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24.

$W(E_7)$–invariant polynomial of degree 10 and 28 bitangents of plan equartic curves

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序文

ルート系に対する複比多様体という概念を筆者は定義した (cf.[Se4]) が, E_7型ルート系の場合にそれを詳しく調べる. 平面の非特異 4 次曲線には 28 本の複接線が存在するが, この古典的話題と関係がある.

本文の結果を説明する.

•E_7型ルート系に対する複比多様体の E_6型部分ルート系に対して定義される部分多様体.

•射影平面の 7 点の配置空間の特別な配置 (与えられた 7 点に対して, この中のある点で cusp になるようなこれらの 7 点を通る 3 次曲線が存在するような配置).

•平面の非特異 4 次曲線には 28 本の複接線が存在するが, それらの複接線の接点は普通 2 点あるがそれらが一致するような複接線が存在する.

以上 3 つの条件がワイル群 $W(E_7)$ のある 10 次の不変式を使って記述できる. この主張を示すことが本文の目的だが, 証明には数式処理システム risa/asir を利用する.

§1. The root system of type E_7.

We first recall the definition of the root system $\Delta(E_7)$ of type E_7. We always denote it by Δ for simplicity in this paper. Let $\tilde{E}$ be an inner product space of dimension 8 with an orthonormal basis $\{\varepsilon_j; 1 \le j \le 8\}$ with respect to an inner product $< \cdot, \cdot >$ and let E be its linear subspace orthogonal to $\varepsilon_7 + \varepsilon_8$. As in [Se4], §4, we define the following 63 vectors of E:

$$\gamma_1 = \varepsilon_8 - \varepsilon_7, \quad \gamma_j = \varepsilon_{j-1} - \gamma_0 + \gamma_1, \quad \gamma_{1j} = -\varepsilon_{j-1} + \gamma_0, \quad (1 < j < 8)$$

$$\gamma_{jk} = \varepsilon_{j-1} - \varepsilon_{k-1}, \quad \gamma_{1jk} = -\varepsilon_{j-1} - \varepsilon_{k-1}, \quad (1 < j < k < 8)$$

$$\gamma_{ijk} = -\varepsilon_{i-1} - \varepsilon_{j-1} - \varepsilon_{k-1} + \gamma_0, \quad (1 < i < j < k < 8)$$

where $\gamma_0 = \frac{1}{2}\sum_{j=1}^{8} \varepsilon_j - \varepsilon_7$. The totality Δ of $\pm\gamma_j, \pm\gamma_{jk}, \pm\gamma_{ijk}$ is a root system of type E_7 (cf. [B]). As a fundamental set of roots of Δ, we may take

$$\alpha_1 = \gamma_{12}, \ \alpha_2 = \gamma_{123}, \ \alpha_3 = \gamma_{23}, \ \alpha_4 = \gamma_{34}, \ \alpha_5 = \gamma_{45}, \ \alpha_6 = \gamma_{56}, \ \alpha_7 = \gamma_{67}.$$

Then the corresponding Dynkin diagram is:

$$\begin{array}{ccccccccccccc} \alpha_1 & - & \alpha_3 & - & \alpha_4 & - & \alpha_5 & - & \alpha_6 & - & \alpha_7 \\ & & & & | & & & & & & \\ & & & & \alpha_2 & & & & & & \end{array}$$

We denote by Δ^+ the set of positive roots in Δ. It is easy to see that Δ^+ consists of $\gamma_i, \gamma_{ij}, \gamma_{ijk}$.

If g_j is the reflection on E with respect to the root α_j, the group generated by $g_1, \cdots, g_7$ is the Weyl group $W(E_7)$ of type E_7. In the sequel, we frequently identify $W(A_6) \simeq \Sigma_7$ (resp. the Weyl group $W(E_6)$ of type E_6) with the subgroup of $W(E_7)$ generated by g_1, g_j $(j = 3,4,5,6,7)$ (resp. g_j $(j = 1,2,3,4,5,6)$).

Using the 63 positive roots defined above, we define linear forms on E by

$$h_j = \gamma_j(t), \quad h_{jk} = \gamma_{jk}(t), \quad h_{ijk} = \gamma_{ijk}(t), \qquad (t \in E).$$

§2. The configuration space of 7 points in $\mathbf{P}^2$.

We briefly review the definition of the configuration space of 7 points of $\mathbf{P}^2$ which we denote by $\mathbf{P}(2,7)$. We first define the vector space $M_{3,7}$ of 3×7 matrices. Then $M_{3,7}$ admits $GL(3)\times GL(7)$-action in a natural manner. Let $D(7)$ be the maximal torus of $GL(7)$ consisting of diagonal matrices. Let $D_{ijk}(X)$ be the determinant of the 3×3 matrix consisting of the i,j,k-th column vectors of $X \in M_{3,7}$. If $M'_{3,7}$ is the subset of $M_{3,7}$ consisting of X with $D_{ijk}(X) \neq 0 \quad \forall(i,j,k)(i < j < k)$, we denote by $\mathbf{P}(2,7)$ the quotient of $M'_{3,7}$ by the action $GL(3) \times D(7)$. It is possible to choose as a representative of any element of $\mathbf{P}(2,7)$ a matrix of the form

$$X = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & x_3 \\ 0 & 0 & 1 & 1 & y_1 & y_2 & y_3 \end{pmatrix}$$

In this way, $\mathbf{P}(2,7)$ is regarded as a quasi-affine subset of $\mathbf{C}^6$ by the correspondence

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & x_3 \\ 0 & 0 & 1 & 1 & y_1 & y_2 & y_3 \end{pmatrix} \longrightarrow (x_1, x_2, x_3, y_1, y_2, y_3).$$

In fact, $\mathbf{P}(2,7)$ is identified with $\mathbf{C}^6 - S_0(A_6)$, where $S_0(A_6)$ is the union of the 28 hypersurfaces below:

$$x_i = 0, \quad x_i - 1 = 0, \quad y_i = 0, \quad y_i - 1 = 0, \quad x_i - x_j = 0, \quad y_i - y_j = 0, \quad x_i - y_i = 0,$$

$$x_i y_j - x_i y_j = 0, \quad (1 - x_i)(1 - y_j) - (1 - x_j)(1 - y_i) = 0,$$

$$\varphi_1(x_1, x_2, x_3, y_1, y_2, y_3) = \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = 0.$$

We introduce the following seven birational transformations $s_1, \cdots, s_6, s_R$:

$$s_1 : (x_1, x_2, x_3, y_1, y_2, y_3) \longrightarrow (1/x_1, 1/x_2, 1/x_3, y_1/x_1, y_2/x_2, y_3/x_3)$$

$$s_2 : (x_1, x_2, x_3, y_1, y_2, y_3) \longrightarrow (y_1, y_2, y_3, x_1, x_2, x_3)$$

$$s_3 : (x_1, x_2, x_3, y_1, y_2, y_3) \longrightarrow (x_1', x_2', x_3', y_1', y_2', y_3')$$

$$s_4 : (x_1, x_2, x_3, y_1, y_2, y_3) \longrightarrow (1/x_1, x_2/x_1, x_3/x_1, 1/y_1, y_2/y_1, y_3/y_1)$$

$$s_5 : (x_1, x_2, x_3, y_1, y_2, y_3) \longrightarrow (x_2, x_1, x_3, y_2, y_1, y_3)$$

$$s_6 : (x_1, x_2, x_3, y_1, y_2, y_3) \longrightarrow (x_1, x_3, x_2, y_1, y_3, y_2)$$

$$s_R : (x_1, x_2, x_3, y_1, y_2, y_3) \longrightarrow (1/x_1, 1/x_2, 1/x_3, 1/y_1, 1/y_2, 1/y_3)$$

where

$$x_j' = \frac{x_j - y_j}{1 - y_j}, \quad y_j' = \frac{y_j}{y_j - 1}, \quad j = 1, 2, 3.$$

The correspondence

$$g_1 \to s_1, \quad g_2 \to s_R, \quad g_j \to s_{j-1} \ (j = 3, \cdots, 7)$$

induces a group isomorphism of $W(E_7)$ to the group generated by $s_1, \cdots, s_6, s_R$.

We introduce 7 polynomials of $(x_1, x_2, x_3, y_1, y_2, y_3)$ defined by

$$\sigma_j(x_1, x_2, x_3, y_1, y_2, y_3) = \varphi_{6-j}(x_1, x_2, x_3, y_1, y_2, y_3), \quad (j = 1, 2, 3, 4),$$

$$\sigma_5(x_1, x_2, x_3, y_1, y_2, y_3) = x_2 y_3 (1 - x_3)(1 - y_2) - x_3 y_2 (1 - x_2)(1 - y_3),$$

$$\sigma_6(x_1, x_2, x_3, y_1, y_2, y_3) = x_1 y_3 (1 - x_3)(1 - y_1) - x_3 y_1 (1 - x_1)(1 - y_3),$$

$$\sigma_7(x_1, x_2, x_3, y_1, y_2, y_3) = x_1 y_2 (1 - x_2)(1 - y_1) - x_2 y_1 (1 - x_1)(1 - y_2),$$

where φ_j $(j = 2, 3, 4, 5)$ are polynomials introduced in [Se4], §4. In particular,

$$\varphi_2(x_1, x_2, x_3, y_1, y_2, y_3) = x_1 x_2 x_3 y_1 y_2 y_3 \varphi_1(1/x_1, 1/x_2, 1/x_3, 1/y_1, 1/y_2, 1/y_3)$$

Let Q_j be the hypersurface in $\mathbf{C}^6$ defined by $\sigma_j = 0$ $(j = 1, \cdots, 7)$. Then it is easy to see that Σ_7 acts on the set $\{Q_1, \cdots, Q_7\}$ as a permutation group. If $\tilde{\sigma}_7 = \sigma_7$ and $\tilde{\sigma}_j = \tilde{\sigma}_{j+1} \circ s_j$ $(j = 1, 2, \cdots, 6)$, and Q'_j is the hypersurface in $\mathbf{C}^6 - S_0(A_6)$ defined by $\tilde{\sigma}_j = 0$, then Q_j is the Zariski closure of Q'_j in $\mathbf{C}^6$. A geometric meaning of Q_j will be given in §6. In the sequel, we denote by $\mathbf{P}_0(2, 7)$ the complement of the union $S(E_7)$ of $S_0(A_6)$ and $Q_1, \cdots, Q_7$. Clearly all the elements of $W(E_7)$ induce biregular transformations on $\mathbf{P}_0(2, 7)$.

§3. The cross ratio variety $C(\Delta(E_7), D_4)$.

For any subroot system Δ_1 of type D_4 in Δ, we defined a D_4-cross ratio map of the Zariski open subset $Z(\Delta)$ of the projective space $\mathbf{P}^6 = \mathbf{P}(E_{\mathbf{C}})$ associated to the complexification $E_{\mathbf{C}}$ of E to $CR(\mathbf{P}) \simeq \mathbf{P}^1$. There are totally 315 subroot systems of type D_4 in Δ. The corresponding D_4-cross ratio maps are denoted by

$$cr^1_{[i_3, i_6 i_7]} = (h_{i_2 i_4} h_{i_2 i_3 i_4} h_{i_1 i_5} h_{i_1 i_3 i_5} : -h_{i_1 i_4} h_{i_1 i_3 i_4} h_{i_2 i_5} h_{i_2 i_3 i_5} : h_{i_1 i_2} h_{i_1 i_2 i_3} h_{i_4 i_5} h_{i_3 i_4 i_5})$$

$$cr^2_{[i_1 i_2, i_3 i_4, i_5 i_6]} = (h_{i_1 i_3 i_5} h_{i_2 i_4 i_5} h_{i_2 i_3 i_6} h_{i_1 i_4 i_6} : -h_{i_2 i_3 i_5} h_{i_1 i_4 i_5} h_{i_1 i_3 i_6} h_{i_2 i_4 i_6} : h_{i_1 i_2} h_{i_3 i_4} h_{i_5 i_6} h_{i_7})$$

$$cr^3_{[i_1 i_2, i_3 i_4]} = (h_{i_1 i_2 i_7} h_{i_3 i_4 i_7} h_{i_5 i_6} h_{i_7} : -h_{i_1 i_2 i_6} h_{i_3 i_4 i_6} h_{i_5 i_7} h_{i_6} : h_{i_1 i_2 i_5} h_{i_3 i_4 i_5} h_{i_6 i_7} h_{i_5})$$

(cf. [Se4], §4). By taking the product of all the 315 maps above, we obtain a map $cr_{D_4, \Delta}$ of $Z(\Delta)$ to $CR(\mathbf{P})^{315}$. Let $C'(\Delta, D_4)$ be the image $cr_{D_4, \Delta}(Z(\Delta))$ and let $C(\Delta, D_4)$ be its closure in $CR(\mathbf{P})^{315}$.

For any subroot system Δ_1 of Δ, we defined a subvariety $Y_{\Delta, D_4}(\Delta_1)$ in [Se4], §4. There are four kinds of hypersurfaces of $C(\Delta, D_4)$ defined as the form $Y_{\Delta, D_4}(\Delta_1)$ for suitable subroot systems.

§4. Hypersurfaces corresponding to subroot systems of type E_6.

We introduce hypersurfaces of $C(\Delta, D_4)$ which are fixed by $W(E_6)$-actions (cf. [Se4], §4,(4.15.10)). If Δ_1 is a subroot system of type E_6 in Δ, it is easy to show that $Y_{\Delta, D_4}(\Delta_1)$ is a hypersurface of

$C(\Delta, D_4)$. Such a hypersurface is called that of the 5^{th} kind. As a basic property of hypersurfaces of the 5^{th} kind, we have the lemma below.

Lemma 4.1. (cf. [Se4a]) $Y_{\Delta,D_4}(\Delta(E_6)) \simeq C(\Delta(E_6), \{A_3, D_4\})$.

Lemma 4.1 establishes an embedding of the cross ratio variety $C(\Delta(E_6), \{A_3, D_4\})$ into $C(\Delta, D_4)$.

To show an identification of $C(\Delta(E_6), \{A_3, D_4\})$ with the variety defined [L], we need some preparation on cubic curves in $\mathbf{P}^2$ passing through 7 points. For simplicity, we take 7 points $P_1, \cdots, P_7$ of $\mathbf{P}^2$ as follows:

$$P_1 = (1:0:0), \quad P_2 = (0:1:0), \quad P_3 = (0:0:1), \quad P_4 = (1:1:1),$$

$$P_5 = (1:x_1:y_1), \quad P_6 = (1:x_2:y_2), \quad P_7 = (1:x_3:y_3).$$

We assume that the 7 points above are in a general position which means the corresponding matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & x_1 & x_2 & x_3 \\ 0 & 0 & 1 & 1 & y_1 & y_2 & y_3 \end{pmatrix}$$

is a representative of the configuration space $\mathbf{P}(2,7)$.

Let $C(P_1, \cdots, P_6; P_7)$ be the cubic curve in $\mathbf{P}^2$ passing through $P_1, \cdots, P_7$ such that P_7 is a double point (cf. [M],[L]). We now consider the case where $C(P_1, \cdots, P_6; P_7)$ has a cusp at P_7 (cf. [L]). This condition implies a relation $\Psi(x, y) = 0$ among $(x, y) = (x_1, x_2, x_3, y_1, y_2, y_3)$.

The explicit form of the polynomial $\Psi(x, y)$ is too lengthy to write down here. It is provable that $\deg_{x_3} \Psi = \deg_{y_3} \Psi = 8$.

Noting that $C(\Delta, D_4)$ is a compactification of $\mathbf{P}_0(2,7)$, we obtain a hypersurface Y_{cusp} of $C(\Delta, D_4)$ as the Zariski closure of the hypersurface of $\mathbf{P}_0(2,7)$ defined by $\Psi(x, y) = 0$.

Theorem 4.2. $Y_{cusp} = Y_{\Delta,D_4}(\Delta_1) \cap \mathbf{P}_0(2,7)$.

The basic idea of the proof employed here is the comparison between the defining equations of $Y_{\Delta,D_4}(\Delta_1)$ and Y_{cusp}. Before entering the details of its outline, we state a result on the polynomial $\Psi(x, y)$.

Lemma 4.3. We put

$$\Phi(x, y) = \Phi_1(x, y)^2 - 4\Phi_2(x, y),$$

where

$$\begin{aligned}\Phi_1(x,y) &= x_1x_2y_1 - x_1x_2y_3 - x_1x_3y_1 + x_1x_3y_2 - x_1y_1y_2 + x_1y_1y_3 + x_1y_2 \\ &\quad -x_1y_3 + x_2y_1y_3 - x_2y_1 - x_3y_1y_2 + x_3y_1, \\ \Phi_2(x,y) &= (x_1-y_1)(x_2y_3-x_3y_2)(y_1-1)(y_2-y_3)x_1.\end{aligned}$$

Then there is $s \in W(E_7)$ such that $\Phi \circ s = \Psi$.

We are going to explain the outline of the proof of Theorem 4.2.

We first compute the condition that the cubic curve $C(P_1,\cdots,P_6;P_7)$ has a cusp at P_7. For this purpose, we assume that $F(\xi_1,\xi_2,\xi_3)=0$ is the defining equation of $C(P_1,\cdots,P_6;P_7)$, where

$$F = c_1\xi_1^3 + c_2\xi_2^3 + c_3\xi_3^3 + c_4\xi_1^2\xi_2 + c_5\xi_1^2\xi_3 + c_6\xi_1\xi_2^2 + c_7\xi_2^2\xi_3 + c_8\xi_1\xi_3^2 + c_9\xi_2\xi_3^2 + c_{10}\xi_1\xi_2\xi_3.$$

In the discussion above, we have taken $\xi = (\xi_1 : \xi_2 : \xi_3)$ as a homogeneous coordinate of $\mathbf{P}^2$. The condition that $C(P_1,\cdots,P_6;P_7)$ passes through $P_1,\cdots,P_7$ is equivalent to

$$(C.1) \qquad F(P_j)=0, \quad j=1,\cdots,7.$$

The condition that P_7 is a double point of $C(P_1,\cdots,P_6;P_7)$ is equivalent to

$$(C.2) \qquad F_{\xi_i}(P_7)=0, \quad i=1,2,3.$$

The condition that P_7 is moreover a cusp point of $C(P_1,\cdots,P_6;P_7)$ is equivalent to

$$(C.3) \qquad F_{\xi_1\xi_1}(P_7)F_{\xi_2\xi_2}(P_7) - F_{\xi_1\xi_2}(P_7)^2 = 0.$$

¿From (C.1), (C.2), we conclude that the ratio of $c_1,\cdots,c_{10}$ is uniquely determined. Substituting such $c_1,\cdots,c_{10}$ to the equation (C.3), we obtain an algebraic relation

$$(4) \qquad \Psi(x,y)=0$$

if $(x,y) \in \mathbf{C}^6 - S(A_6)$. We need a long computation to obtain (4) and it is hard to reproduce here.

Our next purpose is to compute the defining equation of the hypersurface $Y_{\Delta,D_4}(\Delta_1)$ in $\mathbf{P}(2,7)$.

For this purpose, we first recall the definition of the rational map of $\mathbf{P}^6$ to $\mathbf{P}(2,7)$ in [Se4], Lemma 4.2. We put

$$x_1(t) = \frac{h_{24}\cdot h_{234}\cdot h_{15}\cdot h_{135}}{h_{14}\cdot h_{134}\cdot h_{25}\cdot h_{235}}, \quad x_2(t) = \frac{h_{24}\cdot h_{234}\cdot h_{16}\cdot h_{136}}{h_{14}\cdot h_{134}\cdot h_{26}\cdot h_{236}}, \quad x_3(t) = \frac{h_{24}\cdot h_{234}\cdot h_{17}\cdot h_{137}}{h_{14}\cdot h_{134}\cdot h_{27}\cdot h_{237}},$$

$$y_1(t) = \frac{h_{34}\cdot h_{234}\cdot h_{15}\cdot h_{125}}{h_{14}\cdot h_{124}\cdot h_{35}\cdot h_{235}}, \quad y_2(t) = \frac{h_{34}\cdot h_{234}\cdot h_{16}\cdot h_{126}}{h_{14}\cdot h_{124}\cdot h_{36}\cdot h_{236}}, \quad y_3(t) = \frac{h_{34}\cdot h_{234}\cdot h_{17}\cdot h_{127}}{h_{14}\cdot h_{124}\cdot h_{37}\cdot h_{237}}$$

and define the map F_{E_7} of $Z(\Delta)$ to the (x,y)-space by

$$F_{E_7}(t) = (x_1(t),x_2(t),x_3(t),y_1(t),y_2(t),y_3(t)),$$

where $\mathbf{P}(2,7)$ is identified with a Zariski open subset of the (x,y)-space (cf.[Se4], §4) and h_i, h_{ij}, h_{ijk}

are linear forms on E associated with the roots of Δ. Now we put

$$\zeta_1 = h_{12}, \quad \zeta_2 = h_{123}, \quad \zeta_3 = h_{23}, \quad \zeta_4 = h_{34}, \quad \zeta_5 = h_{45}, \quad \zeta_6 = h_{56}, \quad \zeta_7 = h_{67}.$$

It is clear from the definition that linear forms in question corresponding to the roots of $\Delta(E_6)$ are expressed as linear combinations of ζ_j $(j = 1, \cdots, 6)$. We may take $\zeta = (\zeta_1 : \cdots : \zeta_6 : \zeta_7)$ as a homogeneous coordinate of $\mathbf{P}^6$. Now we write $\zeta_j = \zeta_j' \tau$ $(j = 1, \cdots, 6)$. Then $\tilde{\zeta} = ((\zeta_1' : \cdots : \zeta_6'), \tau)$ is also a local coordinate of an affine open subset defined by $\zeta_7 \neq 0$ in $\mathbf{P}^6$

Noting this, we write $\tilde{x}_j(\tilde{\zeta}) = x_j(t)$, $\tilde{y}_j(\tilde{\zeta}) = y_j(t)$ $(j = 1, 2, 3)$. Now we put

$$u_j(\zeta_1' : \cdots : \zeta_6') = \lim_{\tau \to 0} \tilde{x}_j(\tilde{\zeta}), \quad v_j(\zeta_1' : \cdots : \zeta_6') = \lim_{\tau \to 0} \tilde{y}_j(\tilde{\zeta}) \qquad (j = 1, 2, 3).$$

We define polynomials

$$f_1 = x_1(\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4' + \zeta_5')(\zeta_1' + \zeta_3' + \zeta_4')(\zeta_2' + \zeta_3' + \zeta_4')(\zeta_3' + \zeta_4' + \zeta_5')$$

$$-(\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4')(\zeta_1' + \zeta_3' + \zeta_4' + \zeta_5')(\zeta_2' + \zeta_3' + \zeta_4' + \zeta_5')(\zeta_3' + \zeta_4'),$$

$$f_2 = x_2(\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4' + \zeta_5' + \zeta_6')(\zeta_1' + \zeta_3' + \zeta_4')(\zeta_2' + \zeta_3' + \zeta_4')(\zeta_3' + \zeta_4' + \zeta_5' + \zeta_6')$$

$$-(\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4')(\zeta_1' + \zeta_3' + \zeta_4' + \zeta_5' + \zeta_6')(\zeta_2' + \zeta_3' + \zeta_4' + \zeta_5' + \zeta_6')(\zeta_3' + \zeta_4'),$$

$$f_3 = x_3(\zeta_1' + \zeta_3' + \zeta_4')(\zeta_2' + \zeta_3' + \zeta_4') - (\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4')(\zeta_3' + \zeta_4'),$$

$$f_4 = y_1(\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4' + \zeta_5')(\zeta_1' + \zeta_3' + \zeta_4')(\zeta_2' + \zeta_4')(\zeta_4' + \zeta_5')$$

$$-(\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4')(\zeta_1' + \zeta_3' + \zeta_4' + \zeta_5')(\zeta_2' + \zeta_4' + \zeta_5')\zeta_4',$$

$$f_5 = y_2(\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4' + \zeta_5' + \zeta_6')(\zeta_1' + \zeta_3' + \zeta_4')(\zeta_2' + \zeta_4')(\zeta_4' + \zeta_5' + \zeta_6')$$

$$-(\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4')(\zeta_1' + \zeta_3' + \zeta_4' + \zeta_5' + \zeta_6')(\zeta_2' + \zeta_4' + \zeta_5' + \zeta_6')\zeta_4',$$

$$f_6 = y_3(\zeta_1' + \zeta_3' + \zeta_4')(\zeta_2' + \zeta_4') - (\zeta_1' + \zeta_2' + \zeta_3' + \zeta_4')\zeta_4'.$$

Regarding

$$f_1 = \cdots = f_6 = 0, \tag{5}$$

as a system of equations for $\zeta_1', \cdots, \zeta_6'$ with coefficients in the function field $\mathbf{C}(x, y)$, we are going to solve the system (5). If x_j, y_j $(j = 1, 2, 3)$ satisfy an algebraic equation $\Psi'(x, y) = 0$, the system (5) has a non-trivial solution. ¿From the construction, the hypersurface $\Psi'(x, y) = 0$ in $\mathbf{P}(2, 7) = C'(\Delta, D_4)$ is nothing but the subvariety $Y_{\Delta, D_4}(\Delta(E_6))$ (cf.[Se4], §§1,4). By a little lengthy computation, we conclude that $\Psi'(x, y)$ coincides with $\Psi(x, y)$ up to a constant factor, where $\Psi(x, y)$ is the polynomial introduced before.

In this way, we can prove Theorem 2.

§5. Comparison beween $Y_{\Delta(E_6),D_4}(\Delta(D_4))$ and $Y_{\Delta(E_7),D_4}(\Delta(E_6))$.

It is worthwhile to compare similarities between hypersurfaces of the 5^{th} kind and the subvariety $Y_{\Delta(E_6),D_4}(\Delta(D_4))$ of $C(\Delta(E_6), D_4)$ introduced in [Se4], §3 (cf.[N], [L], [Sh2]).

TABLE I

	$Y_{\Delta(E_6),D_4}(\Delta(D_4))$	$Y_{\Delta(E_7),D_4}(\Delta(E_6))$
(I.1)	a cubic surface with an Eckardt point	a plane quartic curve with a special flex
(I.2)	$\lambda - 1 = 0$	$\Psi(x, y) = 0$
(I.3)	a cross ratio variety for $\Delta(D_4)$ (?)	$C(\Delta(E_6), \{A_3, D_4\})$
(I.4)	associated quintic	a $W(E_7)$-invariant of 10th degree
(I.5)	configuration of 6 points	configuration of 7 points

We give here an explanation on TABLE I.

(6.1) Let S be a non-singular cubic surface in $\mathbf{P}^3$. An Eckardt point on S is the intersection of three lines on S (cf. [N]). Every cubic surface does not have an Eckardt point. On the other hand, a flex of a non-singular plane quartic C is a point $p \in C$ such that there is a line l triply tangent to C at p (cf. [Sh2]). A flex is ordinary if $l \cap C$ consists of two points and a flex is special if $l \cap C = \{p\}$. Every plane quartic does not have a special flex.

(6.2) In [N], the parameter λ was introduced. It was shown in [Se3] (cf.[H]) that λ is regarded as a rational function on $\mathbf{P}(2, 6)$. In fact, using the notation in [Se3], we have

$$\lambda = \frac{x_2(x_1 - 1)(y_1 - y_2)(y_2 - 1)}{y_2(x_1 - x_2)(x_2 - 1)(y_1 - 1)}.$$

(6.3) Is it possible to regard $Y_{\Delta(E_6),D_4}(\Delta(D_4))$ as a cross ratio variety for the root system $\Delta(D_4)$ of type D_4 ?

(6.4) If $\delta_5(t)$ is a $W(E_6)$-invariant polynomial of degree 5 (which is unique up to a constant factor), it is shown in [Se3] that the polynomial $P_5(t) = \delta_5(t_1, t_2, t_3, t_4, t_6, -3t_5)$ is $W(F_4)$-semi-invariant under the notation there. Hence, by $W(F_4)$-action, we obtain totally 45 quintic polynomials on the standard representation space of $W(E_6)$. For the sake of convenience, we call these polynomials *associated quintics*. There is a 1-1 correspondence between the set of associated quintics and that of the 45 triple tangent planes.

Similarly, there is a $W(E_7)$-invariant polynomial $\delta_{E_7}(t)$ of degree 10 which plays a role analogues to δ_5. (The construction of $\delta_{E_7}(t)$ will be given later.)

(6.5) (cf. [L]) Let $P_1, \cdots, P_6$ be 6 points of $\mathbf{P}^2$. We consider a conic C passing through five points $P_1, \cdots, P_5$ and a line L passing through P_5, P_6. The condition corresponding to $\lambda - 1 = 0$ is that the line L is also a tangent of C at P_5. Let $P_1, \cdots, P_7$ be 7 points of $\mathbf{P}^2$. The condition corresponding to $\Psi(x, y) = 0$ is the main subject in the previous section. Namely, let $P_1, \cdots, P_7$ be 7 points of $\mathbf{P}^2$. We consider a cubic curve C passing through seven points $P_1, \cdots, P_7$ such that P_7 is a double point. The condition corresponding to $\Psi(x, y) = 0$ is that C has a cusp at P_7.

We are going to explain the construction on $\delta_{E_7}(t)$.

Let ω_j be a fundamental weight of $\Delta(E_7)$ such that $< \omega_j, \alpha_k >= \delta_{jk}$. Then ω_7 belongs to the set of weights of the 56 dimensional irreducible representation of the simple Lie algebra of type E_7. By definition, $\omega_7 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7$. The totality Π of $\alpha \in \Delta$ such that $< \omega_7, \alpha >= 0$ form a root system of type E_6. Let Ω_{27} be the set of weights of 27 irreducible representation of the simple Lie algebra of type E_6 corresponding to the root system Π. Then $z_p = \sum_{\omega \in \Omega_{27}} \omega^p \quad (p = 1, 2, \cdots)$ are $W(\Pi)$-invariant polynomials. (From the definition, $W(\Pi) \simeq W(E_6)$.)

Lemma 5.1. Under the notation above,

$$\begin{aligned} P(t) \;=\; & 43545600\omega_7^{10} - 3628800 z_2 \omega_7^8 + 100800 z_2^2 \omega_7^6 + 725760 z_5 \omega_7^5 \\ & + (4200 z_2^3 - 604800 z_6)\omega_7^4 - 20160 z_2 z_5 \omega_7^3 + (175 z_2^4 - 40320 z_2 z_6 + 181440 z_8)\omega_7^2 \\ & + (3360 z_2^2 z_5 - 57600 z_9)\omega_7 + 1008 z_5^2 \end{aligned}$$

is a $W(E_7)$-invariant polynomial of degree 10.

For simplicity, we put $P = P(\omega_7, z_2, z_5, z_6, z_8, z_9)$. ¿From the definition, P is defined on the standard representation space E of $W(E_7)$. Therefore $P = 0$ is a hypersurface on $\mathbf{P}^6 = \mathbf{P}(E_{\mathbf{C}})$. Similarly, $P(-2\omega_7, z_2, z_5, z_6, z_8, z_9) = 0$ defines a hypersurface H_{ω_7} in $\mathbf{P}^6$.

Theorem 5.2. The closure of $cr_{D_4, \Delta}(H_{\omega_7})$ is isomorphic to $Y_{\Delta(E_7), D_4}(\Delta(E_6))$ by $W(E_7)$-action.

¿From the theorem above, $\delta_{E_7}(t) = P(\omega_7, z_2, z_5, z_6, z_8, z_9)$ is a required $W(E_7)$-invariant polynomial. Therefore $P(-2\omega_7, z_2, z_5, z_6, z_8, z_9)$ is an *associated $W(E_7)$-invariant polynomila of degree 10.* There are totally 28 associated hypersurfaces in $C(\Delta, D_4)$.

It is interesting to characterize the polynomial $\delta_{E_7}(t)$ among $W(E_7)$-invariant polynomials of homogeneous degree 10.

Appendix. Theorem 4.2, Theorem 5.2 について — 数式処理システムの利用 —

(1) §4 の式 $\Psi(x,y)$ の計算はかなり大変．数式処理システム REDUCE3.4 を Toshiba J3100 で利用して計算して得られた 2 通りの式が等しいことを示した．10,000 行程度の式を因数分解して結果的に 1,800 行程度の式になった．最近，富士通の開発した risa/asir を PC9801 NS/R で利用して計算し直してみたが，これならかなり楽に結論を導けた．

(2) $Y_{\Delta,D_4}(\Delta(E_6))$ の形の $C(\Delta,D_4)$ の部分多様体は 56 次元表現の weight と対応する．56 次元表現の weight は

$$\pm(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4+\varepsilon_5+\varepsilon_6),\quad \pm(\varepsilon_{i_1}+\varepsilon_{i_2}+\varepsilon_{i_3}+\varepsilon_{i_4}-\varepsilon_{i_5}-\varepsilon_{i_6}),\quad \pm(\gamma_1\pm 2\varepsilon_j)$$

である．$\omega_7=(\gamma_1-2\varepsilon_6)$ および$-\omega_7$ に対応するのが $\Psi(x,y)=0$ で定義される超平面．$\pm(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4-\varepsilon_5-\varepsilon_6)$ に対応するのが Lemma 4.3 で導入した $\Phi(x,y)$ で定義される超平面．この計算は risa/asir に頼った．simple reflection で weight を次々に移すのと平行して 1,800 行程度の式を双有理変換で座標変換していけばいい．結果を確認することは容易．

(3) x_1,x_2,x_3,y_1,y_2,y_3 を t の斉次有理式で表して，$\Phi(x,y)$ に代入すると t の有理式が得られるが，その分子の自明でない因子は 22 次斉次式である．この式をさらに因数分解すると 1 次の因子が 12 個あり，残りの因子が $F(t)=P(-2\omega_7,z_2,z_5,z_6,z_8,z_9)$ になる．一次の因子はすべて鏡映面に対応する．$F(t)$ は t の式とみると 5,000 行程度である．適当な変数変換によって $F(t)$ から $W(E_7)$ 不変式を導き出すのは自明ではないようにかんじる．E_6の場合にそうなっていたというのが唯一の根拠．それにもとづいて，$P(p\omega_7,z_2,z_5,z_6,z_8,z_9)$ が $W(E_7)$ 不変になるような定数 p を求めよ，という問題を考えた．もし正しいとするならば，t として特殊値を代入しても成り立つはずであると考え，適当な値を代入して調べてみた．正しいか正しくないか computer に計算させたところ，$p=1$ ならば少なくともある特殊値では正しいことがわかった．それで，いくつか特殊値を代入して確かめたところいずれも正しいことが示せた．それで t を変数として計算して主張を示した．奇妙なことだが $P(-2\omega_7,z_2,z_5,z_6,z_8,z_9)$ は t 式とみて 5,000 行程度とかなり長い式だが $P(\omega_7,z_2,z_5,z_6,z_8,z_9)$ は 270 行程度のもとと比べればかなり短い式になった．

§6. Relations between $C(\Delta,D_4)$ and cubic surfaces.

At a meeting organized by H. Yamada held in RIMS, Kyoto University (December, 1993), I. Naruki and J. Matsuzawa gave talks on a root system construction of universal cubic surfaces. They constructed a fibre space $\tilde{C}$ of cubic surfaces over Naruki's cross ratio variety $C(=C(\Delta(E_6),D_4))$ so that the natural projection $\varpi:\tilde{C}\longrightarrow C$ is $W(E_6)$-equivariant.

In this section, we discuss a relation between $C(\Delta,D_4)$ and $\tilde{C}$.

For this purpose, we introduce the set $\tilde{\mathbf{P}}(2,k)$ of k points $(P_1,\cdots,P_k)$ in $\mathbf{P}^2$ such that $P_i\neq P_j$ if $i\neq j$. By definition, $\tilde{\mathbf{P}}(2,k)$ admits a $PGL(3)$-action. Let $\mathbf{P}(2,k)$ be the quotient of $\tilde{\mathbf{P}}(2,k)$

by $PGL(3)$. It is clear that $\mathbf{P}(2,7)$ is nothing but the one introduced in §2. There is a natural projection p of $\mathbf{P}(2,k+1)$ to $\mathbf{P}(2,k)$ defined by $p((P_1,\cdots,P_k,P_{k+1}))=(P_1,\cdots,P_k)$.

¿From now on, we focus our attention to the cases $k=6,7$. It is known (cf. [Se4]) that there is a birational $W(E_k)$-action on $\mathbf{P}(2,k)$ $(k=6,7)$. This easily implies that the projection $p:\mathbf{P}(2,7)\longrightarrow\mathbf{P}(2,6)$ is $W(E_6)$-equivariant. Denoting by $\tilde{p}$ the extension of p to $C(\Delta,D_4)$, we obtain a biratioanl $W(E_6)$-equivariant map $\tilde{p}:C(\Delta,D_4)\longrightarrow C(\Delta(E_6),D_4)$.

We consider the $W(E_6)$-orbits of the set of hypersurfaces of the 1^{st} kind in $C(\Delta,D_4)$. There are two orbits. The first one denoted by Ω_1 consists of those corresponding to roots contained in $\Delta(E_6)$:

$$Y_7,\qquad Y_{ij}\quad(1\le i<j<7),\qquad Y_{ijk}\quad(1\le i<j<k<7).$$

The second one denoted by Ω_2 consists of the remaining 27 hypersurfaces:

$$Y_i\quad(1\le i<7),\qquad Y_{i7}\quad(1\le i<7),\qquad Y_{ij7}\quad(1\le i<j<7).$$

For any $(P_1,\cdots,P_6)\in\mathbf{P}(2,6)$, the closure $S(P_1,\cdots,P_6)=\overline{p^{-1}((P_1,\cdots,P_6))}$ of its fibre in $C(\Delta,D_4)$ is of dimension 2. The surface $S(P_1,\cdots,P_6)$ intersects with all the hypersurfaces of Ω_2. By the intersection relations among hypersurfaces of the 1^{st} kind, we easily find that the intersection relations among the 27 curves $S(P_1,\cdots,P_6)\cap Y$ $(\forall Y\in\Omega_2)$ on $S(P_1,\cdots,P_6)$ are same as those of the 27 lines on a non-singular cubic surface.

If the interpretation of the work of Naruki and Matsuzawa is correct, $\tilde{C}$ coincides with $C(\Delta,D_4)$ and $\tilde{p}:C(\Delta,D_4)\longrightarrow C(\Delta(E_6),D_4)$ defined above is the natural projection ϖ. As an easy consequence (?), $S(P_1,\cdots,P_6)$ is a cubic surface. Therefore it is hopeful that $S(P_1,\cdots,P_6)\cap Y$ $(\forall Y\in\Omega_2)$ are the 27 lines on it. If this is true, hypersurfaces of Ω_2 are global sections of 27 lines of cubic surfaces in the total space $\tilde{C}$.

¿From the definition, $\mathbf{P}(2,7)$ is identified with the open subset of $(x_1,x_2,x_3,y_1,y_2,y_3)$-space outside the union $S_0(A_6)$ of 28 hyperplanes (cf. §2). Moreover, we introduced 7 hypersurfaces $Q_1,\cdots,Q_7$ of the $(x_1,x_2,x_3,y_1,y_2,y_3)$-space in order to define $\mathbf{P}_0(2,7)$. It is clear that the closure of the hypersurface $D_{ijk}=0$ in $C(\Delta,D_4)$ is nothing but Y_{ijk} and that of Q_j is Y_j.

We now take seven points $P_1,\cdots,P_6,P_7$ as in §4 and fix P_j $(j=1,\cdots,6)$ for the moment. Then P_7 is regarded as a point on $\mathbf{P}^2-\{P_1,\cdots,P_6\}$. Therefore (x_3,y_3) are interpreted as an inhomogeneous coordinate of $\mathbf{P}^2$. Under this identification, the defining equation $D_{ij7}=0$ corresponds to the line on $\mathbf{P}^2$ passing through P_i and P_j for i,j $(1\le i<j<7)$. On the other hand, the defining equation of Q_i corresponds to the conic on $\mathbf{P}^2$ passing through the five points $\{P_j;j=1,\cdots,6,j\neq i\}$. This is a geometric interpretation of hypersurfaces $Q_1,\cdots,Q_7$ (cf. [M], Theorem 26.2).

In [Se4] §3, we have studied the structure of subvarieties of the form $Y(M)$ in $C(\Delta, D_4)$, where M is a subset of Δ consisting of mutually orthogonal positive roots. In particular, we now treat the intersection $Y_\alpha \cap Y_\beta$ for $\alpha, \beta \in \Delta$ such that $Y_\alpha, Y_\beta \in \Omega_2$. The intersection $Y_\alpha \cap Y_\beta$ may be regarded as a global section of the intersection of two lines of cubic surfaces $S(P_1, \cdots, P_6)$, being assumed that $S(P_1, \cdots, P_6)$ is a cubic surface for any $(P_1, \cdots, P_6) \in \mathbf{P}(2,6)$. It is interesting ot make clear a relation between cubic surfaces and the global section of the intersections of two lines on them.

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