocks (or the space of vacua) in genus 0 attached to $(z\,;\vec{\lambda})$.

For a vector space V and a complex manifold M, we denote by $V[M]$ the set of multi-valued, holomorphic V-valued functions on M.

Definition 2.1.2. For $\vec{\lambda} \in (P_\ell)^N$, an element $\langle\Phi|$ of $\mathcal{H}_{\vec{\lambda}}^\dagger[R_N]$ is called an N-point function in genus 0 attached to $\vec{\lambda}$ if the following conditions are satisfied:

(A1) For each $z \in R_N$,

$$\langle\Phi(z)| \in \mathcal{V}_0^\dagger(z;\vec{\lambda})$$

(A2) For $j = 1, \ldots, N$,

$$\partial_{z_j}\langle\Phi(z)| = \langle\Phi(z)|\rho_j(L_{-1}).$$

By $\mathfrak{F}_0(\vec{\lambda})$ we denote the set of N-point functions in genus 0 attached to $\vec{\lambda}$.

Remark. The condition (A1) implies the following:

(A1′) For each $z \in R_N$,

$$\langle\Phi(z)|T[g] = 0$$

for any $g \in H^0(\mathbb{P}^1, \Theta_{\mathbb{P}^1}(*\sum_{j=1}^N z_j))$.

2.2. Restrictions of the N-point functions to $V_{\vec{\lambda}}$.

A remarkable property of the space of conformal blocks in genus 0 is the following:

Lemma 2.2.1. *The composition map*

$$V_{\vec{\lambda}} \hookrightarrow \mathcal{H}_{\vec{\lambda}} \to \mathcal{V}_0(z\,;\vec{\lambda})$$

is surjective. In other words, the restriction map

$$\mathcal{V}_0^\dagger(z\,;\vec{\lambda}) \to V_{\vec{\lambda}}^\dagger$$

is injective.

This lemma implies that, for an N-point function $\langle\Phi|$, we can calculate $\langle\Phi|u\rangle$ for any $|u\rangle \in \mathcal{H}_{\vec{\lambda}}$, from the data $\{\, \langle\Phi|v\rangle \,;\, |v\rangle \in V_{\vec{\lambda}} \,\}$. By $\mathfrak{F}_0^r(\vec{\lambda})$ we denote the image of $\mathfrak{F}_0(\vec{\lambda})$ in $V_{\vec{\lambda}}^\dagger[R_N]$ under the restriction map. It is natural to ask how the set $\mathfrak{F}_0^r(\vec{\lambda})$ is characterized in $V_{\vec{\lambda}}^\dagger[R_N]$, and the answer is given as follows:

Proposition 2.2.2. [TK] *The space $\mathfrak{F}_0^r(\vec{\lambda})$ coincides with the solution space of the following system of equations:*

(B1) *For each $X \in \mathfrak{g}$,*

$$\sum_{j=1}^{N} \langle \phi(z) | \rho_j(X) = 0.$$

(B2) [the Knizhnik-Zamolodchikov equations] *For each $j = 1, \ldots, N$,*

$$(\ell + 2)\partial_{z_j} \langle \phi(z) | = \sum_{i \neq j} \langle \phi(z) | \frac{\Omega_{i,j}}{z_i - z_j},$$

where

$$\Omega_{i,j} = \frac{1}{2}\rho_i(H)\rho_j(H) + \rho_i(E)\rho_j(F) + \rho_i(F)\rho_j(E).$$

(B3) *For each $j = 1, \ldots, N$,*

$$\sum_{n_1+\cdots+n_N=\ell_j} \binom{\ell_j}{\vec{n}_j} \prod_{i\neq j} (z_i - z_j)^{-n_i} \langle \phi(z) | E^{n_1} v_1 \otimes \cdots \otimes v(\lambda_j) \otimes \cdots \otimes E^{n_N} v_N \rangle$$
$$= 0$$

for any $|v_i\rangle \in V_{\lambda_i}$ $(i \neq j)$. Here $\ell_j = \ell - 2\lambda_j + 1$, $\vec{n}_j = (n_1, \ldots, n_{j-1}, n_{j+1}, \ldots, n_N)$ and $\binom{\ell_j}{\vec{n}_j}$ is the multinomial coefficient. □

Remark. The equation (B3) is a consequence of the integrability condition

$$E_{-1}^{\ell-2\lambda_j+1} |\bar{v}(\lambda_j)\rangle = 0 \ (j = 1, \ldots, N), \tag{2.2.1}$$

for the highest weight vector $|\bar{v}(\lambda_j)\rangle \in \mathcal{H}_{\lambda_j}$.

2.3. Vertex operators.

We review the description of N-point functions by vertex operators.

Definition 2.3.1. For $(\nu, \lambda, \mu) \in (P_\ell)^3$ a multi-valued, holomorphic, operator valued function $\varphi(z_1)$ on the manifold $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is called a vertex operator of type (ν, λ, μ), if

$$\varphi(z_1) \ : \ V_\lambda \otimes \mathcal{H}_\mu \to \hat{\mathcal{H}}_\nu$$

satisfies the following conditions:

(C1) For $X \in \mathfrak{g}$, $|v\rangle \in V_\lambda$ and $m \in \mathbb{Z}$,

$$[X_m, \varphi(|v\rangle; z_1)] \ = \ z_1^m \varphi(X|v\rangle; z_1).$$

(C2) For $|v\rangle \in V_\lambda$ and $m \in \mathbb{Z}$,

$$[L_m, \varphi(|v\rangle; z_1)] \ = \ z_1^m \left\{ z_1 \frac{d}{dz_1} + (m+1)\Delta_\lambda \right\} \varphi(|v\rangle; z_1).$$

Here $\varphi(|u\rangle; z_1) : \mathcal{H}_\nu \to \hat{\mathcal{H}}_\mu$ is the operator defined by $\varphi(|u\rangle; z_1)|v\rangle = \varphi(z_1)|u \otimes v\rangle$ for $|u\rangle \in V_\lambda$ and $|v\rangle \in \mathcal{H}_\nu$.

For vertex operators $\varphi_j(z_j)$ $(j = 1, \ldots, N)$, the composition $\varphi_1(z_1) \cdots \varphi_N(z_N)$ makes sense for $|z_1| > \cdots > |z_N|$ and analytically continued to R_N.

Proposition 2.3.2. [TK] *The space $\mathfrak{F}_0^r(\vec{\lambda})$ is spanned by the following $V_{\vec{\lambda}}^{\dagger}$-valued functions:*

$$\langle v(0)|\varphi_1(z_1)\cdots\varphi_N(z_N)|v(0)\rangle,$$

where $\varphi_j\,(j = 1,\ldots,N)$ is the vertex operator of type $(\mu_{j-1},\lambda_j,\mu_j)$ for some $\mu_i \in P_\ell\ (i = 0,\ldots,N)$ with $\mu_0 = \mu_N = 0$.

Proposition 2.3.3. [TK] *Any nonzero vertex operator*

$$\varphi(z_1) : V_\lambda \otimes \mathcal{H}_\mu \to \hat{\mathcal{H}}_\nu$$

is uniquely extended to the operator

$$\hat{\varphi}(z_1) : \mathcal{M}_\lambda \otimes \mathcal{H}_\mu \to \hat{\mathcal{H}}_\nu$$

by the following condition:

$$\hat{\varphi}(X_n|u\rangle; z_1) = \operatorname*{Res}_{w=z}(w - z_1)^n\hat{\varphi}(|u\rangle; z_1)X(w)dw, \tag{2.3.1}$$

for each $|u\rangle \in \mathcal{M}_\lambda$, $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$.

Moreover, $\hat{\varphi}$ has the following properties:

$$\partial_z\hat{\varphi}(|u\rangle; z_1) = \hat{\varphi}(L_{-1}|u\rangle; z_1) \quad \text{for any } |u\rangle \in \mathcal{M}_\lambda, \tag{2.3.2}$$

$$\hat{\varphi}(|u\rangle; z_1) = 0 \quad \text{for any } |u\rangle \in \mathcal{J}_\lambda = U(\hat{\mathfrak{p}}_-)E_{-1}^{\ell-2\lambda+1}|v(\lambda)\rangle. \tag{2.3.3}$$

The property (2.3.3) implies that $\hat{\varphi}$ reduces to the operator

$$\hat{\varphi}(z_1) : \mathcal{H}_\lambda \otimes \mathcal{H}_\mu \to \hat{\mathcal{H}}_\nu.$$

§3 The WZNW model in genus 1.

In this section we consider the elliptic analogue of the story in the previous section. Our aim is to embed the set of N-point functions in genus 1 (Definition 3.1.3) into the set of $V_{\vec{\lambda}}^{\dagger}$-valued functions, and to characterize its image by a system of differential equations. We also show that the N-point functions are given by the traces of vertex operators.

3.1 Functions with quasi-periodicity.

First, we prepare some functions for the later use. Put $D^* = \{\, q \in \mathbb{C}^* \ ;\ |q| < 1 \,\}$ and introduce the following functions on $\mathbb{C}^* \times D^*$:

$$\begin{aligned}\Theta(z,q) &= \sum_{n\in\mathbb{Z}+\frac{1}{2}} (-1)^{n+1}q^{\frac{1}{2}n^2}z^n \\ &= -\sqrt{-1}z^{\frac{1}{2}}q^{\frac{1}{8}}\prod_{n\geq 1}(1-q^n)(1-zq^n)(1-z^{-1}q^{n-1}),\end{aligned} \tag{3.1.1}$$

$$\zeta(z,q) = \frac{z\partial_z\Theta(z,q)}{\Theta(z,q)}. \tag{3.1.2}$$

$$\wp(z,q) = -z\partial_z\zeta(z,q) + 2\frac{q\partial_q\eta(q)}{\eta(q)}, \tag{3.1.3}$$

where $\eta(q)$ is the Dedekind eta function

$$\eta(q) = q^{\frac{1}{24}} \prod_{n\geq 1}(1-q^n).$$

The function $\Theta(z,q)$ satisfies the heat equation

$$2q\partial_q \Theta(z,q) = (z\partial_z)^2 \Theta(z,q).$$

The function $\wp(z,q)$ satisfies $\wp(qz,q) = \wp(z,q)$, and $\zeta(z,q)$ have the following quasi-periodicity:

(3.1.4) $$\zeta(qz,q) = \zeta(z,q) - 1.$$

For $(z,q) \in \mathbb{C}^* \times D^*$ and $\xi \in \mathbb{C}^*$, we put

(3.1.5) $$\sigma_{\pm}(z,q,\xi) = \frac{\Theta(z^{-1}\xi^{\pm 1},q)\Theta'(1,q)}{\Theta(z,q)\Theta(\xi^{\pm 1},q)}$$

Here $\Theta'(z,q) = z\partial_z\Theta(z,q)$. The function $\sigma_{\pm}(z,q,\xi)$ have the following properties:

(3.1.6) $$\begin{aligned} \sigma_{\pm}(qz,q,\xi) &= \xi^{\pm 1}\sigma_{\pm}(z,q,\xi), \\ \sigma_{\pm}(z^{-1},q,\xi) &= -\sigma_{\mp}(z,q,\xi). \end{aligned}$$

For $\zeta(z,q)$ and $\sigma_{\pm}(z,q,\xi)$, we have the following expansion at $z=1$:

(3.1.7) $$\zeta(z,q) = \frac{1}{z-1} + \frac{1}{2} - 2\alpha(q)(z-1) + O(z-1)^2,$$

(3.1.8) $$\begin{aligned} \sigma_{\pm}(z,q,\xi) = &\frac{1}{z-1} \mp \zeta(\xi,q) + \frac{1}{2} \\ &- \sum_{n\geq 1}\left(\frac{n\xi^{-1}q^n}{1-\xi^{-1}q^n} + \frac{n\xi q^n}{1-\xi q^n}\right)(z-1) + O(z-1)^2, \end{aligned}$$

where $\alpha(q)$ is given by

(3.1.9) $$\alpha(q) = -\frac{q\partial_q\eta(q)}{\eta(q)} + \frac{1}{24}.$$

3.2. Twisting the space of conformal blocks.

In the case of genus 1 (or > 0), if we work with the formulation of [TUY], an N-point function is not determined by its restriction on $V_{\vec{\lambda}}$. In order to resolve this difficulty we "twist" the space of conformal blocks following [Be1,2][EO1][FW].

For $q \in D^*$, we consider the elliptic curve $\mathcal{E}_q = \mathbb{C}^*/\langle q\rangle$, where $\langle q\rangle$ is the infinite cyclic group of automorphisms generated by $z \mapsto qz$. We denote by $[z]_q$ the image of a point $z \in \mathbb{C}^*$ on $\mathcal{E}_q$ and put

$$T_N = \{\ (z,q) = (z_1,\dots,z_N,q) \in (\mathbb{C}^*)^N \times D^*\ ;\ [z_i]_q \neq [z_j]_q \text{ if } i \neq j\ \}.$$

In the following we omit the subscript q in $[z]_q$. For $(z,q) \in T_N$ and $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in P_\ell$ we can define the space of conformal blocks attached to the elliptic curve $\mathcal{E}_q$:

$$\mathcal{V}_1^\dagger([z],q;\vec{\lambda}) = \{ \langle\Psi| \in \mathcal{H}_{\vec{\lambda}}^\dagger \ ; \ \langle\Psi|\mathfrak{g}([z],q) = 0 \},$$

where

$$\widehat{\mathfrak{g}}([z],q) = H^0(\mathcal{E}_q, \mathfrak{g} \otimes \mathcal{O}_{\mathcal{E}_q}(*\sum_{j=1}^N [z_j])),$$

but for our purpose we need to twist it as follows. We introduce a new variable $\xi \in \mathbb{C}^*$, and put

$$\widehat{\mathfrak{g}}([z],q,\xi) = \left\{ a(t) \in H^0(\mathbb{C}^*, \mathfrak{g} \otimes \mathcal{O}_{\mathbb{C}^*}(*\sum_{j=1}^N \sum_{n\in\mathbb{Z}} q^n z_j)) \ ; \ a(qt) = \xi^{\frac{H}{2}}(a(t))\xi^{-\frac{H}{2}} \right\}.$$

This space is regarded as the space of meromorphic sections of the $\mathfrak{g}$-bundle which is twisted by $\xi^{\frac{H}{2}}$ along the cycle $\{ [w] \in \mathcal{E}_q \ ; \ w \in \mathbb{R}, q \le w < 1 \}$. For $\xi = 1$, we have

$$\widehat{\mathfrak{g}}([z],q,1) = H^0(\mathcal{E}_q, \mathfrak{g} \otimes \mathcal{O}_{\mathcal{E}_q}(*\sum_{j=1}^N [z_j])).$$

As in the previous section we have the following injection:

$$\begin{aligned} \widehat{\mathfrak{g}}([z],q,\xi) &\to \widehat{\mathfrak{g}}_N \\ X \otimes f &\mapsto X[f]. \end{aligned}$$

By this map we regard $\widehat{\mathfrak{g}}([z],q,\xi)$ as a subspace of $\widehat{\mathfrak{g}}_N$. Furthermore we can easily have the following lemma.

Lemma 3.2.1. *The vector space $\widehat{\mathfrak{g}}([z],q,\xi)$ is a Lie subalgebra of $\widehat{\mathfrak{g}}_N$.* □

Definition 3.2.2. Put

$$\begin{aligned} \mathcal{V}_1([z],q,\xi;\vec{\lambda}) &= \mathcal{H}_{\vec{\lambda}}/\widehat{\mathfrak{g}}([z],q,\xi)\mathcal{H}_{\vec{\lambda}}, \\ \mathcal{V}_1^\dagger([z],q,\xi;\vec{\lambda}) &= \{ \langle\Psi| \in \mathcal{H}_{\vec{\lambda}}^\dagger \ ; \ \langle\Psi|\widehat{\mathfrak{g}}([z],q,\xi) = 0 \} \\ &\cong \mathrm{Hom}_{\mathbb{C}}(\mathcal{V}_1([z],q,\xi;\vec{\lambda}),\mathbb{C}). \end{aligned}$$

We call $\mathcal{V}_1^\dagger([z],q,\xi;\vec{\lambda})$ the space of conformal blocks in genus 1 attached to $([z],q,\xi;\vec{\lambda})$.

Following [TUY][FW], we define the N-point functions in genus 1 as follows:

Definition 3.2.3. An element $\langle\Phi|$ of $\mathcal{H}_{\vec{\lambda}}^\dagger[T_N \times \mathbb{C}^*]$ is called an N-point function in genus 1 attached to $\vec{\lambda}$ if the following conditions are satisfied:

(D1) For each $(z,q,\xi) \in T_N \times \mathbb{C}^*$,

$$\langle\Phi(z,q,\xi)| \in \mathcal{V}_1^\dagger([z],q,\xi;\vec{\lambda}).$$

(D2) For $j = 1, \dots, N$,

$$\partial_{z_j} \langle \Phi(z,q,\xi)| = \langle \Phi(z,q,\xi)| \rho_j(L_{-1})$$

(D3)

$$\left(q\partial_q + \frac{c_v}{24}\right) \langle \Phi(z,q,\xi)| = \langle \Phi(z,q,\xi)| T\left[\zeta(t/z_1,q)t\frac{d}{dt}\right],$$

where $\zeta(t,q)$ is the function given by (3.1.2).

(D4)

$$\xi\partial_\xi \langle \Phi(z,q,\xi)| = \langle \Phi(z,q,\xi)| \frac{1}{2} H[\zeta(t/z_1,q)].$$

We denote by $\mathfrak{F}_1(\vec{\lambda})$ the set of N-point functions attached to $\vec{\lambda}$.

Remark. (i) The condition (D1) implies the following:

(D1′) For each $(z,q,\xi) \in T_N \times \mathbb{C}^*$,

$$\langle \Phi(z,q,\xi)| T[g] = 0$$

for any $g \in H^0(\mathcal{E}_q, \Theta_{\mathcal{E}_q}(*\sum_{j=1}^N z_j)) = H^0(\mathcal{E}_q, \mathcal{O}_{\mathcal{E}_q}(*\sum_{j=1}^N z_j)t\frac{d}{dt})$.

(ii) The equations (D1)–(D4) are compatible with each other due to (3.1.4), e.g.

$$\left[\xi\partial_\xi - \frac{1}{2}H[\zeta(t/z_j)]\,,\, X[f(t,q,\xi)]\right] = \tag{3.2.1}$$

$$X[\xi\partial_\xi f(t,q,\xi)] - \frac{1}{2}[H,X][\zeta(t/z_j,q)f(t,q,\xi)] \in \hat{\mathfrak{g}}([z],q,\xi).$$

for $X[f] \in \hat{\mathfrak{g}}([z],q,\xi)$. Conversely, the compatibility condition demands (3.1.4) for ζ.

(iii) In (D3) and (D4) we can replace $\zeta(t/z_1,q)$ with $\zeta(t/z_j,q)$ $(j = 2,\dots,N)$ provided (D1) since

$$\zeta(t/z_1,q) - \zeta(t/z_j,q) \in H^0(\mathcal{E}_q, \mathcal{O}_{\mathcal{E}_q}(*\sum_{i=1}^N z_j)). \tag{3.2.2}$$

The finite-dimensionality of the space $\mathcal{V}_1^\dagger([z],q,\xi;\vec{\lambda})$ can be shown in a similar way as in [TUY]. The compatibility of (D1)–(D4) implies that there exists a vector bundle $\tilde{\mathcal{V}}_1^\dagger(\vec{\lambda})$ over a domain $U \subset T_N \times \mathbb{C}^*$ which has $\mathcal{V}_1^\dagger([z],q,\xi;\vec{\lambda})$ as a fiber at $([z],q,\xi) \in U$, with the integrable connections defined by the differential equations (D2)–(D4). In particular the dimension of the fiber $\mathcal{V}_1^\dagger([z],q,\xi;\vec{\lambda})$ does not depend on $([z],q,\xi)$.

3.3 Restrictions of N-point functions to $V_{\vec{\lambda}}$.

In this subsection we see that, as a consequence of the twisting, an N-point function in genus 1 is determined from its restriction to $V_{\vec{\lambda}}$ (Proposition 3.3.2). We also give the characterization of N-point functions as $V_{\vec{\lambda}}^\dagger$-valued functions (Theorem 3.3.4).

Lemma 3.3.1. *Let $\mathcal{S}$ be the subspace of $\mathcal{H}_{\vec{\lambda}}$ spanned by the vectors*

$$\rho_1(H_{-1})^k|v\rangle \quad (|v\rangle \in V_{\vec{\lambda}},\ k \in \mathbb{Z}_{\geq 0}).$$

Then for $(z,q,\xi) \in T_N \times \mathbb{C}^*$ such that $\xi \neq q^n$ $(n \in \mathbb{Z})$, the natural map

$$\mathcal{S} \ \to \mathcal{V}_1([z],q,\xi;\vec{\lambda})$$

is surjective. In other words the restriction map

$$\mathcal{V}_1^{\dagger}([z],q,\xi;\vec{\lambda}) \ \to \ \mathrm{Hom}_{\mathbb{C}}(\mathcal{S},\mathbb{C})$$

is injective.

Proof. This is shown by noting the fact that, for $\xi \neq q^n$ $(n \in \mathbb{Z})$, the space $\widehat{\mathfrak{g}}([z],q,\xi)$ is spanned by the following $\mathfrak{g}$-valued functions

$$H\otimes 1,\ H\otimes(\zeta(t/z_i,q)-\zeta(t/z_j,q)),\ H\otimes(t\partial_t)^n\wp(t/z_j,q),$$
$$E\otimes(t\partial_t)^n\sigma_+(t/z_j,q,\xi),\ F\otimes(t\partial_t)^n\sigma_-(t/z_j,q,\xi)\ (i,j=1,\dots,N,\ n=0,1,\dots).$$

□

Let $\langle\Phi|$ be an N-point function in genus 1 and $|u\rangle$ be a vector in $\mathcal{H}_{\vec{\lambda}}$. By Lemma 3.3.1 we can express $\langle\Phi(z,q,\xi)|u\rangle$ as a combination of

$$\langle\Phi(z,q,\xi)|\rho_1(H_{-1})^n|v\rangle \ \ (n\in\mathbb{Z}_{\geq 0}, |v\rangle \in V_{\vec{\lambda}}).$$

Combining with (D4) we have the procedure to rewrite $\langle\Phi(z,q,\xi)|u\rangle$ as a combination of

$$(\xi\partial_\xi)^n\langle\Phi(z,q,\xi)|v\rangle \ \ (n\in\mathbb{Z}_{\geq 0}, |v\rangle \in V_{\vec{\lambda}}).$$

Furthermore it is easily seen that we need finitely many data for each $|u\rangle$:

Proposition 3.3.2. *For $|u\rangle \in \mathcal{F}_p\mathcal{H}_{\vec{\lambda}}$, there exist functions*

$$a_{i,n}(z,q,\xi)\ (i=1,\dots,\dim V_{\vec{\lambda}},\ n=1,\dots,p)$$

on $T_N \times \mathbb{C}^*$ such that

$$\langle\Phi(z,q,\xi)|u\rangle = \sum_{i,n} a_{i,n}(z,q,\xi)\,(\xi\partial_\xi)^n\langle\Phi(z,q,\xi)|b_i\rangle$$

for any $\langle\Phi| \in \mathfrak{F}_1(\vec{\lambda})$, where $\{\ |b_i\rangle\ ;\ i=1,\dots,\dim V_{\vec{\lambda}}\ \}$ is a basis of $V_{\vec{\lambda}}$.

By $\mathfrak{F}_1^r(\vec{\lambda})$ we denote the image of $\mathfrak{F}_1(\vec{\lambda})$ in $V_{\vec{\lambda}}^{\dagger}[T_N\times\mathbb{C}^*]$ under the restriction map to $V_{\vec{\lambda}}$, which is injective by the above proposition.

Next, as in the case of genus 0, we consider the characterization of $\mathfrak{F}_1^r(\vec{\lambda})$ in $V_{\vec{\lambda}}^{\dagger}[T_N\times\mathbb{C}^*]$. First, we have the following.

Proposition 3.3.3. *The restriction $\langle\phi|$ of an N-point function satisfies the following equations.*

(E1)

$$\sum_{j=1}^{N}\langle\phi(z,q,\xi)|\rho_j(H)=0.$$

(E2) For each $j=1,\dots,N$,

$$(\ell+2)\left(z_j\partial_{z_j}+\Delta_{\lambda_j}\right)\left(\Theta(\xi,q)\langle\phi(z,q,\xi)|\right)=$$
$$\xi\partial_\xi\left(\Theta(\xi,q)\langle\phi(z,q,\xi)|\right)\rho_j(H)+\sum_{i\neq j}\Theta(\xi,q)\langle\phi(z,q,\xi)|\Omega_{i,j}(z_j/z_i,q,\xi),$$

where

$$\Omega_{i,j}(t,q,\xi)=$$
$$\frac{1}{2}\zeta(t,q)\rho_i(H)\rho_j(H)+\sigma_+(t,q,\xi)\rho_i(F)\rho_j(E)+\sigma_-(t,q,\xi)\rho_i(E)\rho_j(F).$$

(E3)

$$(\ell+2)q\partial_q\left(\Theta(\xi,q)\langle\phi(z,q,\xi)|\right)=$$
$$(\xi\partial_\xi)^2\left(\Theta(\xi,q)\langle\phi(z,q,\xi)|\right)+\sum_{i,j=1}^{N}\Theta(\xi,q)\langle\phi(z,q,\xi)|\Lambda_{i,j}(z_i/z_j,q,\xi).$$

Here

$$\Lambda_{i,j}(t,q,\xi)=\frac{1}{4}\left(\zeta(t,q)^2-\wp(t,q)\right)\rho_i(H)\rho_j(H)$$
$$+\omega_+(t,q,\xi)\rho_i(E)\rho_j(F)+\omega_-(t,q,\xi)\rho_i(F)\rho_j(E),$$

where $\omega_\pm(t,q,\xi)$ denote the functions defined by

$$\omega_\pm(t,q,\xi)=\frac{1}{2}\left\{\partial_t\sigma_\pm(t,q,\xi)+(\zeta(t,q)\pm\zeta(\xi,q))\sigma_\pm(t,q,\xi)\right\},$$

which are holomorphic at $t=1$.

For the proof of Proposition 3.3.3, we refer the reader to [FW].

Remark. The equation (E2) is derived by Bernard as a equation for the trace of the vertex operators (see §§3.4), he also derived (E3) in a special case. The equations (E2)(E3) are called the Knizhnik-Zamolodchikov-Bernard (KZB) equations in [Fe][FW].

Note that the system of equations (E1)–(E3) is not holonomic since we have $j+2$ parameters $z_1,\dots,z_N,q,\xi$, but have only $j+1$ differential equations, which are compatible each other.

The differential equations (E2) and (E3) are of order 1 with respect to z_j ($j = 1, \ldots, N$) and q respectively. Hence to characterize $\mathfrak{F}_1^r(\vec{\lambda})$ in $V_{\vec{\lambda}}[T_N \times \mathbb{C}^*]$, it is sufficient to obtain equations which determine the ξ-dependence of the restricted N-point functions and they are obtained as follows.

Let $\langle\Phi|$ be an N-point function and $\langle\phi|$ its restriction to $V_{\vec{\lambda}}$. We put $\mathcal{M}_{\vec{\lambda}}^\dagger = \mathrm{Hom}_{\mathbb{C}}(\mathcal{M}_{\lambda_1} \otimes \cdots \otimes \mathcal{M}_{\lambda_N}\, \mathbb{C})$ and regard $\langle\Phi|$ as an $\mathcal{M}_{\vec{\lambda}}^\dagger$-valued function. Then as a special case of integrability condition, we have for each non negative integer k

$$\langle\Phi|v_1 \otimes \cdots \otimes F^k E_{-1}^{\ell-2\lambda_j+1} v(\lambda_j) \otimes \cdots \otimes v_N\rangle = 0 \tag{3.3.1}$$

for any $|v_i\rangle \in V_{\lambda_i}$ $(i \neq j)$, where $|v(\lambda_j)\rangle$ denotes the highest weight vector in $\mathcal{M}_{\lambda_j}$.

On the other hand, by Proposition 3.3.2 we can rewrite the left hand side of (3.3.1) as a combination of

$$(\xi\partial_\xi)^n\langle\Phi|v\rangle = (\xi\partial_\xi)^n\langle\phi|v\rangle \quad (n = 0, 1, \ldots, \ell - 2\lambda_j + 1, |v\rangle \in V_{\vec{\lambda}}).$$

Now the equality (3.3.1) implies the differential equation for $\langle\phi|$ with respect to ξ of order at most $\ell - 2\lambda + 1$. We denote this differential equation by

$$\langle\phi|v_1 \otimes \cdots \otimes F^k E_{-1}^{\ell-2\lambda_j+1} v(\lambda_j) \otimes \cdots \otimes v_N\rangle = 0.$$

Theorem 3.3.4. *The space $\mathfrak{F}_1^r(\vec{\lambda})$ coincides with the solution space of the system of equations (E1)–(E4), where (E4) is given by*

(E4) For each $j = 1, \ldots, N$ and nonnegative integer $k \leq \sum_{i=1}^N \lambda_i + \ell - 2\lambda_j + 1$,

$$\langle\phi(z, q, \xi)|v_1 \otimes \cdots \otimes F^k E_{-1}^{\ell-2\lambda_j+1} v(\lambda_j) \otimes \cdots \otimes v_N\rangle = 0,$$

for any $|v_i\rangle \in V_{\lambda_i} (i \neq j)$.

Proof. It is enough to prove that the dimension of the solution space of the system (E1)–(E4) is not larger than $\dim_{\mathbb{C}} \mathfrak{F}_1(\vec{\lambda}) = \dim_{\mathbb{C}} \mathfrak{F}_1^r(\vec{\lambda})$.

Fix $(z, q) \in T_N$ and let $\langle\phi(\xi)| = \langle\phi(z, q, \xi)|$ be a $V_{\vec{\lambda}}^\dagger$-valued function on $\mathbb{C}^*$ which satisfies (E1) and (E4). From $\langle\phi(\xi)|$, we construct an element $\langle\Phi(\xi)|$ of $\mathcal{M}_{\vec{\lambda}}^\dagger[\mathbb{C}^*]$ which satisfies

(i) $\qquad \langle\Phi(\xi)|v\rangle = \langle\phi(\xi)|v\rangle$ for $|v\rangle \in V_{\vec{\lambda}}$,

(ii) $\qquad \langle\Phi(\xi)| \in \mathcal{V}_1^\dagger([z], q, \xi; \vec{\lambda})$ for each $\xi \in \mathbb{C}^*$,

(iii) $\qquad \xi\partial_\xi\langle\Phi(\xi)| = \langle\Phi(\xi)|\frac{1}{2}H[\zeta(t/z_1)]$,

The well-definedness is proved by induction with respect to the filtration $\{\mathcal{F}_\bullet\}$ using Lemma 3.2.1 and the compatibility condition (3.2.1). Moreover we can show that $\langle\Phi(\xi)|$ belongs to $\mathcal{H}_{\vec{\lambda}}^\dagger$, that is,

$$\langle\Phi(\xi)|u_1 \otimes \cdots \otimes a \cdot E_{-1}^{\ell-2\lambda+1} v(\lambda_j) \otimes \cdots \otimes u_N\rangle = 0$$

for any $j = 1, \dots, N$, $|u_i\rangle \in \mathcal{M}_{\lambda_i}$ and $a \in U(\widehat{\mathfrak{p}}_-)$. This is reduced to (E4) also by induction.

Now we have the injective homomorphism from the solution space of (E1)–(E4) to the space of functions on $\mathbb{C}^*$ satisfying (ii) and (iii); the latter space has the same dimension as $\mathcal{V}_1^\dagger([z], q, \xi; \vec{\lambda})$. □

In the case of $N = 1$ we can write down the differential equations (E4) explicitly as we will see in §4.

3.4 Sewing procedure.

In this subsection we show that the N-point functions in genus 1 are given by the traces of vertex operators and hence Bernard's approach is equivalent to ours. For this purpose we construct an N-point function in genus 1 from an $N+2$-point function in genus 0. This construction is known as the sewing procedure.

Fix $\mu \in P_\ell$ and $\vec{\lambda} = (\lambda_1, \dots, \lambda_N) \in (P_\ell)^N$, and consider a sequence of vertex operators $\varphi_j(z_j) : V_{\lambda_j} \otimes \mathcal{H}_{\mu_j} \to \hat{\mathcal{H}}_{\mu_{j-1}}$ for some $\mu_{j-1}, \mu_j \in P_\ell$ with $\mu_0 = \mu_N = \mu$. For $|u\rangle = |u_1 \otimes \cdots \otimes u_N\rangle \in \mathcal{H}_{\vec{\lambda}}$, we put

$$\Phi_0(|u\rangle; z) = \hat{\varphi}_1(|u_1\rangle; z_1)\hat{\varphi}_2(|u_2\rangle; z_2) \cdots \hat{\varphi}_N(|u_N\rangle; z_N) : \mathcal{H}_\mu \to \hat{\mathcal{H}}_\mu,$$

where $\hat{\varphi}_j(z_j)$ means the extended vertex operator in the sense of Proposition 2.3.3. We define a $\mathcal{H}^\dagger_{\vec{\lambda}}$-valued function on $T_N \times \mathbb{C}^*$ by

$$\langle \Phi_1(z, q, \xi)|u\rangle = \mathrm{Tr}_{\mathcal{H}_\mu} \left(\Phi_0(|u\rangle; z) q^{L_0 - \frac{c_v}{24}} \xi^{\frac{H}{2}} \right) \tag{3.4.1}$$

for $|u\rangle \in \mathcal{H}_{\vec{\lambda}}$.

Proposition 3.4.1. *The element $\langle \Phi_1|$ of $\mathcal{H}_{\vec{\lambda}}[T_N \times \mathbb{C}^*]$ defined by (3.4.1) is an N-point function in genus 1.*

Proof. First we prove that $\langle \Phi_1|$ satisfies the condition (D1).

Fix any $X \otimes f \in \widehat{\mathfrak{g}}([z], q, \xi; \vec{\lambda})$ and $|u\rangle \in \mathcal{H}_{\vec{\lambda}}$, and put

$$\langle \Phi_1 | X(t) |u\rangle dt = \mathrm{Tr}_{\mathcal{H}_\mu} \Phi_0(|u\rangle; z) X(t) q^{L_0 - \frac{c_v}{24}} \xi^{\frac{H}{2}} dt.$$

This is a holomorphic 1-form on $\mathbb{C}^* \setminus \{ q^n z_j \in \mathbb{C}^* \; ; \; n \in \mathbb{Z}, j = 1, \dots, N \}$. Then by (2.3.1), what we should show is the following.

$$\sum_{j=1}^{N} \operatorname*{Res}_{t=z_j} f(t) \langle \Phi_1 | X(t) |u\rangle dt = 0. \tag{3.4.2}$$

But we have

$$\begin{aligned} f(t)\langle \Phi_1 | X(t) |u\rangle dt &= f(t) \mathrm{Tr}_{\mathcal{H}_\mu} X(t) \Phi_0(|u\rangle; z) q^{L_0 - \frac{c_v}{24}} \xi^{\frac{H}{2}} dt \\ &= f(t) \mathrm{Tr}_{\mathcal{H}_\mu} \Phi_0(|u\rangle; z) q^{L_0 - \frac{c_v}{24}} \xi^{\frac{H}{2}} X(t) dt \\ &= f(qt) \mathrm{Tr}_{\mathcal{H}_\mu} \Phi_0(|u\rangle; z) X(qt) q^{L_0 - \frac{c_v}{24}} \xi^{\frac{H}{2}} d(qt) \\ &= f(qt) \langle \Phi_1 | X(qt) |u\rangle d(qt), \end{aligned}$$

where we used the commutativity of vertex operators and currents, and

$$f(t)\xi^{\frac{H}{2}}(X(t))\xi^{-\frac{H}{2}} = f(qt)X(t), \; q^{L_0}(X(t))q^{-L_0} = X(qt)q.$$

Therefore we have $f(t)\langle\Phi_1|X(t)|u\rangle dt \in H^0(\mathcal{E}_q, \omega_{\mathcal{E}_q}(\sum_{j=1}^N *[z_j]))$, where $\omega_{\mathcal{E}_q}$ denotes the sheaf of 1-forms on $\mathcal{E}_q$. This implies (3.4.2).

Next we prove that $\langle\Phi|$ satisfies the equation (D2)–(D4). It is obvious that $\langle\Phi|$ satisfies (D2) from (2.3.2). We give a proof of (D4). The equation (D3) is proved in a similar way. We chose (z, q) from the region $1 > |z_1| > |z_2| > \cdots > |z_N| > |q|$, where $\langle\Phi_1|$ is a convergent power series. Let $Z_r = \{|w| = r\}$ be a cycle with anticlockwise orientation. We have

$$\begin{aligned}
&2\pi\sqrt{-1}\langle\Phi_1|H[\zeta(t/z_1)]|u\rangle \\
&= \mathrm{Tr}_{\mathcal{H}_\mu}\int_{Z_1}\zeta(t/z_1)H(t)\Phi_0(|u\rangle;z)q^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}}dt \\
&- \mathrm{Tr}_{\mathcal{H}_\mu}\int_{Z_q}\zeta(t/z_1)\Phi_0(|u\rangle;z)H(t)q^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}}dt \\
&= \mathrm{Tr}_{\mathcal{H}_\mu}\left\{\int_{Z_q}\zeta(q^{-1}t/z_1) - \int_{Z_q}\zeta(t/z_1)\right\}\Phi_0(|u\rangle;z)H(t)q^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}}dt.
\end{aligned}$$

By $\zeta(t) = \zeta(q^{-1}t) - 1$, we conclude

$$\begin{aligned}
\langle\Phi_1|H[\zeta(t/z_1)]|u\rangle &= \frac{1}{2\pi\sqrt{-1}}\mathrm{Tr}_{\mathcal{H}_\mu}\int_{Z_q}\Phi_0(|u\rangle;z)H(t)q^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}}dt \\
&= \mathrm{Tr}_{\mathcal{H}_\mu}\left(\Phi_0(|u\rangle;z)Hq^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}}\right).
\end{aligned}$$

This proves (D4). □

By Proposition 3.4.1 we have the mapping from $\mathfrak{F}_0(\mu, \vec{\lambda}, \mu)$ to $\mathfrak{F}_1(\vec{\lambda})$. We denote this mapping by s_μ. The following proposition follows from "the factorization property" proved in [TUY].

Proposition 3.4.2. *The following map is bijective.*

$$\bigoplus_{\mu\in P_\ell} s_\mu : \bigoplus_{\mu\in P_\ell}\mathfrak{F}_0(\mu, \vec{\lambda}, \mu) \to \mathfrak{F}_1(\vec{\lambda}). \quad \square$$

By Proposition 2.3.2 and Proposition 3.4.2 we get the following.

Theorem 3.4.3. *The space $\mathfrak{F}_1^r(\vec{\lambda})$ is spanned by the functions*

$$\mathrm{Tr}_{\mathcal{H}_\mu}\varphi_1(z_1)\cdots\varphi_N(z_N)q^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}},$$

where $\varphi_j(z_j)$ $(j = 1, \dots, N)$ is the vertex operator of type $(\mu_{j-1}, \lambda_j, \mu_j)$ for some $\mu_i \in P_\ell$ $(i = 1, \dots, N+1)$ with $\mu := \mu_0 = \mu_N$.

Remark. The integral representations of the above functions are obtained in [BF].

§4 Explicit formulas for 1-point functions in genus 1.

In this section, we see how the system (E1)–(E4) determine the 1-point function explicitly (Theorem 4.2.4). We also solve the system in a few cases.

4.1. The 1-point functions in genus 1.

Fix a weight λ and consider the set $\mathfrak{F}_1^r(\lambda)$ of restricted 1-point functions in genus 1, which is, by Theorem 3.4.3, spanned by the following $V_\lambda^\dagger$-valued functions:

$$\langle \phi_\mu(z_1, q, \xi)| := \mathrm{Tr}_{\mathcal{H}_\mu} \varphi(z_1) q^{L_0 - \frac{c_v}{24}} \xi^{\frac{H}{2}} \quad (\mu \in P_\ell),$$

where $\varphi(z_1)$ is the vertex operator of type (μ, λ, μ). We put

$$L = \ell - 2\lambda.$$

Note that a nonzero vertex operator of type (μ, λ, μ) exists if and only if λ and μ satisfy

$$\lambda \in \mathbb{Z}, \ \frac{\lambda}{2} \leq \mu \leq \frac{\lambda + L}{2},$$

and the vertex operators are unique up to constant multiples. In particular we have

$$\dim_{\mathbb{C}} \mathfrak{F}_1(\lambda) = L + 1.$$

As we have seen in Theorem 3.3.4, the restrictions of 1-point functions $\langle \phi|$ are characterized by (E1)–(E4). The equation (E2) now implies

$$\langle \phi(z_1, q, \xi)| = z_1^{-\Delta_\lambda} \langle \phi(1, q, \xi)|.$$

Hence in the following we specialize $z_1 = 1$ and put $\langle \phi(\xi, q)| = \langle \phi(1, q, \xi)|$. By the condition (E1), we can identify $\mathfrak{F}_1^r(\lambda)$ with the space spanned by the following function:

$$\phi_\mu(\xi, q) \quad \mu = \frac{\lambda}{2}, \frac{\lambda+1}{2}, \dots, \frac{\lambda + L}{2},$$

where $|0_\lambda\rangle$ is the weight 0 vector in V_λ defined by

$$|0_\lambda\rangle = \frac{1}{\lambda!} F^\lambda |v(\lambda)\rangle.$$

From the equation (E3) we immediately have the following heat equation.

Proposition 4.1.1. *For $\phi \in \mathfrak{F}_1^r(\lambda)$,*

$$(\ell + 2) q\partial_q(\Theta(\xi, q)\phi(\xi, q)) =$$
$$\left\{ (\xi\partial_\xi)^2 - \lambda(\lambda + 1) \left(\wp(\xi, q) - 2\frac{q\partial_q \eta(q)}{\eta(q)} \right) \right\} (\Theta(\xi, q)\phi(\xi, q)).$$

Remark. The heat equations for 1-point functions are studied by Etingof and Kirillov in more general cases: $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $V_\lambda = S^{\lambda n}\mathbb{C}^n$ ($\lambda \in \mathbb{Z}$), where S^m denotes m-th symmetric product and $\mathbb{C}^n$ the defining representation of $\mathfrak{g}$ [EK].

4.2 The differential equation with respect to ξ.

This subsection is devoted to write down differential equations for by $\phi \in \mathfrak{F}_1^r(\lambda)$ derived from (E4):

$$\langle\phi|F^k E_{-1}^{L+1}|v(\lambda)\rangle = 0 \quad (0 \le k \le \lambda + L + 1).$$

Among them the only nontrivial equality is the following:

$$\langle\phi|F^{\lambda+L+1} E_{-1}^{L+1}|v(\lambda)\rangle = 0, \tag{4.2.1}$$

because other equalities fall into trivial by (E1).

To rewrite (4.2.1) as a differential equation with respect to ξ, we consider the following set of vectors in $\mathcal{M}_\lambda$

$$\left\{ |u^k\rangle = \frac{1}{(\lambda+k)!} F^{\lambda+k} E_{-1}^k |v(\lambda)\rangle \ ; \ k = 0, 1, \ldots \right\}.$$

Note that $|u^0\rangle = |0_\lambda\rangle$. The following lemma plays a key role in the following discussions.

Lemma 4.2.1. *For $k \in \mathbb{Z}_{\ge 0}$, we have*

$$\frac{1}{2} H[\zeta(z,q)]|u^k\rangle \equiv \tag{4.2.2}$$
$$-\frac{1}{2}|u^{k+1}\rangle + (k+\lambda)\zeta(\xi,q)|u^k\rangle + k(L-k+1)\beta(\xi,q)|u^{k-1}\rangle \ mod\ \widehat{\mathfrak{g}}([z],q,\xi)\mathcal{M}_\lambda,$$

where $\beta(\xi,q)$ is given by

$$\beta(\xi,q) = \frac{q\partial_q \Theta(\xi,q)}{\Theta(\xi,q)} - 3\frac{q\partial_q \eta(q)}{\eta(q)}. \tag{4.2.3}$$

For $\langle\Phi| \in \mathfrak{F}_1(\lambda)$, we put $\vec{\Phi} = {}^t(\langle\Phi|u^0\rangle, \ldots, \langle\Phi|u^L\rangle)$. Then by $\langle\Phi|u^{L+1}\rangle = 0$ and Lemma 4.2.1, we obtain the following differential equation for $\vec{\Phi}$.

Proposition 4.2.2. *For $\langle\Phi| \in \mathfrak{F}_1(\lambda)$, we have*

$$\xi\partial_\xi \vec{\Phi}(\xi,q) \ = \ \mathcal{A}_{L+1}(\xi,q)\vec{\Phi}(\xi,q) \ + \ \lambda\zeta(\xi,q)\vec{\Phi}(\xi,q). \tag{4.2.4}$$

Here, $\mathcal{A}_{L+1}$ is an $(L+1)\times(L+1)$ tri-diagonal matrix given by

$$\begin{pmatrix} 0 & -\frac{1}{2} & & & & \\ 1\cdot L\cdot\beta & \zeta & -\frac{1}{2} & & & \\ & 2\cdot(L-1)\cdot\beta & 2\zeta & -\frac{1}{2} & & \\ & & \cdots & \cdots & \cdots & \\ & & & (L-1)\cdot 2\cdot\beta & (L-1)\zeta & -\frac{1}{2} \\ & & & & L\cdot 1\cdot\beta & L\zeta \end{pmatrix},$$

where the functions $\zeta(\xi, q)$ *and* $\beta(\xi, q)$ *are given by* (3.1.2) *and* (4.2.3).

It is remarkable that the equation (4.2.4) can be written in the following form:

$$\xi\partial_\xi\left(\Theta^{-\lambda}\vec{\Phi}\right) = \mathcal{A}_{L+1}\left(\Theta^{-\lambda}\vec{\Phi}\right).$$

By Proposition 4.2.2 we can write $\langle\Phi|u^k\rangle$ as a combination of differentials of $\phi :=$ $\langle\Phi|u^0\rangle$ with respect to ξ; e.g.

$$\begin{aligned}\langle\Phi|u^1\rangle &= -2\xi\partial_\xi\phi\,,\\ \langle\Phi|u^2\rangle &= -2\left(\xi\partial_\xi - \zeta\right)\langle\Phi|u^1\rangle - 2L\beta\phi\\ &= 4\left(\xi\partial_\xi - \zeta\right)\xi\partial_\xi\phi - 2L\beta\phi\,,\\ &\text{etc...}\end{aligned}$$

In general, we have the following lemma by simple calculations.

Lemma 4.2.3. *For* $k = 1, \dots, L+1$, *we have*

$$\Theta^{-\lambda}\langle\Phi|u^k\rangle = (-2)^k\mathbf{Det}\left[\xi\partial_\xi\cdot I_k - \mathcal{A}^{(k)}_{L+1}\right]\left(\Theta^{-\lambda}\phi\right)\,,$$

where I_k *is the* $k\times k$*-identity matrix and* $\mathcal{A}^{(k)}_{L+1}$ *is the* $k\times k$*-matrix given by the first* $k\times k$ *block of* $\mathcal{A}_{L+1}$*:*

$$\mathcal{A}^{(k)}_{L+1} = \begin{pmatrix} 0 & -\frac{1}{2} & & & \\ L\beta & \zeta & -\frac{1}{2} & & \\ & 2(L-1)\beta & 2\zeta & -\frac{1}{2} & \\ & & \cdots & \cdots & \cdots \\ & & & (k-1)(L-k+2)\beta & (k-1)\zeta \end{pmatrix}.$$

Here, for an $n\times n$*-matrix* $A = (a_{i,j})$ *with elements in some, possibly non-commutative, ring,* $\mathbf{Det}A$ *is defined inductively as follows:*

$$\begin{aligned}\mathbf{Det}A &= a_{1,1}\quad \text{for } n = 1,\\ \mathbf{Det}A &= \mathbf{Det}A_{1,1}\cdot a_{1,1} - \mathbf{Det}A_{1,2}\cdot a_{1,2} + \cdots + (-1)^{n-1}\mathbf{Det}A_{1,n}\cdot a_{1,n}\,,\end{aligned}$$

where $A_{i,j}$ *is the matrix given by removing the* i*-th row and* j*-th column from* A.
□

Through this lemma, we can rewrite (4.2.1) explicitly as a differential equation for $\phi\in\mathfrak{F}_1^r(\lambda)$ of order $L+1$ with respect to ξ. Combining with Proposition 4.1.1 we get the following.

Theorem 4.2.4. *The space $\mathfrak{F}_1^r(\lambda)$ coincides with the solution space of the following system of differential equations.*

(*F*1)

$$(\ell+2)q\partial_q\left(\Theta(\xi,q)^{-\lambda}\phi(\xi,q)\right) = \left\{(\xi\partial_\xi)^2 + 2(\lambda+1)\zeta(\xi,q)\xi\partial_\xi - L(\lambda+1)\frac{q\partial_q\Theta(\xi,q)}{\Theta(\xi,q)}\right\}\left(\Theta(\xi,q)^{-\lambda}\phi(\xi,q)\right).$$

(*F*2)

$$\mathbf{Det}\left[\xi\partial_\xi\cdot I_{L+1} - \mathcal{A}_{L+1}(\xi,q)\right]\left(\Theta(\xi,q)^{-\lambda}\phi(\xi,q)\right) = 0.$$

Remark. (i) It can be easily checked directly that the solution space of (F1)(F2) is $(L+1)$-dimensional.

(ii) For $\lambda=0$, the vertex operator $\varphi_\mu(|0\rangle_0;z)$ is equal to the identity operator on $\mathcal{H}_\mu$ up to a constant multiple. Thus the 1-point function ϕ_μ is nothing but the character

$$\chi_\mu^{(\ell)}(\xi,q) = \mathrm{Tr}_{\mathcal{H}_\mu} q^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}} = \frac{\Theta_{2\mu+1,\ell+2}(\xi,q)-\Theta_{-2\mu-1,\ell+2}(\xi,q)}{\sqrt{-1}\Theta(\xi,q)},$$

where $\Theta_{m,k}(\xi,q)$ is the theta function of level k defined by

$$\Theta_{m,k}(\xi,q) = \sum_{n\in\mathbb{Z}+\frac{m}{2k}} q^{kn^2}\xi^{kn}.$$

In the case of $\ell=1,2$, the system (F1)(F2) coincides with the one obtained in [EO2].

We can easily solve (F2) by noting the above remark (ii).

Proposition 4.2.5. *For $\lambda\in P_\ell$ and $q\in D^*$, the functions*

$$\Theta(\xi,q)^\lambda\chi_\mu^{(\ell-2\lambda)}(\xi,q)\quad\left(\mu=0,\frac{1}{2},\ldots,\frac{\ell-2\lambda}{2}\right)$$

form a basis of the solution space of (F2). □

4.3. Some solutions.

In this subsection we determine the trace of vertex operators explicitly when $L=\ell-2\lambda\le 1$, by solving the differential equations (F1) and (F2).

Case $L=0$:
In this case the space $\mathfrak{F}_1^r(\lambda)$ is spanned by the single function

$$\phi_{\frac{\lambda}{2}}(\xi,q) = \mathrm{Tr}_{\mathcal{H}_{\frac{\lambda}{2}}}\varphi(|0\rangle_{\frac{\lambda}{2}};1)q^{L_0-\frac{c_v}{24}}\xi^{\frac{H}{2}}.$$

On the other hand, by Proposition 4.2.5, any solution of (F2) is given in the following form:

$$\phi(\xi,q) = a(q)\Theta(\xi,q)^\lambda\chi_0^{(0)}(\xi,q) = a(q)\Theta(\xi,q)^\lambda$$

with some function $a(q)$, and the equation (F1) now implies $\partial_q a(q) = 0$. Therefore, we have

$$\phi_{\frac{\lambda}{2}}(\xi, q) = \Theta(\xi, q)^{\lambda} \tag{4.3.1}$$

under the appropriate normalization.

Case $L = 1$:

The space $\mathfrak{F}_1^{\tau}(\lambda)$ has dimension 2 and it is spanned by

$$\phi_{\mu}(\xi, q) = \mathrm{Tr}_{\mathcal{H}_{\mu}} \varphi(|0\rangle_{\mu}; 1) q^{L_0 - \frac{c_v}{24}} \xi^{\frac{H}{2}} \quad \left(\mu = \frac{\lambda}{2}, \frac{\lambda + 1}{2}\right).$$

On the other hand, by substituting

$$a_0(q)\Theta(\xi, q)^{\lambda} \chi_0^{(1)}(\xi, q) + a_1(q)\Theta(\xi, q)^{\lambda} \chi_{\frac{1}{2}}^{(1)}(\xi, q)$$

for $\phi(\xi, q)$ in (F1), and using (F1) for $L = 1, \lambda = 0$, we find that the functions

$$\eta(q)^{-\frac{\lambda}{2\lambda+3}} \Theta(\xi, q)^{\lambda} \chi_{\nu}^{(1)}(\xi, q) \quad \left(\nu = 0, \frac{1}{2}\right)$$

are solutions of the system. By comparing the exponents of q, we conclude

$$\begin{aligned} \phi_{\frac{\lambda}{2}}(\xi, q) &= \eta(q)^{-\frac{\lambda}{2\lambda+3}} \Theta(\xi, q)^{\lambda} \chi_0^{(1)}(\xi, q), \\ \phi_{\frac{\lambda+1}{2}}(\xi, q) &= \eta(q)^{-\frac{\lambda}{2\lambda+3}} \Theta(\xi, q)^{\lambda} \chi_{\frac{1}{2}}^{(1)}(\xi, q). \end{aligned} \tag{4.3.2}$$

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