数理解析研究所講究録
919 巻 1995 年 27-43

Separation of variables in the A_2 type Jack polynomials

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Abstract

An integral operator M is constructed performing a separation of variables for the 3-particle quantum Calogero-Sutherland (CS) model. Under the action of M the CS eigenfunctions (Jack polynomials for the root system A_2) are transformed to the factorized form $\varphi(y_1)\varphi(y_2)$, where $\varphi(y)$ is a trigonometric polynomial of one variable expressed in terms of the ${}_3F_2$ hypergeometric series. The inversion of M produces a new integral representation for the A_2 Jack polynomials.

1 Quantum Calogero-Sutherland model

Define N differential operators $\{H_k\}_{k=1}^N$, acting on functions of N variables $\vec{q} = \{q_1, \ldots, q_N\}$ and depending on a parameter g, by the formula [1]

$$H_k = \sum_{0\le l\le \frac{k}{2}} \sum_{\sigma\in\mathfrak{S}_N} \frac{1}{\#G(l,k-2l)} D^{\sigma}_{l,k-2l} \tag{1}$$

where

$$D_{m,n} = u(q_1-q_2)u(q_3-q_4)\ldots u(q_{2m-1}-q_{2m}) \frac{(-i)^n\partial^n}{\partial q_{2m+1}\partial q_{2m+2}\ldots\partial q_{2m+n}}. \tag{2}$$

Here we denote $u(q) = -g(g-1)/\sin^2 q$, whereas $\mathfrak{S}_N$ is the permutation group of the set $\{1,\ldots,N\}$, and $G(m,n) = \{\sigma\in\mathfrak{S}_N | D^{\sigma}_{m,n} = D_{m,n}\}$.

Note that, when $g\to 0$, the operators H_k behave as

$$H_k = (-i)^k \sum_{j_1<\cdots<j_k} \frac{\partial^k}{\partial q_{j_1}\ldots\partial q_{j_k}} + \mathcal{O}(g), \tag{3}$$

providing thus a one-parameter deformation of the elementary symmetric polynomials in $\partial/\partial q_j$.

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It is known [1] that the operators H_k generate a commutative ring which contains, in particular, the quantum Calogero-Sutherland [2, 3, 4, 5] Hamiltonian

$$H = \frac{1}{2}H_1^2 - H_2 = -\frac{1}{2}\sum_{j=1}^{N}\frac{\partial^2}{\partial q_j^2} + \sum_{j_1<j_2}\frac{g(g-1)}{\sin^2(q_{j_1} - q_{j_2})}. \tag{4}$$

To describe the quantum problem more precisely, define the space of quantum states $\mathcal{H}^{(N)}$ as the complex Hilbert space of functions Ψ on the torus $T^{(N)} = \mathbb{R}^N/\pi\mathbb{Z}^N \ni \vec{q}$ which are symmetric w.r.t. the permutations of q_j, the scalar product being defined as

$$\langle\Psi, \Phi\rangle = \int_0^\pi dq_1 \ldots \int_0^\pi dq_N\, \bar{\Psi}(\vec{q})\Phi(\vec{q}). \tag{5}$$

Note that for the real g the operators (1) are formally Hermitian w.r.t. the above sesquilinear form. Let the vacuum (ground state) function Ω be defined as

$$\Omega(\vec{q}) = \left|\prod_{j<k}\sin(q_j - q_k)\right|^g. \tag{6}$$

Though $\Omega \in \mathcal{H}^{(N)}$ for $g > -\frac{1}{2}$, we shall assume a more strong condition $g > 0$ which simplifies description of the eigenvectors. Let $\mathcal{T}^{(N)}$ be the space of symmetric trigonometric polynomials in variables $\vec{q}$, that is the symmetric Laurent polynomials in variables $t_j = e^{2iq_j}$. The simplest way to fix the "boundary conditions" for the operators H_k is to restrict them first on the dense linear subset $\mathcal{D}_g^{(N)} = \Omega\mathcal{T}^{(N)} \subset \mathcal{H}^{(N)}$. Since $\mathcal{D}_g^{(N)}$ consists of common analytical vectors of operators H_k, the latter can be extended uniquely to commuting self-adjoint operators in $\mathcal{H}^{(N)}$.

The complete set of orthogonal eigenvectors to the self-adjoint H_k

$$H_k\Psi_{\vec{n}} = h_k\Psi_{\vec{n}} \tag{7}$$

is well known [3, 5]. The eigenvectors are parametrized by the sequences $\vec{n} = \{n_1 \leq n_2 \leq \ldots \leq n_N\}$ of integers $n_j \in \mathbb{Z}$. The corresponding eigenvalues h_k are

$$h_k = 2^k \sum_{j_1<\cdots<j_k} m_{j_1}\ldots m_{j_k}, \qquad m_j = n_j + g\left(j - \frac{N+1}{2}\right). \tag{8}$$

The eigenfunctions allow the factorization

$$\Psi_{\vec{n}}(\vec{q}) = \Omega(\vec{q})J_{\vec{n}}(\vec{q}), \qquad J_{\vec{n}} \in \mathcal{T}^{(N)}. \tag{9}$$

In particular, for the ground state $\Omega = \Psi_{0\ldots0}$ and $J_{0\ldots0} = 1$. The symmetric trigonometric polynomials $J_{\vec{n}}$ are known as Jack polynomials corresponding to the root system A_{N-1} or simple Lie algebra sl_N, see [6] and also [7] for the A_2 case. Our notation differs slightly from the conventional one: our parameter g relates to α used in [6] as $g = \alpha^{-1}$, and we do not impose the restriction $n_j \geq 0$.

The problem of finding square integrable eigenfunctions $\Psi \in \mathcal{H}^{(N)}$ of the operators H_k turns out thus to be equivalent to the purely algebraic problem of finding the polynomial eigenfunctions $J \in \mathcal{T}^{(N)}$ of the differential operators $\widetilde{H}_k$ obtained by conjugation of H_k with Ω

$$\widetilde{H}_k = \Omega^{-1} H_k \Omega. \tag{10}$$

Jack polynomials can be considered as a one-parametric deformation of elementary symmetric polynomials $S_{\vec{n}}(\vec{q}) = \sum t_1^{\nu_1} \dots t_N^{\nu_N}$ where the sum is taken over all distinct permutations $\vec{\nu}$ of $\vec{n}$, such that

$$J_{\vec{n}} = S_{\vec{n}} + \sum_{\vec{n}' \preceq \vec{n}} \kappa_{\vec{n}\vec{n}'} S_{\vec{n}'}, \tag{11}$$

where $\kappa_{\vec{n}\vec{n}'}$ is a rational function in g vanishing for $g = 0$, and the dominant order for sequences $\vec{n}$ is defined as

$$\vec{n} \succeq \vec{n}' \quad \Longleftrightarrow \quad \left\{ \sum_{j=1}^{N} n_j = \sum_{j=1}^{N} n'_j; \quad \sum_{j=k}^{N} n_j \geq \sum_{j=k}^{N} n'_j, \quad k = 2, \dots, N \right\} \tag{12}$$

Another important property of Jack polynomials is the orthogonality with the weight Ω^2,

$$\int_0^{\pi} dq_1 \dots \int_0^{\pi} dq_N \, \bar{J}_{\vec{n}}(\vec{q}) J_{\vec{n}'}(\vec{q}) \Omega^2(\vec{q}) = 0, \quad \vec{n} \neq \vec{n}' \tag{13}$$

For the generalization of Jack polynomials for other root systems see [8].

2 Separation of variables: conjectures

In the classical case, when the differentiation $-i\partial/\partial q_j$ is replaced by the momentum p_j canonically conjugated to q_j, the Calogero-Sutherland system is completely integrable in the Liouville's sense [2, 4]. It is thus natural to speak of its quantum version described above as a quantum integrable system. The common property to be expected from an integrable system (classical or quantum one) is the *separability of variables* [9, 10, 11, 12] which suggests the following conjecture.

Conjecture 1. *There exists a linear operator*

$$K : \Psi_{\vec{n}}(\vec{q}) \longmapsto \widetilde{\Psi}_{\vec{n}}(y_1, \dots, y_{N-1}; Q) \tag{14}$$

such that any eigenfunction $\Psi_{\vec{n}}$ is transformed into the factorized function

$$\widetilde{\Psi}_{\vec{n}}(y_1, \dots, y_{N-1}; Q) = e^{ih_1 Q} \prod_{k=1}^{N-1} \psi_{\vec{n}}(y_k). \tag{15}$$

The distinguished variable $Q \equiv q_N$ is simply the coordinate canonically conjugated to the total momentum H_1.

The study of the low-dimensional cases $N = 2, 3$ allows to formulate an even more detailed conjecture about the structure of the separated eigenfunction $\widetilde{\Psi}$.

Conjecture 2. *The factor $\psi_{\vec{n}}(y)$ in (15) allows further factorization*

$$\psi_{\vec{n}}(y) = (\sin y)^{(N-1)g} \varphi_{\vec{n}}(y) \tag{16}$$

where $\varphi_{\vec{n}}(y)$ *is a Laurent polynomial in* $t = e^{2iy}$

$$\varphi_{\vec{n}}(y) = \sum_{k=n_1}^{n_N} t^k c_k(\vec{n}; g). \tag{17}$$

The coefficients $c_k(\vec{n}; g)$ *are rational functions of* k, n_j *and* g. *Moreover,* $\varphi_{\vec{n}}(y)$ *can be expressed explicitely in terms of the hypergeometric function* ${}_NF_{N-1}$ *as*

$$\varphi_{\vec{n}}(y) = t^{n_1}(1-t)^{1-Ng}{}_NF_{N-1}(a_1, \ldots, a_N; b_1, \ldots, b_{N-1}; t) \tag{18}$$

where

$$a_j = n_1 - n_{N-j+1} + 1 - (N-j+1)g, \qquad b_j = a_j + g, \tag{19}$$

$${}_NF_{N-1}(a_1, \ldots, a_N; b_1, \ldots, b_{N-1}; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_N)_k t^k}{(b_1)_k \ldots (b_{N-1})_k k!}, \tag{20}$$

and $(a)_k$ *is the standard Pochhammer symbol:*

$$(a)_0 = 1, \qquad (a)_k = a(a+1)\ldots(a+k-1) = \frac{\Gamma(a+k)}{\Gamma(a)}. \tag{21}$$

The conjectures 1 and 2 are proved in the next section for the $N = 2$ case and in the sections 4 and 5 for the $N = 3$ case. Section 5 contains also a more detailed discussion of the conjecture 2 for $N > 3$, see theorem 3. Further support to the conjectures is given by the study of the case $g = 1$ when Jack polynomials degenerate into Schur functions (section 7).

3 A_1 case

It is a well known fact that in the A_1 case Jack polynomials are reduced to hypergeometric polynomials of one variable [8]. Nevertheless, we review the derivation briefly in order to prepare the stage for the discussion of the A_2 case.

For $N = 2$ the commuting operators (1) are

$$H_1 = -i(\partial_1 + \partial_2), \qquad H_2 = -\partial_1\partial_2 - g(g-1)\sin^{-2} q_{12}. \tag{22}$$

(we denote $\partial_j = \partial/\partial q_j$ and $q_{jk} = q_j - q_k$). Respectively,

$$\widetilde{H}_1 = -i(\partial_1 + \partial_2), \qquad \widetilde{H}_2 = -\partial_1\partial_2 + g\cot q_{12}(\partial_1 - \partial_2) - g^2,$$

the vacuum vector being

$$\Omega(\vec{q}) = |\sin q_{12}|^g. \tag{23}$$

The eigenvectors $\Psi_{\vec{n}}$, resp. $J_{\vec{n}}$, according to (8), are parametrized by the pairs of integers $\vec{n} = \{n_1 \le n_2\}$, the corresponding eigenvalues being

$$h_1 = 2(m_1 + m_2) = 2(n_1 + n_2), \qquad h_2 = 4m_1m_2 = (2n_1 - g)(2n_2 + g) \tag{24}$$

where

$$m_1 = n_1 - \frac{g}{2}, \qquad m_2 = n_2 + \frac{g}{2}. \tag{25}$$

The separation of variables is given by the simple change of coordinates

$$K : \Psi(q_1, q_2) \longmapsto \tilde{\Psi}(y, Q) = \Psi(y + Q, Q). \tag{26}$$

Actually, the calculations would be simpler for the more symmetric definition $Q = (q_1 + q_2)/2$ rather than $Q = q_2$ but we wish to preserve here the coherence of notation for the study of $N = 3$ case.

The spectral problem $H_k\Psi = h_k\Psi$ rewritten in terms of the function $\tilde{\Psi}$ reads

$$[\partial_Q - ih_1]\,\tilde{\Psi} = 0, \qquad \left[\partial_y^2 - \partial_y\partial_Q - \frac{g(g-1)}{\sin^2 y} - h_2\right]\tilde{\Psi} = 0, \tag{27}$$

allowing immediate separation of variables of the form (15)

$$\tilde{\Psi}(y, Q) = e^{ih_1 Q}\psi(y), \tag{28}$$

the function ψ satisfying the second order differential equation

$$\left[\partial_y^2 - ih_1\partial_y - \left(h_2 + \frac{g(g-1)}{\sin^2 y}\right)\right]\psi = 0 \tag{29}$$

which, via the transformation $\psi(y) = \sin^g y\,\varphi(y)$, can be rewritten as

$$\left[\partial_y^2 + (2g\cot y - ih_1)\partial_y - (g^2 + igh_1\cot y + h_2)\right]\varphi = 0. \tag{30}$$

The last equation, after the substitution $t = e^{2iy}$, is reduced to the standard Fuchsian form

$$\left[\partial_t^2 + \left(-\frac{g - 1 + \frac{1}{2}h_1}{t} + \frac{2g}{t-1}\right)\partial_t + \left(\frac{\frac{1}{4}(g^2 + gh_1 + h_2)}{t^2} - \frac{\frac{1}{2}gh_1}{t(t-1)}\right)\right]\varphi = 0. \tag{31}$$

The equation (31) has 3 regular singularities: $\{0, 1, \infty\}$ with the characteristic exponents:

$$\begin{array}{lll} t \sim 1 & \varphi \sim (t-1)^\mu & \mu \in \{-2g+1, 0\} \\ t \sim 0 & \varphi \sim t^\rho & \rho \in \{n_1, n_2 + g\} \\ t \sim \infty & \varphi \sim t^{-\sigma} & -\sigma \in \{n_1 - g, n_2\} \end{array}$$

Moreover, by the substitution $\varphi(t) = t^{n_1}(1-t)^{1-2g}f(t)$ the equation (31) is reduced to the standard hypergeometric equation

$$[t\partial_t(t\partial_t + b_1 - 1) - t(t\partial_t + a_1)(t\partial_t + a_2)]\,f = 0, \tag{32}$$

the parameters a_1, a_2, b_1 being given by the formulas (19) which for $N = 2$ read

$$a_1 = n_1 - n_2 + 1 - 2g, \qquad a_2 = 1 - g, \qquad b_1 = n_1 - n_2 + 1 - g. \tag{33}$$

¿From $J_{n_1n_2} \in \mathcal{T}^{(2)}$ it follows immediately that the corresponding $\varphi_{n_1n_2}(t)$ is a Laurent polynomial in t.

Proposition 1 *The Laurent polynomial $\varphi_{n_1 n_2}(t)$ is given, up to a constant factor, by the formula (18) which, for $N=2$ takes the form*

$$\varphi_{n_1 n_2}(t) = t^{n_1}(1-t)^{1-2g}\,{}_2F_1(a_1, a_2; b_1; t) \tag{34}$$

the parameters a_1, a_2, b_1 being given by (33).

Proof. Define the function $F_{n_1 n_2}(t)$ by the right hand side of the formula (34). Strictly speaking, the hypergeometric series converges only for $|t| < 1$ but in few moments we shall see that $F_{n_1 n_2}(t)$ continues analytically to the whole complex plane. Using the well known formula

$$(1-t)^{a+b-c}\,{}_2F_1(a, b; c; t) = {}_2F_1(c-a, c-b; c; t)$$

we can rewrite $F_{n_1 n_2}(t)$ as folllows

$$F_{n_1 n_2}(t) = t^{n_1}\,{}_2F_1(n_1 - n_2, g; n_1 - n_2 + 1 - g; t)$$

It is easy to observe now that the hypergeometric series in the right hand side terminates for integer $\{n_1 \leq n_2\}$ and $F_{n_1 n_2}$ is thus a Laurent polynomial

$$F_{n_1 n_2} = \sum_{k=n_1}^{n_2} t^k c_k(\vec{n}; g),$$

of the form (17). Since $F_{n_1 n_2}$ satisfies the same differential equation (31) as $\varphi_{n_1 n_2}$ and the linearly independent solution to (31) is obviously not polynomial, the functions $F_{n_1 n_2}(t)$ and $\varphi_{n_1 n_2}(t)$ are identical up to a constant factor, which finishes the proof of the proposition and of the conjectures 1 and 2 for $N = 2$. ∎

4 A_2 case: Integral transform

For $N = 3$ the commuting differential operators (1) read

$$\begin{aligned}
H_1 &= -i(\partial_1 + \partial_2 + \partial_3), \\
H_2 &= -(\partial_1\partial_2 + \partial_1\partial_3 + \partial_2\partial_3) - g(g-1)\left(\sin^{-2} q_{12} + \sin^{-2} q_{13} + \sin^{-2} q_{23}\right), \\
H_3 &= i\partial_1\partial_2\partial_3 + ig(g-1)\left(\sin^{-2} q_{23}\,\partial_1 + \sin^{-2} q_{13}\,\partial_2 + \sin^{-2} q_{12}\,\partial_3\right),
\end{aligned}$$

and, respectively,

$$\begin{aligned}
\widetilde{H}_1 &= -i(\partial_1 + \partial_2 + \partial_3) \\
\widetilde{H}_2 &= -(\partial_1\partial_2 + \partial_1\partial_3 + \partial_2\partial_3) \\
&\quad g[\cot q_{12}(\partial_1 - \partial_2) + \cot q_{13}(\partial_1 - \partial_3) + \cot q_{23}(\partial_2 - \partial_3)] \\
&\quad -4g^2 \\
\widetilde{H}_3 &= i\partial_1\partial_2\partial_3 \\
&\quad -ig[\cot q_{12}(\partial_1 - \partial_2)\partial_3 + \cot q_{13}(\partial_1 - \partial_3)\partial_2 + \cot q_{23}(\partial_2 - \partial_3)\partial_1] \\
&\quad +2ig^2[(1 + \cot q_{12}\cot q_{13})\partial_1 + (1 - \cot q_{12}\cot q_{23})\partial_2 + (1 + \cot q_{13}\cot q_{23})\partial_3]
\end{aligned}$$

the vacuum function being

$$\Omega(\vec{q}) = |\sin q_{12} \sin q_{13} \sin q_{23}|^g . \tag{35}$$

The eigenvectors $\Psi_{\vec{n}}$, resp. $J_{\vec{n}}$, according to (8), are parametrized by the triplets of integers $\{n_1 \le n_2 \le n_3\} \in \mathbb{Z}^3$, the corresponding eigenvalues being

$$h_1 = 2(m_1 + m_2 + m_3), \quad h_2 = 4(m_1 m_2 + m_1 m_3 + m_2 m_3), \quad h_3 = 8 m_1 m_2 m_3, \tag{36}$$

where,

$$m_1 = n_1 - g, \qquad m_2 = n_2, \qquad m_3 = n_3 + g. \tag{37}$$

The structure of the operator K performing separation of variables in the A_2 case is more complicated than in the A_1 case. In contrast with the A_1 case, K is given by an integral operator rather then by simple change of coordinates. To describe K, let us introduce the following notation.

$$x_1 = q_1 - q_3, \qquad x_2 = q_2 - q_3, \qquad Q = q_3,$$

$$x_\pm = x_1 \pm x_2, \qquad y_\pm = y_1 \pm y_2.$$

We shall study the action of K locally, assuming that $q_1 > q_2 > q_3$ and hence $x_+ > x_-$.

The operator $K : \Psi(q_1, q_2, q_3) \mapsto \tilde{\Psi}(y_1, y_2; Q)$ is defined as an integral operator

$$\tilde{\Psi}(y_1, y_2; Q) = \int_{y_-}^{y_+} d\xi \, \mathcal{K}(y_1, y_2; \xi) \Psi \left(\frac{y_+ + \xi}{2} + Q, \frac{y_+ - \xi}{2} + Q, Q \right) \tag{38}$$

with the kernel

$$\mathcal{K} = \kappa \left[\frac{\sin\left(\frac{\xi + y_-}{2}\right) \sin\left(\frac{\xi - y_-}{2}\right) \sin\left(\frac{y_+ + \xi}{2}\right) \sin\left(\frac{y_+ - \xi}{2}\right)}{\sin y_1 \sin y_2 \sin \xi} \right]^{g-1} \tag{39}$$

where κ is a normalization coefficient to be fixed later. It is assumed in (38) and (39) that $y_- < x_- = \xi < y_+ = x_+$. The integral converges when $g > 0$ which will always be assumed henceforth.

The motivation for such a choice of K takes its origin from considering the problem in the classical limit ($g \to \infty$) where there exists effective prescription for constructing a separation of variables for an integrable system from the poles of the so-called Baker-Akhiezer function. See [12], §7, for a detailed explanation.

Theorem 1 *Let $H_k \Psi_{n_1 n_2 n_3} = h_k \Psi_{n_1 n_2 n_3}$. Then the function $\tilde{\Psi}_{\vec{n}} = K \Psi_{\vec{n}}$ satisfies the differential equations*

$$\mathcal{Q} \tilde{\Psi}_{\vec{n}} = 0, \qquad \mathcal{Y}_j \tilde{\Psi}_{\vec{n}} = 0, \quad j = 1, 2 \tag{40}$$

where

$$\mathcal{Q} = -i\partial_Q - h_1, \tag{41}$$

$$\mathcal{Y}_j = i\partial^3_{y_j} + h_1\partial^2_{y_j} - i\left(h_2 + 3\frac{g(g-1)}{\sin^2 y_j}\right)\partial_{y_j}$$
$$- \left(h_3 + \frac{g(g-1)}{\sin^2 y_j}h_1 + 2ig(g-1)(g-2)\frac{\cos y_j}{\sin^3 y_j}\right). \tag{42}$$

The proof is based on the following proposition.

Proposition 2 *The kernel K satisfies the differential equations*

$$[-i\partial_Q - H_1^*]K = 0,$$

$$\left[i\partial^3_{y_j} + H_1^*\partial^2_{y_j} - i\left(H_2^* + \frac{3g(g-1)}{\sin^2 y_j}\right)\partial_{y_j}\right.$$
$$\left. - \left(H_3^* + H_1^*\frac{g(g-1)}{\sin^2 y_j} + 2ig(g-1)(g-2)\frac{\cos y_j}{\sin^3 y_j}\right)\right] K = 0,$$

where H_n^* is the Lagrange adjoint of H_n

$$\int \varphi(q)(H\psi)(q)\, dq = \int (H^*\varphi)(q)\psi(q)\, dq$$

$$\begin{aligned} H_1^* &= i(\partial_{q_1} + \partial_{q_2} + \partial_{q_3}), \\ H_2^* &= -\partial_{q_1}\partial_{q_2} - \partial_{q_1}\partial_{q_3} - \partial_{q_2}\partial_{q_3} - g(g-1)[\sin^{-2} q_{12} + \sin^{-2} q_{13} + \sin^{-2} q_{23}], \\ H_3^* &= -i\partial_{q_1}\partial_{q_2}\partial_{q_3} - ig(g-1)[\sin^{-2} q_{23}\,\partial_{q_1} + \sin^{-2} q_{13}\,\partial_{q_2} + \sin^{-2} q_{12}\partial_{q_3}]. \end{aligned}$$

The proof is given by a direct, though tedious calculation.

To complete the proof of the theorem 1, consider the expressions $\mathcal{Q}K\Psi_{\vec{n}}$ and $\mathcal{Y}_j K\Psi_{\vec{n}}$ using the formulas (38) and (39) for K. The idea is to use the fact that $\Psi_{\vec{n}}$ is an eigenfunction of H_k and replace $h_k\Psi_{\vec{n}}$ by $H_k\Psi_{\vec{n}}$. After integration by parts in the variable ξ the operators H_k are replaced by their adjoints H_k^* and the result is zero by virtue of proposition 2.

The caution is needed however when handling the limits of integration $y_\pm$ in (38). The following argument allows to circumvent the problem of boundary terms. One can hide the limits of integration into the definition of the kernel $\mathcal{K}$ considering the factors containing $(\xi - y_\pm)$ as the generalized functions similar to x_+^λ, see [13]. It is known that x_+^λ defined through the linear functional

$$\left\langle f, x_+^\lambda \right\rangle = \int_0^\infty dx\, f(x) x_+^\lambda$$

is analytic in λ on the complex plane excluding the poles $x = -1, -2, \ldots$ and can be differentiated just as usual power function $\partial_x x_+^\lambda = \lambda x_+^{\lambda-1}$. Therefore, we can safely ignore the boundary of integral (38) while integrating by parts. The only possible obstacle may present the integer points $g = 1, 2, 3$ (no more, since we need to differentiate $\mathcal{K}$ maximum 3 times) where the boundary may contribute delta-function terms. The direct calculation shows, however, that all such terms cancel.
■

The following theorem validates the conjectures 1 and 2 for the A_2 case.

Theorem 2 *The function $\widetilde{\Psi}_{n_1n_2n_3}$ is factorized*

$$\widetilde{\Psi}_{n_1n_2n_3}(y_1,y_2;Q) = e^{ih_1Q}\psi_{n_1n_2n_3}(y_1)\psi_{n_1n_2n_3}(y_2) \tag{43}$$

according to (15). The separated function $\psi_{n_1n_2n_3}(y_2)$ has the structure (16).

Note that, by virtue of the theorem 1, the function $\widetilde{\Psi}_{\vec{n}}(y_1,y_2;Q)$ satisfies an ordinary differential equation in each variable. Since $\mathcal{Q}f=0$ is a first order differential equation having a unique, up to a constant factor, solution $f(Q)=e^{ih_1Q}$, the dependence on Q is factorized. However, the differential equations $\mathcal{Y}_j\psi(y_j)=0$ are of third order and have three linearly independent solutions. To prove the theorem 2 one needs thus to study the ordinary differential equation

$$\begin{aligned}\Bigg[i\partial_y^3 + h_1\partial_y^2 - i\left(h_2 + 3\frac{g(g-1)}{\sin^2 y}\right)\partial_y \\ - \left(h_3 + \frac{g(g-1)}{\sin^2 y}h_1 + 2ig(g-1)(g-2)\frac{\cos y}{\sin^3 y}\right)\Bigg]\psi = 0.\end{aligned} \tag{44}$$

and to select its special solution corresponding to $\widetilde{\Psi}$.

The proof will take several steps. First, let us eliminate from Ψ and $\widetilde{\Psi}$ the vacuum factors Ω, see (9), and, respectively

$$\widetilde{\Psi}(y_1,y_2;Q) = \omega(y_1)\omega(y_2)\tilde{J}(y_1,y_2;Q), \qquad \omega(y) = \sin^{2g} y. \tag{45}$$

Conjugating the operator K with the vacuum factors

$$M = \omega_1^{-1}\omega_2^{-1}K\Omega : J \mapsto \tilde{J} \tag{46}$$

we obtain the integral operator

$$\tilde{J}(y_1,y_2;Q) = \int_{y_-}^{y_+} d\xi\, \mathcal{M}(y_1,y_2;\xi) J\left(\frac{y_+ + \xi}{2} + Q, \frac{y_+ - \xi}{2} + Q, Q\right) \tag{47}$$

with the kernel

$$\begin{aligned}\mathcal{M}(y_1,y_2;\xi) &= \mathcal{K}(y_1,y_2;\xi)\frac{\Omega\left(\frac{y_+ + \xi}{2} + Q, \frac{y_+ - \xi}{2} + Q, Q\right)}{\omega(y_1)\omega(y_2)} \\ &= \kappa \sin\xi \frac{\left[\sin\left(\frac{\xi + y_-}{2}\right)\sin\left(\frac{\xi - y_-}{2}\right)\right]^{g-1}\left[\sin\left(\frac{y_+ + \xi}{2}\right)\sin\left(\frac{y_+ - \xi}{2}\right)\right]^{2g-1}}{[\sin y_1 \sin y_2]^{3g-1}}.\end{aligned} \tag{48}$$

Proposition 3 *Let S be a trigonometric polynomial in q_j, i.e. Laurent polynomial in $t_j = e^{2iq_j}$, which is symmetric w.r.t. the transpositon $q_1 \leftrightarrow q_2$. Then $\widetilde{S} = MS$ is a trigonometric polynomial symmetric w.r.t. $y_1 \leftrightarrow y_2$.*

Proof. It is more convenient to use variables $x_\pm$, Q and, respectively, $y_\pm$, Q. Since the kernel $\mathcal{M}$ does not depend on Q it is safe to omit the dependence on Q

in S. The polynomiality and symmetry of S are expressed now as $S = S(x_+, x_-) = \sum_{k,n} s_{kn} e^{ikx_+} \cos nx_-$ where k, n are integers of the same parity, and $n \geq 0$.

¿From (47), (48) we obtain

$$\tilde{S}(y_+, y_-) = \kappa \left(\sin^2 \frac{y_+}{2} - \sin^2 \frac{y_-}{2}\right)^{-3g+1} \times$$
$$\times \int_{y_-}^{y_+} dx_-\ \sin x_- \left(\sin^2 \frac{x_-}{2} - \sin^2 \frac{y_-}{2}\right)^{g-1} \left(\sin^2 \frac{y_+}{2} - \sin^2 \frac{x_-}{2}\right)^{2g-1} S(y_+, x_-).$$

Let us make now the change of variables

$$\xi_\pm = \sin^2 \frac{x_\pm}{2}, \quad d\xi_\pm = \frac{1}{2} \sin x_\pm\, dx_\pm, \qquad \eta_\pm = \sin^2 \frac{y_\pm}{2}, \tag{49}$$

denoting $\check{S}(x_+, \xi_-) = S(x_+, x_-)$. It is easy to see that $\check{S}(x_+, \xi_-)$ is polynomial in ξ_- and that

$$\tilde{S}(y_+, y_-) = 2\kappa(\eta_+ - \eta_-)^{-3g+1} \int_{\eta_-}^{\eta_+} d\xi_-\ (\xi_- - \eta_-)^{g-1} (\eta_+ - \xi_-)^{2g-1} \check{S}(y_+, \xi_-). \tag{50}$$

Now put

$$\xi_- = (\eta_+ - \eta_-)\xi + \eta_-$$

and choose

$$\kappa = \frac{1}{2B(g, 2g)} = \frac{\Gamma(3g)}{2\Gamma(g)\Gamma(2g)}. \tag{51}$$

Then, finally

$$\tilde{S}(y_+, y_-) = \frac{\Gamma(3g)}{\Gamma(g)\Gamma(2g)} \int_0^1 d\xi\, \xi^{g-1} (1 - \xi)^{2g-1} \check{S}(y_+, (\eta_+ - \eta_-)\xi + \eta_-). \tag{52}$$

It is sufficient to calculate the integral (52) for the monomials

$$\check{S} = e^{iky_+} \eta_-^l (\eta_+ - \eta_-)^m \xi^m$$

such that $k, l, m \in \mathbb{Z}$, $l, m \geq 0$ and $k \equiv l + m \pmod 2$. Evaluating the beta-function integral

$$\int_0^1 d\xi\, \xi^{g-1+m} (1 - \xi)^{2g-1} = \frac{\Gamma(g + m)\Gamma(2g)}{\Gamma(3g + m)}$$

one obtains

$$\tilde{S}(y_+, y_-) = \frac{\Gamma(3g)\Gamma(g + m)}{\Gamma(3g + m)\Gamma(g)} e^{iky_+} \eta_-^l (\eta_+ - \eta_-)^m. \tag{53}$$

It is easy to verify that the result is a symmetric trigonometric polynomial in y_1, y_2. ■

Note that the normalization constant κ is chosen in such a way that $M : 1 \mapsto 1$.

The formula (53) shows that the operator M can in fact be continued analytically in g on the whole complex plane excluding the points $g = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots$ coming from the poles of the gamma functions in (53) and also $g = -\frac{1}{3}, -\frac{2}{3}, \ldots$ coming from the poles of $\Gamma(3g)$ in the normalization constant κ (51).

5 A_2: Separated equation

To complete the proof of the theorem 2 we need to learn more about the separated equation (44).

Eliminating from ψ the vacuum factor $\omega(y) = \sin^{2g} y$ via the substitution $\psi(y) = \varphi(y)\omega(y)$ one obtains

$$\Big[i\partial_y^3 + (h_1 + 6ig\cot y)\partial_y^2 + (-i(h_2 + 12g^2) + 4gh_1 \cot y + 3ig(3g-1)\sin^{-2} y)\partial_y + (-(h_3 + 4g^2h_1) - 2ig(h_2 + 4g^2)\cot y + g(3g-1)h_1 \sin^{-2} y)\Big]\varphi = 0. \quad (54)$$

The change of variable $t = e^{2iy}$ brings the last equation to the Fuchsian form:

$$\left[\partial_t^3 + w_1\partial_t^2 + w_2\partial_t + w_3\right]\varphi = 0 \quad (55)$$

where

$$\begin{aligned} w_1 &= -\frac{3(g-1) + \frac{1}{2}h_1}{t} + \frac{6g}{t-1}, \\ w_2 &= \frac{(3g^2 - 3g + 1) + \frac{1}{2}(2g-1)h_1 + \frac{1}{4}h_2}{t^2} + \frac{3g(3g-1)}{(t-1)^2} - \frac{g(9(g-1) + 2h_1)}{t(t-1)}, \\ w_3 &= -\frac{g^3 + \frac{1}{2}g^2h_1 + \frac{1}{4}gh_2 + \frac{1}{8}h_3}{t^3} + \frac{\frac{1}{2}g((h_2 + 4g^2)(t-1) - (3g-1)h_1)}{t^2(t-1)^2}. \end{aligned}$$

The points $t = 0, 1, \infty$ are regular singularities with the exponents

$$\begin{array}{lll} t \sim 1 & \varphi \sim (t-1)^\mu & \mu \in \{-3g+2, -3g+1, 0\} \\ t \sim 0 & \varphi \sim t^\rho & \rho \in \{n_1, n_2 + g, n_3 + 2g\} \\ t \sim \infty & \varphi \sim t^{-\sigma} & -\sigma \in \{n_1 - 2g, n_2 - g, n_3\} \end{array}$$

Like in the A_2 case, the equation (55) is reduced by the substitution $\varphi(t) = t^{n_1}(1-t)^{1-3g}f(t)$ to the standard ${}_3F_2$ hypergeometric form [14]

$$[t\partial_t(t\partial_t + b_1 - 1)(t\partial_t + b_2 - 1) - t(t\partial_t + a_1)(t\partial_t + a_2)(t\partial_t + a_3)]\, f = 0, \quad (56)$$

the parameters a_1, a_2, a_3, b_1, b_2 being given by the formulas (20) which for $N = 3$ read

$$a_1 = n_1 - n_3 + 1 - 3g, \qquad a_2 = n_1 - n_2 + 1 - 2g, \qquad a_3 = 1 - g,$$

$$b_1 = n_1 - n_3 + 1 - 2g, \qquad b_2 = n_1 - n_2 + 1 - g.$$

Proposition 4 *Let the parameters h_k be given by (36), (37) for a triplet of integers $\{n_1 \le n_2 \le n_3\}$ and $g \ne 1, 0, -1, -2, \ldots$. Then the equation (55) has a unique, up to a constant factor, Laurent-polynomial solution*

$$\varphi(t) = \sum_{k=n_1}^{n_3} t^k c_k(\vec{n}; g), \quad (57)$$

the coefficients $c_k(\vec{n}; g)$ being rational functions of k, n_j and g.

The above proposition follows from a more general statement.

Theorem 3 *Let the function $F_{n_1,\dots,n_N}(t)$ be given for $|t| < 1$ by the right hand side of the formula (18), the parameters a_j and b_j being given by (19) for some sequence of integers $\vec{n} = \{n_1 \le n_2 \le \dots \le n_N\}$. Let $g \ne 1, 0, -1, -2, \dots$. Then $F_{\vec{n}}(t)$ is a Laurent polynomial*

$$F_{\vec{n}}(t) = \sum_{k=n_1}^{n_N} t^k c_k(\vec{n}; g), \tag{58}$$

the coefficients $c_k(\vec{n}; g)$ being rational functions of k, n_j and g.

Proof. Consider first the hypergeometric series (20) for ${}_NF_{N-1}$ which converges for $|t| < 1$. Using for a_j and b_j the expressions (19) one notes that $a_{j+1} = b_j + n_{N-j+1} - n_{N-j}$ and therefore

$$\frac{(a_{j+1})_k}{(b_j)_k} = \frac{(b_j + k)_{n_{N-j+1}-n_{N-j}}}{(b_j)_{n_{N-j+1}-n_{N-j}}}.$$

The expression

$$\frac{(a_2)_k \dots (a_N)_k}{(b_1)_k \dots (b_{N-1})_k} = \frac{(b_1 + k)_{n_N - n_{N-1}} \cdots (b_{N-1} + k)_{n_2 - n_1}}{(b_1)_{n_N - n_{N-1}} \cdots (b_{N-1})_{n_2 - n_1}} = P_{n_N - n_1}(k)$$

is thus a polynomial in k of degree $n_N - n_1$. So we have

$${}_NF_{N-1}(a_1, \dots, a_N; b_1, \dots, b_{N-1}; t) = \sum_{k=0}^{\infty} \frac{(a_1)_k}{k!} P_{n_N - n_1}(k)$$

from which it follows that

$${}_NF_{N-1}(a_1, \dots, a_N; b_1, \dots, b_{N-1}; t) = \widetilde{P}_{n_N - n_1}(t)(1 - t)^{Ng-1}$$

where $\widetilde{P}_{n_N - n_1}(t)$ is a polynomial of degree $n_N - n_1$ in t. ■

To prove now the proposition 4 it is sufficient to notice that in the case $N = 3$ the hypergeometric series ${}_3F_2(a_1, a_2, a_3; b_1, b_2; t)$ satisfies the same equation (56) as $f(t)$ and therefore the Laurent polynomial $F_{\vec{n}}(t)$ constructed above satisfies the equation (55). The uniqueness follows from the fact that all the linearly independent solutions to (55) are nonpolynomial which is seen from the characteristic exponents. ■

Now everything is ready to finish the proof of the theorem 2. Since the function $\widetilde{J}_{n_1 n_2 n_3}(y_1, y_2; Q)$ satisfies (54) in variables $y_{1,2}$ and is a Laurent polynomial it inevitably has the factorized form

$$\widetilde{J}_{n_1 n_2 n_3}(y_1, y_2; Q) = e^{ih_1 Q} \varphi_{n_1 n_2 n_3}(y_1) \varphi_{n_1 n_2 n_3}(y_2) \tag{59}$$

by virtue of the proposition 4. ■

6 Integral representation for Jack polynomials

The formula (59) presents an interesting opportunity to construct a new integral representation of the Jack polynomial $J_{\vec{n}}$ in terms of the ${}_3F_2$ hypergeometric polynomials $\varphi_{\vec{n}}(y)$ constructed above. To achieve this goal, it is necessary to invert explicitely the operator $M : J \mapsto \widetilde{J}$.

Let us examine again the integral (50). Assume that $x_+ = y_+$ and respectively $\xi_+ = \eta_+$ are fixed whereas ξ_-, y_- are variables. Then, denoting

$$\tilde{s}(\eta_-) = \frac{1}{2\kappa\Gamma(g)}\widetilde{S}(y_+, y_-)(\eta_+ - \eta_-)^{3g-1}, \quad s(\xi_-) = (\eta_+ - \xi_-)^{2g-1}\breve{S}(y_+, \xi_-)$$

we face the problem of inverting the integral transform

$$\tilde{s}(\eta_-) = \int_{\eta_-}^{\eta_+} d\xi_- \frac{(\xi_- - \eta_-)^{g-1}}{\Gamma(g)} s(\xi_-) \tag{60}$$

which is known as Riemann-Liouville integral of fractional order g [15]. Its inversion is formally given by changing sign of g

$$s(\xi_-) = \int_{\xi_-}^{\xi_+} d\eta_- \frac{(\eta_- - \xi_-)^{-g-1}}{\Gamma(-g)} \tilde{s}(\eta_-) \tag{61}$$

and is called fractional differentiation operator. However, by our assumption $g > 0$, so the integrand becomes singular at $\xi_- = \eta_-$ and the integral should be regularized in the standard way [13].

Retracing all the intermediate transformations we obtain

$$S(x_+, x_-) = \frac{\Gamma(2g)}{\Gamma(-g)\Gamma(3g)}(\xi_+ - \xi_-)^{-2g+1} \int_{\xi_-}^{\xi_+} d\eta_- \, (\eta_- - \xi_-)^{-g-1}(\xi_+ - \eta_-)^{3g-1}\widetilde{S}(x_+, y_-)$$

and finally come to the formula for $M^{-1} : \widetilde{J} \mapsto J$

$$J(x_+, x_-; Q) = \int_{x_-}^{x_+} dy_- \, \breve{\mathcal{M}}(x_+, x_-; y_-)\widetilde{J}(x_+, y_-; Q) \tag{62}$$

$$\breve{\mathcal{M}} = \breve{\kappa} \frac{\sin y_- \left[\sin\left(\frac{x_+ + y_-}{2}\right) \sin\left(\frac{x_+ - y_-}{2}\right)\right]^{3g-1}}{\left[\sin\left(\frac{y_- + x_-}{2}\right) \sin\left(\frac{y_- - x_-}{2}\right)\right]^{g+1} [\sin x_1 \sin x_2]^{2g-1}} \tag{63}$$

where

$$\breve{\kappa} = \frac{\Gamma(2g)}{2\Gamma(-g)\Gamma(3g)}. \tag{64}$$

For K^{-1} we have respectively

$$\breve{\mathcal{K}} = \breve{\kappa} \frac{\sin^g x_- \sin y_- \left[\sin\left(\frac{x_+ + y_-}{2}\right) \sin\left(\frac{x_+ - y_-}{2}\right)\right]^{g-1}}{\left[\sin\left(\frac{y_- + x_-}{2}\right) \sin\left(\frac{y_- - x_-}{2}\right)\right]^{g+1} [\sin x_1 \sin x_2]^{g-1}}. \tag{65}$$

The formulas (59), (62), (63) provide a new integral representation for Jack polynomial $J_{\vec{n}}$ in terms of the ${}_3F_2$ hypergeometric polynomials $\varphi_{\vec{n}}(y)$. The representation would acquire more satisfactory form if one could describe explicitely the normalization of φ corresponding to the standard normalization (11) of J. We intend to study this question in a subsequent paper.

It is remarkable that for positive integer g the operators K^{-1}, M^{-1} become differential operators of order g. In particular, for $g = 1$ we have $K^{-1} = \partial/\partial y_-$.

7 Separation of variables in the Schur polynomials

For the generic g the separation of variables in Jack polynomials is so far unknown for $N > 3$. However, the problem simplifies drastically in the case $g = 1$, when Jack polynomials are reduced to the Schur polynomials [6], and allows quite simple solution. In the present section we have changed notation to make it more convenient for handling Schur polynomials.

Let

$$P_{n_1 \ldots n_N}(t_1, \ldots, t_N) = \det \begin{vmatrix} t_1^{n_1} & t_2^{n_1} & \ldots & t_N^{n_1} \\ t_1^{n_2} & t_2^{n_2} & \ldots & t_N^{n_2} \\ \ldots & \ldots & \ldots & \ldots \\ t_1^{n_N} & t_2^{n_N} & \ldots & t_N^{n_N} \end{vmatrix}. \tag{66}$$

Schur polynomial is defined as the ratio of two antisymmetric polynomials:

$$S_{\vec{n}}(\vec{t}) = \frac{P_{n_1, n_2+1, \ldots, n_N+N-1}(\vec{t})}{P_{0,1,2,\ldots,N-1}(\vec{t})}. \tag{67}$$

Denominator (corresponding to Ω in the previous sections)

$$P_{0,1,2,\ldots,N-1}(\vec{t}) = \prod_{k>j} (t_k - t_j) \tag{68}$$

is the elementary antisymmetric polynomial (Vandermonde determinant).

The separated equation

$$\prod_{j=1}^{N} (t\partial_t - n_j)\, \psi(t) = 0. \tag{69}$$

has as the general solution the polynomial $\psi(t) = \sum_{j=1}^N c_j t^{n_j}$. The boundary condition

$$\left. \frac{\partial^k}{\partial t^k} \psi(t) \right|_{t=1} = 0, \qquad k = 0, 1, \ldots, N-2 \tag{70}$$

selects the solution

$$c_j \sim \det \begin{vmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 \\ n_1 & \ldots & n_{j-1} & n_{j+1} & \ldots & n_N \\ n_1^2 & \ldots & n_{j-1}^2 & n_{j+1}^2 & \ldots & n_N^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ n_1^{N-2} & \ldots & n_{j-1}^{N-2} & n_{j+1}^{N-2} & \ldots & n_N^{N-2} \end{vmatrix} = \prod_{\substack{k>l \\ k,l \neq j}} (n_k - n_l). \tag{71}$$

In case of Schur polynomials it is easier to construct the inverse operator K^{-1} rather than K. Let

$$\widetilde{\Psi}(t_1,\ldots,t_{N-1}) = \psi(t_1)\ldots\psi(t_{N-1}) = \prod_{j=1}^{N-1}\psi(t_j) \tag{72}$$

and

$$K^{-1} = \prod_{k>j}\left(t_k\partial_{t_k} - t_j\partial_{t_j}\right). \tag{73}$$

Theorem 4 *The operator K^{-1} transforms the symmetric polynomial $\widetilde{\Psi}$ into an antisymmetric polynomial $\Psi(t_1,\ldots,t_{N-1}) = K^{-1}\widetilde{\Psi}$ which is none other than the numerator of Schur polynomial*

$$\Psi\left(\frac{t_1}{t_N},\ldots,\frac{t_{N-1}}{t_N}\right)t^{n_1+\cdots+n_N} \sim P_{n_1\ldots n_N}(t_1,\ldots,t_N). \tag{74}$$

The proof consists in an elementary calculation.

Since we have already seen in the $N=3$ case that K^{-1} becomes a differential operator for integer $g>0$, it is not surprising that here K^{-1} is also a differential operator.

8 Discussion

The construction of the operator M performing the separation of variables for Jack polynomials originates from mathematical physics (Inverse Scattering Method) and contains a lot of guesswork. A generalization of our results to the case of higher rank $N>3$ could probably throw some light on the algebraic and geometric meaning of the whole construction which remains still obscure. The only available results in this direction are so far the case $g=1$ (Schur polynomials) and theorem 3 which allows to formulate conjecture 2 about the structure of separated polynomials in the general case.

Among other challenging problems one should mention generalizations to other root systems, first of all BC_N, and also to the q-finite-difference case (Macdonald polynomials).

Acknowledgments. VK acknowledges support by the Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO). ES wishes to thank K. Aomoto and S.G. Gindikin for their interest in the work and valuable remarks.

References

[1] T. Oshima and H. Sekiguchi. Commuting families of differential operators invariant under the action of a Weyl group. Preprint Univ. of Tokyo, UTMS 93-43, November 5, 1993.

[2] F. Calogero. Solution of a three-body problem in one dimension, J. Math. Phys. **10** (1969) 2191–2196.

[3] B. Sutherland. Quantum many-body problem in one dimension, I, II. J. Math. Phys. **12** (1971) 246–250, 251–256.

[4] M.A. Olshanetsky and A.M. Perelomov. Classical integrable finite-dimensional systems related to Lie algebras, Phys. Rep. **71** (1981) 313–400.

[5] M.A. Olshanetsky and A.M. Perelomov. Quantum integrable systems related to Lie algebras, Phys. Rep. **94** (1983) 313–404.

[6] I.G. Macdonald, Symmetric functions and Hall polynomials, Oxford University Press (1979).
——— Commuting differential operators and zonal spherical functions, in "Algebraic groups, Utrecht 1986" (A.M. Cohen et al., Eds.), Lecture Notes in Math., vol. 1271, pp. 189–200, Springer, 1987.

[7] T.H. Koornwinder, Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators I, II, II, IV, Nederl. Akad. Wetensch. Proc. Ser. A **77** (1974) 48–58, 59–66, 357–369, 370–381.
J. Sekiguchi, Zonal spherical functions on some symmetric spaces, Publ. RIMS Kyoto Univ. **12 Suppl.** (1977) 455–459.

[8] G.J. Heckman (part I with E.M. Opdam). Root systems and hypergeometric functions I, II, Compositio Math. **64** (1987) 329–352, 353–373.

[9] E.G. Kalnins. Separation of variables for Riemannian spaces of constant curvature, Pitman Monographs and Surveys in Pure and Applied Mathematics 28, Longman Scientific and Technical, Essex, England, 1986.

[10] V.B. Kuznetsov. Equivalence of two graphical calculi, J. Phys. A: Math. Gen. **25** (1992) 6005–6026.

[11] E.K. Sklyanin. *Quantum Inverse Scattering Method. Selected Topics.* In: "Quantum Group and Quantum Integrable Systems" (Nankai Lectures in Mathematical Physics), ed. by Mo-Lin Ge, Singapore: World Scientific, 1992, p.63–97.

[12] E.K. Sklyanin. *Separation of variables. New trends.* Preprint, Univ. of Tokyo, UTMS 95-9; to be published in: "Quantum Field Theory, Integrable Models and Beyond", eds. T. Inami and R. Sasaki, (International Workshop, YITP, Kyoto, 14–17 Feb, 1994), Supplement of Progress in Theoretical Physics (1995).

[13] I.M. Gel'fand and G.E. Shilov, *Generalized functions*, Academic Press, N.Y. (1964), vol. I.

[14] A. Erdélyi, ed., *"Higher Transcendental Functions. "*, McGraw-Hill Book Company (1953), vol. I.

[15] A. Erdélyi, ed., *"Tables of Integral Transforms."*, McGraw-Hill Book Company (1954), vol. II.