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Functional Integral Representation of a Model in QED

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Abstract

This article presents functional integral representations for the heat semigroups with the infinitesimal generators given by self-adjoint Hamiltonians describing an interaction of a non-relativistic charged particle and a quantized radiation field in the Coulomb gauge without the dipole approximation. Special attention is paid to definition of the "time-ordered Hilbert space-valued stochastic integrals associated with a family of isometries from a Hilbert space to another one" and semigroup techniques. Some inequalities are derived, which are infinite degree versions of those known for finite dimensional Schrödinger operators with classical vector potentials.

1 INTRODUCTION

The purpose of this paper is to construct a functional integral representation for the heat semigroup with the infinitesimal generator given by a Hamiltonian which describes an interaction of a non-relativistic charged particle in a scalar potential and a quantized radiation field in the Coulomb gauge. This model plays an important role for interpretations of some physical phenomena, for example, "Lamb shift" ([1,32]).

There are many literatures which deal with models describing interactions of non-relativistic particles and a quantized field. For example, the Pauli-Fierz model of non-relativistic QED([1,2,3,6,16,23,32]), the Nelson model([7,20]), and persistent model([10,11]) etc.. For this kind of models, the problems of the removal of an infrared cut-off ([7,10,11,20]), asymptotic behaviors ([1,4,9,16]), resonance ([23]), scattering states ([3]), and dressed one electron states ([10,11]) have been discussed by many authors. These examples especially play an important role as interaction models of non-relativistic particles with quantized fields.

The Wiener path integral method has been studied extensively. In particular, with the help of stochastic integral, path integral representations for the heat semigroup generated by the Schrödinger Hamiltonian

$$\mathbf{H}_{cl} = \frac{1}{2}\sum_{\mu=1}^{d}(-iD_\mu - A_\mu)^2 + V \tag{1. 1}$$

with a vector potential A_μ and a scalar potential V were investigated. These are well known as the Feynman-Kac-Itô (FKI) formulas. The Hamiltonian $\mathbf{H}_{cl}$ has been studied extensively by many authors ([5]), who used the path integral method.

On the other hand, E.Nelson ([21,22]) introduced the "generalized path space" in connection with the construction of quantum field models from markoff fields (so called the functional integral method). In [14], the authors introduced a natural embedding of the relativistic Boson Fock space in d space dimensions into a constant time subspace in the L^2 space over the "generalized path space" in $d+1$ dimensions, by which, the Feynman-Kac-Nelson (FKN) formula relating the relativistic $P(\phi)_{1+1}$ theory to the Euclidean $P(\phi)_2$ was obtained. The "generalized path space" was studied more generally and abstracted by [19].

The classical path integral and the functional integral methods have been applied simultaneously to interaction models of non-relativistic particles and quantized fields. In [4], weak coupling limits for Hamiltonians describing a quantum system of finite number of non-relativistic particles interacting with a massive or massless bose field was studied, where the FKN formula and the Wiener path integrals were applied. And in [12,13], analyzing the Pauli-Fiertz model of non-relativistic QED by using the functional integrals and stochastic integrals was suggested. Our main problem is to give functional integral representations for the Pauli-Fiertz model.

The Hamiltonian, $\mathbf{H}_{\rho,B} + V \otimes I$, of the model which we consider is defined as an operator acting in the tensor product $\mathcal{M}_B$ of two Hilbert spaces $L^2(\mathbb{R}^d)$ and $\mathcal{F}(\mathcal{W})$ by

$$\mathbf{H}_{\rho,B} = \frac{1}{2}\sum_{\mu=1}^{d}(-iD_\mu \otimes I - A_\mu(\rho(\cdot)))^2 + I \otimes d\Gamma_B(\tilde{\omega}_B). \tag{1. 2}$$

Here $\mathcal{F}(\mathcal{W})$ denotes the Boson Fock space over $\mathcal{W} = \underbrace{L^2(\mathbb{R}^d) \oplus ... \oplus L^2(\mathbb{R}^d)}_{d-1}$, $A_\mu(\rho(\cdot))$ the μ-th direction time-zero radiation field with an ultraviolet cut-off function ρ in the Coulomb gauge, $d\Gamma_B(\tilde{\omega}_B)$ is the free Hamiltonian of the quantized radiation field and V is a scalar potential (see section 3). Comparing (1.1) and (1.2), functional integral representations for $e^{-t\mathbf{H}_{\rho,B}}$ seem to rely on the FKN and the FKI formulas heavily. Actually, as it will become clear later, these formulas are fundamental in this article.

In [1,2,3,6,16], instead of $\mathbf{H}_{\rho,B}$, the Hamiltonian $\mathbf{H}^D_{\rho,B}$ defined by taking the dipole approximation for $\mathbf{H}_{\rho,B}$ was studied. This approximation implies replacing $\rho(x)$ in $\mathbf{H}_{\rho,B}$ with $\rho(0)$;

$$\mathbf{H}^D_{\rho,B} = \frac{1}{2}\sum_{\mu=1}^{d}(-iD_\mu \otimes I - I \otimes A_\mu(\rho(0)))^2 + I \otimes d\Gamma_B(\tilde{\omega}_B).$$

However, for the original Hamiltonian $\mathbf{H}_{\rho,B}$, there are few mathematically rigorous results ([23]). It can be thought that mathematical difficulties come from the coupling term of the derivative D_μ and $A_\mu(\rho(x))$ in the Hamiltonian $\mathbf{H}_{\rho,B}$.

The main strategy to achieve our goal will be certain semigroup idea and introducing the "time-ordered Hilbert space-valued stochastic integrals associated with a family of isometries". As in the method used in [14,21,22,26], we construct a unitary operator from $\mathcal{M}_B \equiv L^2(\mathbb{R}^d) \otimes \mathcal{F}(\mathcal{W})$ to the tensor product $\mathcal{M}$ of $L^2(\mathbb{R}^d)$ and the L^2-space over generalized path space. We define $\mathbf{H}_\rho$ as an operator acting in $\mathcal{M}$ by the unitary transform of $\mathbf{H}_{\rho,B}$ restricted to some domain. Supposing some regularity conditions for ultraviolet cut-off functions ρ's, we shall show that the contraction semigroup generated by a self-adjoint extension of $\mathbf{H}_{\rho,0}$(see below) can be constructed on $\mathcal{M}$. Applying the FKN, the FKI formulas and the time ordered stochastic integral, the functional integral representation for $\left\langle F, e^{-t\mathbf{H}_\rho} G\right\rangle_{\mathcal{M}}$, $F, G \in \mathcal{M}$, is obtained.

The outline of the present paper is as follows. In Section II, following the standard stochastic integral procedure, we extend stochastic integrals to Hilbert space valued one and define "time-ordered Hilbert space-valued stochastic integral associated with a family of isometries from a Hilbert space to another one " (Theorem 2.5). In Section III, we introduce polarization vectors $e^r, r = 1, ..., d-1$. Two Hilbert spaces $[\widetilde{\mathcal{H}}_{-1}]$ and $[\widetilde{\mathcal{H}}_{-2}]$ are defined for given polarization vectors, and we construct a unitary operator from $\mathcal{M}_B$ to $\mathcal{M} = L^2(\mathbb{R}^d)\otimes L^2(Q_{-1}, d\mu_{-1})$(Theorem 3.1). The Hilbert space $L^2(Q_{-1}, d\mu_{-1})$ is the L^2-space over the underlying measure space for the Gaussian random process indexed by $[\widetilde{\mathcal{H}}_{-1}]$. Moreover, using a natural embedding of $L^2(Q_{-1}, d\mu_{-1})$ into a constant time subspace in $L^2(Q_{-2}, d\mu_{-2})$, and the Markoff property for some projection operators on $L^2(Q_{-2}, d\mu_{-2})$, we derive a simple extension of the FKN formula(Proposition 3.4). The Hilbert space $L^2(Q_{-2}, d\mu_{-2})$ is the L^2-space over the underlying measure espace for the Gaussian random process indexed by $[\widetilde{\mathcal{H}}_{-2}]$. In a formal definition, the Hamiltonian $\mathbf{H}_\rho$ shall be given as an operator acting in $\mathcal{M}$((3.4));

$$\begin{aligned}\mathbf{H}_\rho &\equiv \mathbf{H}_{\rho,0} + I\otimes\mathbf{H}_0,\\ \mathbf{H}_{\rho,0} &\equiv \frac{1}{2}\sum_{\mu=1}^{d}\left(-iD_\mu\otimes I - \phi^\rho_{\mathcal{F},\mu}\right)^2.\end{aligned}$$

Moreover it is shown that $\mathbf{H}_\rho$ is the unitary transform of $\mathbf{H}_{\rho,B}$ restricted to some domain (Theorem 3.1). In Section IV, we construct the contraction C_0-semigroup $G_\rho(t)$ on $\mathcal{M}$ such that the infinitesimal generator $\widetilde{\mathbf{H}}_{\rho,0}$ is a self-adjoint extension of the formally defined Hamiltonian $\mathbf{H}_{\rho,0}$ (Lemmas 4.6,4.7 and 4.8). We give a rigorous definition of $\mathbf{H}_\rho$ in terms of the form sum $\dot{+}$ of $\widetilde{\mathbf{H}}_{\rho,0}$ and $I\otimes\mathbf{H}_0$. Applying the Trotter product formula([18]), the time-ordered stochastic integral, and the FKN formula, a functional integral representation for the heat semigroup generated by an extended self-adjoint Hamiltonian of $\mathbf{H}_\rho + I\otimes V$ are derived in Theorem 4.10, where V is a suitable scalar potential. Moreover, they are extended for a more general class of potentials in Theorem 4.12. In Section V, we derive some inequalities which are known in the classical case as a diamagnetic inequality ([5,31]) and an abstract Kato's inequality ([15,27,29]) through the functional integral representation. In Section VI, we give some remarks, comparing our model with the classical one ([31]) and scalar field theory ([26]).

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2 TIME ORDERED STOCHASTIC INTEGRAL

In this section, we extend the standard stochastic integral to a Hilbert space-valued one and introduce the "time-ordered Hilbert space-valued stochastic integral associated with a family of isometries". (A general reference is [31])

For a Hilbert space $\mathcal{X}$ over $\mathbb{C}$, we denote the inner product and the associated norm by $< *, \cdot >_\mathcal{X}$ and $||\cdot||_\mathcal{X}$, respectively. The inner product is linear in $\cdot$ and antilinear in $*$. The domain of an operator A is denoted by $D(A)$. The notation $C(\mathbb{R}^d;\mathcal{X})$ denotes the space of strongly continuous functions from $\mathbb{R}^d$ to the Hilbert space $\mathcal{X}$. For $n = 1, 2, ...$, we denote by $C^n(\mathbb{R}^d;\mathcal{X})$ the subspace of n-times strongly differentiable functions in $C(\mathbb{R}^d;\mathcal{X})$ and define

$$C^n_b(\mathbb{R}^d;\mathcal{X}) = \left\{f\in C^n(\mathbb{R}^d;\mathcal{X})\,\middle|\,\sup_{|k|\le n, x\in\mathbb{R}^d}\left\|\partial^k f(x)\right\|_\mathcal{X} < \infty\right\},$$

$$H^n(\mathbb{R}^d;\mathcal{X}) \;=\; \left\{f\in C^n(\mathbb{R}^d;\mathcal{X}) \,\middle|\, \left\|\partial^k f(\cdot)\right\|_{\mathcal{X}} \in L^2(\mathbb{R}^d), |k|\le n\right\},$$

where $k=(k_1,k_2,...,k_d)$ is a multi-index ,$|k|=k_1+k_2+...+k_d$, and the derivative $\partial^k = \partial_1^{k_1}\partial_2^{k_2}...\partial_d^{k_d}$ is taken in the strong topology in $\mathcal{X}$. We fix probabilistic notations. Let (Ω, Db) be a probability space for d-dimensional Brownian motion $b(t)=(b_\mu(t))_{1\le\mu\le d, t\ge 0}$ and $d\mu$ be the Wiener measure on $\mathbb{R}^d\times\Omega$ defined by $d\mu = dx\otimes Db$. Let E denote the expectation value with respect to (Ω, Db). Following [24, XIII.16], we use the following identification;

$$L^2(M,dm)\otimes\mathcal{X}\cong\int_M^{\oplus}\mathcal{X}dm.$$

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$.

Lemma 2.1 *Let $f\in C_b^1(\mathbb{R}^d;\mathcal{H})$ and define*

$$\mathbf{J}_n^\mu(f,b)=\sum_{k=1}^{2^n} f\left(b\left(\frac{k-1}{2^n}t\right)\right)\left\{b_\mu\left(\frac{k}{2^n}t\right)-b_\mu\left(\frac{k-1}{2^n}t\right)\right\}, t\ge 0, \mu=1,...,d.$$

Then the strong limit

$$s-\lim_{n\to\infty}\mathbf{J}_n^\mu(f,b)\equiv\int_0^t f(b(s))db_\mu$$

exists in $L^2(\Omega;\mathcal{H})$. Moreover, for any $g\in C_b^1(\mathbb{R}^d;\mathcal{H})$,

$$\left\langle\int_0^t f(b(s))db_\mu,\int_0^t g(b(s))db_\nu\right\rangle_{L^2(\Omega;\mathcal{H})}=\delta_{\mu\nu}E\left(\int_0^t\left(f(b(s)),g(b(s))\right)_{\mathcal{H}}ds\right), \tag{2. 1}$$

where $\delta_{\mu\nu}$ is Kroneker's delta.

Proof: In the same way as in the proof of [31,Theorem 14.2], one can see that $\{\mathbf{J}_n^\mu(f,b)\}_{n\ge1}$ is a Cauchy sequence in $L^2(\Omega;\mathcal{H})$. Hence the strong limit of $\mathbf{J}_n^\mu(f,b)$ exists in $L^2(\Omega;\mathcal{H})$. One can see that

$$\langle\mathbf{J}_n^\mu(f,b),\mathbf{J}_n^\nu(g,b)\rangle_{L^2(\Omega;\mathcal{H})}=E\left(\sum_{k=1}^{2^n}\frac{t}{2^n}\left\langle f\left(b\left(\frac{k-1}{2^n}t\right)\right),g\left(b\left(\frac{k-1}{2^n}t\right)\right)\right\rangle_{\mathcal{H}}\right)\delta_{\mu\nu}.$$

Since $(f(b(s)),g(b(s)))_{\mathcal{H}}$ is continuous in s a.s.$b\in\Omega$, we have

$$\lim_{n\to\infty}\sum_{k=1}^{2^n}\frac{t}{2^n}\left\langle f\left(b\left(\frac{k-1}{2^n}t\right)\right),g\left(b\left(\frac{k-1}{2^n}t\right)\right)\right\rangle_{\mathcal{H}} = \int_0^t ds\,(f(b(s)),g(b(s)))_{\mathcal{H}},$$

$$a.s.b\in\Omega.$$

Moreover,

$$\left|\sum_{k=1}^{2^n}\frac{t}{2^n}\left\langle f\left(b\left(\frac{k-1}{2^n}t\right)\right),g\left(b\left(\frac{k-1}{2^n}t\right)\right)\right\rangle_{\mathcal{H}}\right|\le c_0c_0't,$$

where $c_0=\sup_{x\in\mathbb{R}^d}||f(x)||_{\mathcal{H}}$ and $c_0'=\sup_{x\in\mathbb{R}^d}||g(x)||_{\mathcal{H}}$. Hence the Lebesgue dominated convergence theorem yields (2.1). □

We call $\int_0^t f(b(s))db_\mu$ the " $\mathcal{H}$-valued stochastic integral for f".

Remark 2.2 (1) *As in [31,p152], Lemma 2.1 suggests that one can extend the definition of* $\int_0^t f(b(s))db_\mu$ *from* $C_b^1(\mathbb{R}^d;\mathcal{H})$ *to arbitrary functions* f *such that*

$$\begin{aligned}\left\|\int_0^t f(b(s))db_\mu\right\|^2_{L^2(\Omega;\mathcal{H})} &= E\left(\int_0^t \|f(b(s))\|^2_{\mathcal{H}}ds\right)\\ &= \int_0^t \left(\int_{\mathbb{R}^d} dx(2\pi s)^{-\frac{d}{2}}\|f(x)\|^2_{\mathcal{H}}e^{-\frac{x^2}{2s}}\right)ds < \infty.\end{aligned}$$

(2) *In an obvious way, we can extend* $\int_0^t f(b(s))db_\mu$ *to* $\int_t^s f(b(s))db_\mu$. *Then for* $[t_1,t_2)\cap(t_3,t_4]=\phi$ *and* $f,g\in C_b^1(\mathbb{R}^d;\mathcal{H})$

$$\left\langle \int_{t_1}^{t_2} f(b(s))db_\mu, \int_{t_3}^{t_4} g(b(s))db_\nu \right\rangle_{L^2(\Omega,\mathcal{H})} = 0. \tag{2. 2}$$

(3) *From (2.1) and (2.2) it follows that* $\int_0^t f(b(s))db_\mu$ *is strongly continuous in* t *in* $L^2(\Omega;\mathcal{H})$.

Lemma 2.3 *Let* $f\in C_b^2(\mathbb{R}^d;\mathcal{H})$ *and define for* $t\geq 0,\mu=1,...,d,$

$$\mathbf{S}_n^\mu(f,b)=\sum_{k=1}^{2^n}\frac{1}{2}\left\{f\left(b\left(\frac{k}{2^n}t\right)\right)+f\left(b\left(\frac{k-1}{2^n}t\right)\right)\right\}\left\{b_\mu\left(\frac{k}{2^n}t\right)-b_\mu\left(\frac{k-1}{2^n}t\right)\right\}.$$

Then

$$s-\lim_{n\to\infty}\mathbf{S}_n^\mu(f,b)=\int_0^t f(b(s))db_\mu+\frac{1}{2}\int_0^t(\partial_\mu f)(b(s))ds \tag{2. 3}$$

in $L^2(\Omega;\mathcal{H})$, *where* $\int_0^t(\partial_\mu f)(b(s))ds$ *is the Bochner integral of* $L^2(\Omega;\mathcal{H})$*-valued function* $(\partial_\mu f)(b(\cdot))$ *on* $\mathbb{R}^d$.

Proof: We divide $\mathbf{S}_n^\mu(f,b)$ in two parts as follows

$$\begin{aligned}\mathbf{S}_n^\mu(f,b) &= \sum_{k=1}^{2^n} f\left(b\left(\frac{k-1}{2^n}t\right)\right)\left\{b_\mu\left(\frac{k}{2^n}t\right)-b_\mu\left(\frac{k-1}{2^n}t\right)\right\}\\ &\quad+\sum_{k=1}^{2^n}\frac{1}{2}\left\{f\left(b\left(\frac{k}{2^n}t\right)\right)-f\left(b\left(\frac{k-1}{2^n}t\right)\right)\right\}\left\{b_\mu\left(\frac{k}{2^n}t\right)-b_\mu\left(\frac{k-1}{2^n}t\right)\right\}.\end{aligned} \tag{2. 4}$$

Similarly to Lemma 2.1 ([31, p160]), it is not hard to see that the two terms on the right hand side (r.h.s.) of (2.4) strongly converges to the two terms on the r.h.s.of (2.3) in $L^2(\Omega;\mathcal{H})$, respectively. □

Remark 2.4 *One can easily see that for* $f\in C_b^1(\mathbb{R}^d;\mathcal{H})$,

$$s-\lim_{n\to\infty}\sum_{k=1}^{[2^n t]} f\left(b\left(\frac{k-1}{2^n}\right)\right)\left\{b_\mu\left(\frac{k}{2^n}\right)-b_\mu\left(\frac{k-1}{2^n}\right)\right\}=\int_0^t f(b(s))db_\mu.$$

Moreover, for $f \in C_b^2(\mathbb{R}^d;\mathcal{H})$,

$$s-\lim_{n\to\infty}\sum_{k=1}^{[2^n t]}\frac{1}{2}\left\{f\left(b\left(\frac{k-1}{2^n}\right)\right)+f\left(b\left(\frac{k}{2^n}\right)\right)\right\}\left\{b_\mu\left(\frac{k}{2^n}\right)-b_\mu\left(\frac{k-1}{2^n}\right)\right\}$$
$$=\int_0^t f(b(s))db_\mu+\frac{1}{2}\int_0^t(\partial_\mu f)(b(s))ds,$$

where $[\,\cdot\,]$ *denotes the Gauss symbol.*

Let $\mathcal{K}$ be a Hilbert space and $\{I_t\}_{t\geq 0}$ be a family of isometries from $\mathcal{H}$ to $\mathcal{K}$, so that $I_t^* I_t = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ is the identity operator in $\mathcal{H}$. we denote $I_t f$ by f_t for simplicity. For $f \in C_b^1(\mathbb{R}^d;\mathcal{H})$ and the isometries I_t, we define the $\mathcal{K}$-valued stochastic integral $\widehat{\mathbf{J}}_n^\mu(f,b)$ by

$$\widehat{\mathbf{J}}_n^\mu(f,b)=\sum_{k=1}^{2^n}\int_{\frac{k-1}{2^n}t}^{\frac{k}{2^n}t} f_{\frac{k-1}{2^n}t}(b(s))db_\mu.$$

Theorem 2.5 *Let* $f \in C_b^1(\mathbb{R}^d;\mathcal{H})$ *such that for all sufficiently small* $s \geq 0$,

$$\left\|I_{t+s}^* I_t f(x)-f(x)\right\|_{\mathcal{H}} \leq sM(f), \tag{2. 5}$$

where $M(f)$ *is a positive constant independent of* $x \in \mathbb{R}^d$ *and* $t \geq 0$. *Then*

$$s-\lim_{n\to\infty}\widehat{\mathbf{J}}_n^\mu(f,b)\equiv\int_0^t \widehat{I}_{0\to t} f(b(s))db_\mu$$

exists in $L^2(\Omega;\mathcal{K})$.

Proof: Fix $f \in C_b^1(\mathbb{R}^d;\mathcal{H})$ and put $c_0=\sup_{x\in\mathbb{R}^d}\|f(x)\|_{\mathcal{H}}$. It is sufficient to show that the family $\{\widehat{\mathbf{J}}_n^\mu(f,b)\}_{n\geq 1}$ is a Cauchy sequence in $L^2(\Omega;\mathcal{K})$. By Lemma 2.1, (2.2), (2.5) and the fact $I_t^* I_t = I_{\mathcal{H}}$, we can see that

$$\begin{aligned}
&\left\|\widehat{\mathbf{J}}_n^\mu(f,b)-\widehat{\mathbf{J}}_{n+1}^\mu(f,b)\right\|_{L^2(\Omega;\mathcal{K})}^2\\
&=\left\|\sum_{k=1}^{2^n}\int_{\frac{2k-1}{2^{n+1}}t}^{\frac{2k}{2^{n+1}}t}\left(f_{\frac{2k-1}{2^{n+1}}t}(b(s))-f_{\frac{2k-2}{2^{n+1}}t}(b(s))\right)db_\mu\right\|_{L^2(\Omega;\mathcal{K})}^2\\
&=\sum_{k=1}^{2^n}E\left(\int_{\frac{2k-1}{2^{n+1}}t}^{\frac{2k}{2^{n+1}}t}\left\|f_{\frac{2k-1}{2^{n+1}}t}(b(s))-f_{\frac{2k-2}{2^{n+1}}t}(b(s))\right\|_{\mathcal{K}}^2 ds\right)\\
&\leq 2\sum_{k=1}^{2^n}E\left(\int_{\frac{2k-1}{2^{n+1}}t}^{\frac{2k}{2^{n+1}}t}\left\|\left(I_{\mathcal{H}}-I_{\frac{2k-1}{2^{n+1}}t}^* I_{\frac{2k-2}{2^{n+1}}t}\right)f(b(s))\right\|_{\mathcal{H}}\|f(b(s))\|_{\mathcal{H}}ds\right)\\
&\leq 2\sum_{k=1}^{2^n}E\left(\int_{\frac{2k-1}{2^{n+1}}t}^{\frac{2k}{2^{n+1}}t}\frac{tM(f)}{2^{n+1}}c_0 ds\right)\\
&=\frac{M(f)t^2c_0}{2^{n+1}}.
\end{aligned}$$

Then we have

$$\left\|\widehat{\mathbf{J}}^{\mu}_{m}(f,b)-\widehat{\mathbf{J}}^{\mu}_{n}(f,b)\right\|_{L^2(\Omega;\mathcal{K})}\le t\sqrt{M(f)c_0}\sum_{k=n}^{m-1}\left(\frac{1}{\sqrt{2}}\right)^k.$$

Hence $\{\widehat{\mathbf{J}}^{\mu}_{n}(f,b)\}_{n\ge 1}$ is Cauchy in $L^2(\Omega;\mathcal{K})$ as required. □

We call $\int_0^t \widehat{I}_{0\to t}f(b(s))db_\mu$ the "time-ordered $\mathcal{K}$-valued stochastic integral associated with $\{I_t\}_{t\ge 0}$".

We conclude the present section with stochastic integrals over the Wiener paths. Defining $\int_0^t f(b(s))db_\mu$ as a strong limit in $L^2(\Omega;\mathcal{H})$, for $f\in H^1(\mathbb{R}^d;\mathcal{H})$, we can define $\int_0^t f(\omega(s))d\omega_\mu$ as a strong limit in $L^2(\mathbb{R}^d\times\Omega;\mathcal{H})$ as follows

$$s-\lim_{n\to\infty}\sum_{k=1}^{2^n} f\left(\omega\left(\frac{k-1}{2^n}t\right)\right)\left\{\omega_\mu\left(\frac{k}{2^n}t\right)-\omega_\mu\left(\frac{k-1}{2^n}t\right)\right\}\equiv\int_0^t f(\omega(s))d\omega_\mu. \qquad (2.\ 6)$$

The existence of this limit can be proven in the same way as in the proof of Lemma 2.1. For $f,g\in H^1(\mathbb{R}^d;\mathcal{H})$, we have

$$\begin{aligned}\left\langle\int_0^t f(\omega(s))d\omega_\mu,\int_0^t g(\omega(s))d\omega_\nu\right\rangle_{L^2(\mathbb{R}^d\times\Omega;\mathcal{H})} &= \delta_{\mu\nu}\widetilde{E}\left(\int_0^t\langle f(\omega(s)),g(\omega(s))\rangle_{\mathcal{H}}\,ds\right)\\ &= t\delta_{\mu\nu}\int_{\mathbb{R}^d}\langle f(x),g(x)\rangle_{\mathcal{H}}\,dx, \qquad (2.\ 7)\end{aligned}$$

where $\widetilde{E}$ denotes the expectation value with respect to $(\mathbb{R}^d\times\Omega,d\mu)$. Eq.(2.7) allows us to extend the definition of $\int_0^t f(\omega(s))d\omega_\mu$ to f such that the r.h.s. of (2.7) is finite.

3 PROBABILISTIC DESCRIPTION OF THE TIME-ZERO RADIATION FIELD WITH THE COULOMB GAUGE

In this section we define a model which describes a quantum system of a non-relativistic charged particle interacting with a quantized radiation field with the Coulomb gauge.

For mathematical generality, we consider the situation where the charged particle moves in $\mathbb{R}^d$ and the quantized radiation field is over $\mathbb{R}^d$. We define polarization vectors e^r ($r=1,...,d-1$) as measurable functions $e^r:\mathbb{R}^d\to\mathbb{R}^d$ such that

$$e^r(k)\cdot e^s(k)=\delta_{rs},\quad k\cdot e^r(k)=0,\quad a.e.k\in\mathbb{R}^d.$$

In what follows, fix the polarization vectors e^r. We introduce two Hilbert spaces $[\widetilde{\mathcal{H}}_{-1}]$ and $[\widetilde{\mathcal{H}}_{-2}]$ as follows. First we define two real Hilbert spaces $\mathcal{H}_{-1}$ and $\mathcal{H}_{-2}$ by

$$\mathcal{H}_{-1}\equiv\left\{f\in\mathcal{S}'_r(\mathbb{R}^d)\left|\int_{\mathbb{R}^d}\frac{|\hat{f}(k)|^2}{|k|}dk<\infty\right.\right\},$$

$$\mathcal{H}_{-2}\equiv\left\{f\in\mathcal{S}'_r(\mathbb{R}^{d+1})\left|\int_{\mathbb{R}^{d+1}}\frac{|\hat{f}(k)|^2}{|k|^2}dk<\infty\right.\right\},$$

where $\mathcal{S}_r'(\mathbb{R}^n)$ denotes the set of real tempered distributions on $\mathbb{R}^n (n = d, d+1)$ and $\hat{}$ denotes the Fourier transformation ($^\vee$ the inverse Fourier transformation) from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$:

$$\hat{f}(k) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) e^{-ikx} dx.$$

Put

$$\begin{aligned}
\widetilde{\mathcal{H}}_{-1} &= \underbrace{\mathcal{H}_{-1} \oplus \ldots \oplus \mathcal{H}_{-1}}_{d}, \\
\widetilde{\mathcal{H}}_{-2} &= \underbrace{\mathcal{H}_{-2} \oplus \ldots \oplus \mathcal{H}_{-2}}_{d}.
\end{aligned}$$

We introduce bilinear forms $(\cdot,\cdot)_{-1}$ and $(\cdot,\cdot)_{-2}$ in $\widetilde{\mathcal{H}}_{-1}$ and $\widetilde{\mathcal{H}}_{-2}$ by

$$\begin{aligned}
(f,g)_{-1} &= \sum_{\mu,\nu=1}^{d} \int_{\mathbb{R}^d} \frac{d_{\mu\nu}(k) \bar{\hat{f}}_\mu(k) \hat{g}_\nu(k)}{|k|} dk, \\
(f,g)_{-2} &= 2 \sum_{\mu,\nu=1}^{d} \int_{\mathbb{R}^{d+1}} \frac{d_{\mu\nu}(k) \bar{\hat{f}}_\mu(k) \hat{g}_\nu(k)}{|k|^2} dk,
\end{aligned}$$

respectively, where f_μ and g_μ are the μ-th components of f and g, $^-$ denotes the complex conjugate and

$$\begin{aligned}
d_{\mu\nu}(k) &\equiv \sum_{r=1}^{d-1} e_\mu^r(k) e_\nu^r(k), \\
&= \delta_{\mu\nu} - \frac{k_\mu k_\nu}{|k|^2}.
\end{aligned}$$

We denote the associated semi-norms by $|\cdot|_{-1}$ and $|\cdot|_{-2}$ respectively and put

$$\begin{aligned}
N_{-1} &= \left\{ f \in \widetilde{\mathcal{H}}_{-1} \middle| \, |f|_{-1} = 0 \right\}, \\
N_{-2} &= \left\{ f \in \widetilde{\mathcal{H}}_{-2} \middle| \, |f|_{-2} = 0 \right\}.
\end{aligned}$$

Then we define pre-Hilbert spaces by the quotient spaces

$$\begin{aligned}
[\widetilde{\mathcal{H}}_{-1}] &= \widetilde{\mathcal{H}}_{-1} / N_{-1}, \\
[\widetilde{\mathcal{H}}_{-2}] &= \widetilde{\mathcal{H}}_{-2} / N_{-2},
\end{aligned}$$

with inner products $< \cdot,\cdot >_{-1}$ and $< \cdot,\cdot >_{-2}$ defined by

$$\begin{aligned}
\langle \pi_{-1}(f), \pi_{-1}(g) \rangle_{-1} &\equiv (f,g)_{-1}, \\
\langle \pi_{-2}(f), \pi_{-2}(g) \rangle_{-2} &\equiv (f,g)_{-2}.
\end{aligned}$$

Here $\pi_{-1}(f)$ and $\pi_{-2}(f)$ denote the equivalence classes of f in $\widetilde{\mathcal{H}}_{-1}$ and $\widetilde{\mathcal{H}}_{-2}$, respectively. We denote the norms associated with the inner products $<\cdot,\cdot>_{-1}$ and $<\cdot,\cdot>_{-2}$ by $||\cdot||_{-1}$ and $||\cdot||_{-2}$, respectively. The Hilbert spaces constructed by the completions of $[\widetilde{\mathcal{H}}_{-1}]$ and $[\widetilde{\mathcal{H}}_{-2}]$ with respect to $||\cdot||_{-1}$ and $||\cdot||_{-2}$ are denoted by the same symbols.
Let $\{\phi_{-1}(\pi_{-1}(f))|f \in \widetilde{\mathcal{H}}_{-1}\}$ and $\{\phi_{-2}(\pi_{-2}(f))|f \in \widetilde{\mathcal{H}}_{-2}\}$ be the Gaussian random processes indexed by $[\widetilde{\mathcal{H}}_{-1}]$ and $[\widetilde{\mathcal{H}}_{-2}]$ such that the characteristic functions are given by

$$\int_{Q_j} e^{i\phi_j(\pi_j(f))} d\mu_j = e^{-\frac{1}{4}||\pi_j(f)||_j^2}, \quad j=-1,-2,$$

where $(Q_{-1}, d\mu_{-1})$ and $(Q_{-2}, d\mu_{-2})$ denote the underlying measure spaces of these processes, respectively. It is well known that $L^2(Q_j, d\mu_j)$ has the orthogonal decomposition

$$L^2(Q_j, d\mu_j) = \bigoplus_{n=0}^{\infty} \Gamma_n([\widetilde{\mathcal{H}}_j])$$

with

$$\begin{aligned} &\Gamma_0([\widetilde{\mathcal{H}}_j]) = \mathbb{C}, \\ &\Gamma_n([\widetilde{\mathcal{H}}_j]) = L\{:\phi_j(\pi_j(f_1))\phi_j(\pi_j(f_2))...\phi_j(\pi_j(f_n)): |f_k \in \widetilde{\mathcal{H}}_j, k=1,..,n\}^-, \quad n \geq 1, \end{aligned}$$

where $L\{...\}$ denotes the linear span of the vectors in $\{...\}$ over $\mathbb{C}$, $^-$ the closure in $L^2(Q_j, d\mu_j)$ and $:\cdot:$ the "Wick product" ([4]). We denote the complexifications of $[\widetilde{\mathcal{H}}_j]$ by $[\widetilde{\mathcal{H}}_j]_{\mathbb{C}}$. Suppose that T is a contraction operator from $[\widetilde{\mathcal{H}}_i]_{\mathbb{C}}$ to $[\widetilde{\mathcal{H}}_j]_{\mathbb{C}}$. Corresponding to each such T we can define the contraction operator $\Gamma(T): L^2(Q_i; d\mu_i) \longrightarrow L^2(Q_j; d\mu_j)$ by

$$\begin{aligned} \Gamma(T)\Omega_i &= 0, \\ \Gamma(T):\phi_i(\pi_i(f_1))...\phi_i(\pi_i(f_n)): &= :\phi_j(T\pi_j(f_1))\phi_j(T\pi_j(f_2))...\phi_j(T\pi_j(f_n)):. \end{aligned}$$

For a nonnegative self-adjoint operator $A: [\widetilde{\mathcal{H}}_i]_{\mathbb{C}} \longrightarrow [\widetilde{\mathcal{H}}_i]_{\mathbb{C}}$ $(i=-1,-2)$ we define $d\Gamma(A)$ by

$$\begin{aligned} d\Gamma(A)\Omega_i &= 0, \\ d\Gamma(A):\phi_i(\pi_i(f_1))...\phi_i(\pi_i(f_n)): &= :\phi_i(A\pi_i(f_1))\phi_i(\pi_i(f_2))...\phi_i(\pi_i(f_n)): \\ &\quad +:\phi_i(\pi_i(f_1))\phi_i(A\pi_i(f_2))...\phi_i(\pi_i(f_n)): \\ &\quad +...+:\phi_i(\pi_i(f_1))\phi_i(\pi_i(f_2))...\phi_i(A\pi_i(f_n)):, \\ &\quad \pi_i(f_k) \in D(A), k=1,...,n, \end{aligned}$$

where Ω_i denotes the constant function 1 in $L^2(Q_i, d\mu_i)$. It is well known that $d\Gamma(A)$ has unique self-adjoint extension in $L^2(Q_i; d\mu_i)$. We denote it by the same symbol $d\Gamma(A)$. We set $L^2(Q_{-1}, d\mu_{-1}) = \mathcal{F}$, $L^2(Q_{-2}, d\mu_{-2}) = \mathcal{E}$, $\phi_{-1}(\cdot) = \phi_{\mathcal{F}}(\cdot)$, $\phi_{-2}(\cdot) = \phi_{\mathcal{E}}(\cdot)$ and $\Omega_{-1} = \Omega_{\mathcal{F}}$ and $\Omega_{-2} = \Omega_{\mathcal{E}}$. Put

$$\mathcal{F}^N = \bigoplus_{n=0}^{N} \Gamma_n([\widetilde{\mathcal{H}}_{-1}]) \bigoplus_{n>N+1} \{0\}$$

and define $\mathcal{F}^\infty$ by

$$\mathcal{F}^\infty = \bigcup_{N=0}^{\infty} \mathcal{F}^N.$$

The standard Boson Fock space ([28,X.7]) over $\mathcal{W} = \underbrace{L^2(\mathbb{R}^d) \oplus \ldots \oplus L^2(\mathbb{R}^d)}_{d-1}$ is defined by ([2,3,16])

$$\mathcal{F}(\mathcal{W}) = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(\mathcal{W}),$$
$$\mathcal{F}_n(\mathcal{W}) = \otimes_s^n \mathcal{W}, \ n \geq 1, \quad \mathcal{F}_0 = \mathbb{C},$$

where $\otimes_s^n$ denotes the n-fold symmetric tensor product. The vacuum vector Ω in $\mathcal{F}(\mathcal{W})$ is defined by

$$\Omega = \{1, 0, 0, \ldots.\}.$$

The Boson Fock space $\mathcal{F}(\mathcal{W})$ describes a Hilbert space of state vectors for the quantized radiation field with the Coulomb gauge. Let

$$\mathcal{F}^N(\mathcal{W}) = \bigoplus_{n=0}^{N} \mathcal{F}_n(\mathcal{W}) \bigoplus_{n>N+1} \{0\}.$$

Then a finite particle subspace is defined by

$$\mathcal{F}^\infty(\mathcal{W}) = \bigcup_{N=0}^{\infty} \mathcal{F}^N(\mathcal{W}).$$

The annihilation operator $a(f)$ and the creation operator $a^\dagger(f)$ $(f \in \mathcal{W})$ ([25]) act on the finite particle subspace and leave it invariant with the canonical commutation relations (CCR): for $f, g \in \mathcal{W}$

$$\begin{aligned} [a(f), a^\dagger(g)] &= \left\langle \bar{f}, g \right\rangle_{\mathcal{W}}, \\ [a^\sharp(f), a^\sharp(g)] &= 0, \end{aligned}$$

where $[A, B] = AB - BA$, $a^\sharp$ denotes either a or $a^\dagger$. Furthermore,

$$\left\langle a^\dagger(f)\Phi, \Psi \right\rangle_{\mathcal{F}(\mathcal{W})} = \left\langle \Phi, a(\bar{f})\Psi \right\rangle_{\mathcal{F}(\mathcal{W})}, \quad \Phi, \Psi \in \mathcal{F}^\infty(\mathcal{W}).$$

For any contraction operator $A : \mathcal{W} \to \mathcal{W}$, the"second quantization of A" $\Gamma_B(A) : \mathcal{F}(\mathcal{W}) \to \mathcal{F}(\mathcal{W})$ is a bounded operator uniquely determined by

$$\begin{aligned} \Gamma_B(A)\Omega &= 0, \\ \Gamma_B(A)a^\dagger(f_1)a^\dagger(f_2)\ldots a^\dagger(f_n)\Omega &= a^\dagger(Af_1)a^\dagger(Af_2)\ldots a^\dagger(Af_n)\Omega. \end{aligned}$$

For a nonnegative self-adjoint operator σ in $\mathcal{W}$, the "second quantization of σ", $d\Gamma_B(\sigma)$, is defined by the infinitesimal generator of the C_0-semigroup $\{\Gamma_B(e^{-t\sigma})\}_{t\geq 0}$;

$$\Gamma_B(e^{-t\sigma}) = e^{-td\Gamma_B(\sigma)}.$$

We define the maximal multiplication operator ω_B in $L^2(\mathbb{R}^d)$ by

$$(\omega_B f)(k) = h(k)f(k),$$

where $h(k) = |k|$. Put $\tilde{\omega}_B = \underbrace{\omega_B \oplus ... \oplus \omega_B}_{d-1}$. Then $d\Gamma_B(\tilde{\omega}_B)$ will be the free Hamiltonian of the quantized radiation field. The second quantization of the identity operator $I_{\mathcal{W}}$ on $\mathcal{W}$, $d\Gamma(I_{\mathcal{W}})$, is called the number operator. The following inequality is well known

$$||a^\sharp(f)\Phi||_{\mathcal{F}(\mathcal{W})} \leq ||f||_{\mathcal{W}} \times ||(d\Gamma(I_{\mathcal{W}}) + I)^{\frac{1}{2}}\Phi||_{\mathcal{F}(\mathcal{W})}, \quad \Phi \in \mathcal{F}^\infty(\mathcal{W}). \tag{3. 1}$$

For $f \in \mathcal{H}_{-1}$ we define the μ-th direction time-zero radiation field $A_\mu(f)$ ($\mu = 1, ..., d$) by

$$A_\mu(f) = \frac{1}{\sqrt{2}}\left\{a^\dagger\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r \hat{f}}{\sqrt{h}}\right) + a\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r \tilde{\hat{f}}}{\sqrt{h}}\right)\right\}, \tag{3. 2}$$

where $\tilde{g}(k) = g(-k)$. For $g = (g_1, ..., g_d) \in \widetilde{\mathcal{H}}_{-1}$ we put

$$A(g) \equiv \sum_{\mu=1}^{d} A_\mu(g_\mu).$$

We give connection between $\mathcal{F}$ and $\mathcal{F}(\mathcal{W})$. Here we introduce the subspace $\mathcal{D}_0$ in $\widetilde{\mathcal{H}}_{-1}$ by

$$\mathcal{D}_0 = L\left\{f^r = (f_1^r, ..., f_d^r) \in \widetilde{\mathcal{H}}_{-1} \middle| f_\mu^r = \left(e_\mu^r \sqrt{h}\hat{f}\right)^\vee, \hat{f} \in C_0^\infty(\mathbb{R}^d\backslash\{0\}), r = 1, ..., d-1\right\},$$

where $C_0^\infty(\mathbb{R}^d\backslash\{0\})$ denotes the set of infinitely differentiable functions with compact support on $\mathbb{R}^d\backslash\{0\}$. Then it can be easily seen that $\mathcal{D}_0$ is dense in $\widetilde{\mathcal{H}}_{-1}$ with respect to the semi-norm $|\cdot|_{-1}$, which implies that $\pi_{-1}(\mathcal{D}_0)$ is dense in $[\widetilde{\mathcal{H}}_{-1}]$. Hence

$$L\{: \phi_{\mathcal{F}}(\pi_{-1}(f_1))...\phi_{\mathcal{F}}(\pi_{-1}(f_n)) : \Omega_{\mathcal{F}}, \Omega_{\mathcal{F}} | f_j \in \mathcal{D}_0, j = 1, ..., n, n \geq 1\}$$

is dense in $\mathcal{F}$. On the other hand, choosing $\rho^r = \left((e_1^r\sqrt{h}\hat{\rho})^\vee, ..., (e_d^r\sqrt{h}\hat{\rho})^\vee\right) \in \mathcal{D}_0$, it turns out that

$$\begin{aligned} A(\rho^r) &= \sum_{\mu=1}^{d} A_\mu\left(\left(e_\mu^r\sqrt{h}\hat{\rho}\right)^\vee\right) \\ &= \frac{1}{\sqrt{2}}\left\{a^\dagger\left(\overbrace{0 \oplus ... \oplus \underbrace{\hat{\rho}}_{r-th} \oplus ... \oplus 0}^{d-1}\right) + a\left(\overbrace{0 \oplus ... \oplus \underbrace{\tilde{\hat{\rho}}}_{r-th} \oplus ... \oplus 0}^{d-1}\right)\right\}. \end{aligned}$$

Then we see that

$$L\{: A(f_1)...A(f_n) : \Omega, \Omega | f_j \in \mathcal{D}_0, j = 1, ..., n, n \geq 1\}$$

is dense in $\mathcal{F}(\mathcal{W})$, where $: \cdot :$ denotes the "Wick product" in the Boson Fock space ([25,p226]). We define the operator ω in $\mathcal{H}_{-1}$ by

$$\widehat{\omega f}(k) = h(k)\hat{f}(k),$$

and put $\tilde{\omega} = \underbrace{\omega \oplus ... \oplus \omega}_{d}$. Furthermore, $[\tilde{\omega}] : [\widetilde{\mathcal{H}}_{-1}] \to [\widetilde{\mathcal{H}}_{-1}]$ is defined by

$$[\tilde{\omega}]\pi_{-1}(f) = \pi_{-1}(\tilde{\omega}f), \quad D([\tilde{\omega}]) = \{\pi_{-1}(f) \in [\widetilde{\mathcal{H}}_j] | \tilde{\omega}f \in \widetilde{\mathcal{H}}_{-1}\}.$$

Extend $[\tilde{\omega}] : [\widetilde{\mathcal{H}}_{-1}]_{\mathbb{C}} \to [\widetilde{\mathcal{H}}_{-1}]_{\mathbb{C}}$ as follows:

$$[\tilde{\omega}]\left(\pi_{-1}(f_1), \pi_{-1}(f_2)\right) = \left([\tilde{\omega}]\pi_{-1}(f_1), [\tilde{\omega}]\pi_{-1}(f_2)\right), f_1, f_2 \in \widetilde{\mathcal{H}}_1.$$

Then it is easy to see that $Ran\,([\tilde{\omega}] \pm i) = [\widetilde{\mathcal{H}}_{-1}]_{\mathbb{C}}$, which implies that $[\tilde{\omega}]$ is a self-adjoint operator in $[\widetilde{\mathcal{H}}_{-1}]_{\mathbb{C}}$.

Theorem 3.1 *There exists a unitary operator $\mathcal{U}$ from $\mathcal{F}(\mathcal{W})$ to $\mathcal{F}$ such that*

$$\begin{array}{ll} (a) & \mathcal{U}\Omega = \Omega_{\mathcal{F}}, \\ (b) & \mathcal{U}A(f)\mathcal{U}^{-1} = \phi_{\mathcal{F}}(f), \quad f \in \widetilde{\mathcal{H}}_{-1}, \\ (c) & \mathcal{U}\mathcal{F}_n(\mathcal{W}) = \Gamma_n([\widetilde{\mathcal{H}}_{-1}]), \\ (d) & \mathcal{U}d\Gamma_B(\tilde{\omega}_B)\mathcal{U}^{-1} = d\Gamma([\tilde{\omega}]), \\ (e) & \mathcal{U}d\Gamma_B(I_\mathcal{W})\mathcal{U}^{-1} = d\Gamma(I_\mathcal{F}), \end{array}$$

where $I_\mathcal{F}$ is the identity operator in $[\widetilde{\mathcal{H}}_{-1}]$.

Proof: For $f_j \in \mathcal{D}_0, j = 1, ..., n$, we define

$$\begin{aligned} \mathcal{U} : A(f_1)...A(f_n) : \Omega &= : \phi_\mathcal{F}(\pi_{-1}(f_1))...\phi_\mathcal{F}(\pi_{-1}(f_n)) : \Omega_\mathcal{F}, \\ \mathcal{U}\Omega &= \Omega_\mathcal{F}. \end{aligned}$$

One can easily show $\mathcal{U}$ can be uniquely extended to a unitary operator from $\mathcal{F}(\mathcal{W})$ to $\mathcal{F}$ with (a),(b) and (c). We shall show (d). Let

$$\begin{aligned} X_n &= L\{: \phi_\mathcal{F}(\pi_{-1}(f_1))...\phi_\mathcal{F}(\pi_{-1}(f_n)) : \Omega_\mathcal{F} | f_j \in \mathcal{D}_0, j = 1, ..., n\}, \\ Y_n &= L\{: A(f_1)...A(f_n) : \Omega | f_j \in \mathcal{D}_0, j = 1, ..., n\}. \end{aligned}$$

Since, as long as $\hat{\rho} \in C_0^\infty(\mathbb{R}^d \backslash \{0\})$, it follows that $\exp(-th)\hat{\rho} \in C_0^\infty(\mathbb{R}^d \backslash \{0\})$, one can see that $\exp\left(-td\Gamma_B(\tilde{\omega}_B)\right)$ leaves $\bigcup_{n=0}^\infty Y_n$ invariant. Hence $\bigcup_{n=0}^\infty Y_n$ is a core of $d\Gamma_B(\tilde{\omega}_B)$ ([25,Theorem X.49]). Moreover, since

$$\mathcal{U}\exp\left(-td\Gamma_B(\tilde{\omega}_B)\right)\mathcal{U}^{-1} = \exp\left(-t\mathcal{U}d\Gamma_B(\tilde{\omega}_B)\mathcal{U}^{-1}\right),$$

it follows that $\bigcup_{n=0}^{\infty} X_n$ is a core of $\mathcal{U}d\Gamma_B(\tilde{\omega}_B)\mathcal{U}^{-1}$. Noting that on $\bigcup_{n=0}^{\infty} X_n$

$$\mathcal{U}d\Gamma_B(\tilde{\omega}_B)\mathcal{U}^{-1} = d\Gamma([\tilde{\omega}]).$$

Thus (d) holds. The proof of (e) is similar to that of (d). □

We set $\mathbf{H}_0 = d\Gamma([\tilde{\omega}])$, $\mathbf{N} = d\Gamma(I_{\mathcal{F}})$. Following [26,Chapter III], we can give connection between $\mathcal{F}$ and $\mathcal{E}$. For $t \in \mathbb{R}$ we define the operator j_t by

$$\begin{aligned} j_t &: \mathcal{H}_{-1} \longrightarrow \mathcal{H}_{-2}, \\ j_t f &= \delta_t \otimes f, \quad f \in \mathcal{H}_{-1}, \end{aligned}$$

where δ_t is the one-dimensional delta function with mass at $\{t\}$. In momentum space,

$$\left(\widehat{j_t f}\right)(\vec{k}, k_0) = (2\pi)^{-\frac{1}{2}} \hat{f}(\vec{k}) e^{-itk_0},$$

where $(\vec{k}, k_0) \in \mathbb{R}^d \times \mathbb{R} = \mathbb{R}^{d+1}$. We put $\tilde{j}_t = j_t \oplus ... \oplus j_t$ and define

$$\begin{aligned} [\tilde{j}_t] &: [\widetilde{\mathcal{H}}_{-1}] \longrightarrow [\widetilde{\mathcal{H}}_{-2}], \\ [\tilde{j}_t]\pi_{-1}(f) &= \pi_{-2}(\tilde{j}_t f). \end{aligned}$$

It can be easily seen that $[\tilde{j}_t]$ is a linear isometry ([26,Proposition III.2]). Hence the range of $[\tilde{j}_t]$ is a closed subspace of $[\widetilde{\mathcal{H}}_{-2}]$. We denote the projection onto $Ran([\tilde{j}_t])$ by $[e_t]$. Let

$$U_{[a,b]} \equiv L\left\{\pi_{-2}(f) \in [\widetilde{\mathcal{H}}_{-2}] \,\middle|\, \pi_{-2}(f) \in Ran([\tilde{j}_t]), a \le t \le b\right\}.$$

We denote the projection onto the closure $\overline{U_{[a,b]}}$ by $[e_{[a,b]}]$.

Proposition 3.2 *([26,Propositions III.3 and III.4])*
(a) $[\tilde{j}_t][\tilde{j}_t]^* = [e_t]$.
(b) $[\tilde{j}_t]^*[\tilde{j}_s] = e^{-|t-s|[\tilde{\omega}]}$.
(c) Let $a \le b \le c$. Then

$$[e_a][e_b][e_c] = [e_a][e_c].$$

(d) Let $a \le b \le t \le c \le d$. Then

$$[e_{[a,b]}][e_t][e_{[c,d]}] = [e_{[a,b]}][e_{[c,d]}].$$

Proof: (a) is straightforwardly seen. Since we have

$$\begin{aligned} \left\langle [\tilde{j}_t]^*[\tilde{j}_s]\pi_{-1}(f), \pi_{-1}(g)\right\rangle_{-1} &= \left\langle \pi_{-2}(\tilde{j}_s f), \pi_{-2}(\tilde{j}_t g)\right\rangle_{-2} \\ &= \frac{1}{\pi} \sum_{\mu,\nu=1}^{d} \int_{\mathbb{R}^{d+1}} \frac{\bar{\hat{f}}_\mu(\vec{k})\hat{g}_\nu(\vec{k}) d_{\mu\nu}(\vec{k}) e^{i(t-s)k_0}}{|\vec{k}|^2 + k_0^2} d\vec{k} dk_0 \\ &= \sum_{\mu,\nu=1}^{d} \int_{\mathbb{R}^d} \frac{\bar{\hat{f}}_\mu(\vec{k})\hat{g}_\nu(\vec{k}) d_{\mu\nu}(\vec{k}) e^{-|t-s||\vec{k}|}}{|\vec{k}|} d\vec{k}, \end{aligned}$$

(b) holds. Eq.(c) follows from (a) and (b). For any $\pi_{-2}(f)$ and $\pi_{-2}(g)$, by the definition of $[e_{[a,b]}]$ and $[e_{[c,d]}]$, they can be presented as follows

$$\begin{aligned} [e_{[c,d]}]\pi_{-2}(f) &= \lim_{n\to\infty}\sum_{\alpha=1}^{N_n} f_{n_\alpha}, \quad f_{n_\alpha}\in Ran([e_{t_{n_\alpha}}]), t_{n_\alpha}\in[c,d], \\ [e_{[a,b]}]\pi_{-2}(g) &= \lim_{m\to\infty}\sum_{\beta=1}^{M_m} f_{m_\beta}, \quad g_{m_\beta}\in Ran([e_{t_{m_\beta}}]), t_{m_\beta}\in[a,b]. \end{aligned}$$

Hence by (c) we have

$$\begin{aligned} \left\langle [e_{[a,b]}][e_t][e_{[c,d]}]\pi_{-2}(f), \pi_{-2}(g)\right\rangle_{-2} &= \lim_{n,m\to\infty}\sum_{\alpha,\beta=1}^{N_n,M_m}\left\langle [e_t]f_{n_\alpha}, g_{m_\beta}\right\rangle_{-2} \\ &= \lim_{n,m\to\infty}\sum_{\alpha,\beta=1}^{N_n,M_m}\left\langle f_{n_\alpha}, g_{m_\beta}\right\rangle_{-2} \\ &= \left\langle [e_{[a,b]}][e_{[c,d]}]\pi_{-2}(f), \pi_{-2}(g)\right\rangle_{-2}. \end{aligned}$$

Then (d) follows. □

We introduce notations;

$$\begin{aligned} \Gamma([e_{[a,b]}]) &\equiv E_{[a,b]}, \\ \Gamma([\tilde{j}_t]) &\equiv J_t, \\ \Gamma([e_t]) &\equiv E_t. \end{aligned} \qquad (3.\ 3)$$

Proposition 3.3 *([26, Theorem III.5])*
(a) J_t is a linear isometry from $\mathcal{F}$ to $\mathcal{E}$.
(b) $J_tJ_t^* = E_t$.
(c) $J_t^*J_s = e^{-|t-s|\mathbf{H}_0}$.
(d) Let $\Sigma_{[a,b]}$ denote the σ-algebra generated by

$$L\left\{\phi_{\mathcal{E}}(\pi_{-2}(f))\,\middle|\,\pi_{-2}(f)\in U_{[a,b]}\right\}$$

and the set of $\Sigma_{[a,b]}$-measurable functions in $\mathcal{E}$ by $\mathcal{E}_{[a,b]}$. Then

$$Ran\left(E_{[a,b]}\right) = \mathcal{E}_{[a,b]}.$$

(e) (Markoff property) Let $a\le b\le t\le c\le d$. Then

$$E_{[a,b]}E_tE_{[c,d]} = E_{[a,b]}E_{[c,d]}.$$

Proof: Eqs.(a),(b),(c) and (e) follow from Proposition 3.2. Eq.(d) follows from [26,Theorem III.8]. □

As in [26], a FKN formula follows from Proposition 3.3.

Proposition 3.4 *([26, Theorem III.6], FKN formula) Let $f_1, ..., f_n \in \widetilde{\mathcal{H}}_{-1}$ and $G_0, ..., G_k$ be bounded measurable functions on $\mathbb{R}^d$. Let $t_1, ..., t_k \geq 0$ be given. Then*

$$\begin{aligned}&\left\langle \Omega_{\mathcal{F}}, G_0^{\mathcal{F}} e^{-t_1 \mathbf{H}_0} G_1^{\mathcal{F}} ... e^{-t_k \mathbf{H}_0} G_k^{\mathcal{F}} \Omega_{\mathcal{F}} \right\rangle_{\mathcal{F}} \\ &= \langle \Omega_{\mathcal{E}}, G_0^{s_0} ... G_k^{s_k} \Omega_{\mathcal{E}} \rangle_{\mathcal{E}},\end{aligned}$$

where s_0 is arbitrary and

$$\begin{aligned} s_j &= s_0 + \sum_{i=1}^{j} t_i, \\ G_j^{\mathcal{F}} &= G_j \left(\phi_{\mathcal{F}}(\pi_{-1}(f_1)), ..., \phi_{\mathcal{F}}(\pi_{-1}(f_n)) \right), \\ G_j^{s_j} &= G_j \left(\phi_{\mathcal{E}}(\pi_{-2}(\widetilde{j}_{s_j} f_1)), ..., \phi_{\mathcal{E}}(\pi_{-2}(\widetilde{j}_{s_j} f_n)) \right). \end{aligned}$$

□

The Hilbert space of state vectors in the system of a non-relativistic charged particle interacting with a quantized radiation field is given by $\mathcal{M}_B = L^2(\mathbb{R}^d) \otimes \mathcal{F}(\mathcal{W})$. The unitary operator $\mathcal{U}$ given in Theorem 3.1 implements unitary equivalence between $\mathcal{M}_B$ and

$$\mathcal{M} = L^2(\mathbb{R}^d) \otimes \mathcal{F} \cong \int_{\mathbb{R}^d}^{\oplus} \mathcal{F} dx.$$

For an $\mathcal{H}_{-1}$-valued function on $\mathbb{R}^d$, $\rho(\cdot) : \mathbb{R}^d \to \mathcal{H}_{-1}$, we put

$$\widetilde{\rho}_\mu(\cdot) = \overbrace{(0, ..., \underbrace{\rho(\cdot)}_{\mu-th}, ..., 0)}^{d}.$$

Then we define an operator in $\mathcal{M}$ by

$$\phi_{\mathcal{F},\mu}^{\rho} = \int_{\mathbb{R}^d}^{\oplus} \phi_{\mathcal{F}} \left(\pi_{-1} \left(\widetilde{\rho}_\mu(x) \right) \right) dx.$$

Let $D_\mu, (\mu = 1, ..., d)$ be the generalized L^2-derivative in the μ-direction. Then the interaction Hamiltonian of the non-relativistic charged particle with mass 1 and the quantized radiation field is "formally" given as an operator in $\mathcal{M}$ by

$$\mathbf{H}_\rho = \frac{1}{2} \sum_{\mu=1}^{d} \left(-iD_\mu \otimes I - \phi_{\mathcal{F},\mu}^{\rho} \right)^2 + I \otimes \mathbf{H}_0. \tag{3. 4}$$

Here "formally" means that we mention nothing about the domain of $\mathbf{H}_\rho$. The precise definition will be given in the following section. We set

$$\mathbf{H}_{\rho,0} = \frac{1}{2} \sum_{\mu=1}^{d} \left(-iD_\mu \otimes I - \phi_{\mathcal{F},\mu}^{\rho} \right)^2.$$

We conclude this section with giving a typical example of the $\mathcal{H}_{-1}$-valued function $\rho(\cdot)$. One can take

$$\rho(x) = \left(\hat{f}(\cdot)e^{i\cdot x}\right)^{\vee}, \tag{3. 5}$$

where f is a real-valued rapidly decreasing infinitely differentiable function on $\mathbb{R}^n$. In this case, the corresponding standard Boson Fock space element $A(\tilde{\rho}_\mu(x))$ is given by ([2,4,23])

$$\begin{aligned} \mathcal{U}^{-1}\phi_{\mathcal{F}}(\tilde{\rho}_\mu(x))\mathcal{U} &= A(\tilde{\rho}_\mu(x)), \\ &= \frac{1}{\sqrt{2}}\left\{a^\dagger\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r \hat{f} e^{-i\cdot x}}{\sqrt{h}}\right) + a\left(\oplus_{r=1}^{d-1}\frac{e_\mu^r \check{f} e^{i\cdot x}}{\sqrt{h}}\right)\right\}. \end{aligned}$$

Then the function f serves as an ultraviolet cut-off function for photon momenta. Moreover, $\tilde{\rho}_\mu(x)$ satisfies the Coulomb gauge condition (see (4.17)).

4 FUNCTIONAL INTEGRALS

In this section we construct a self-adjoint extension of $\mathbf{H}_\rho$ given formally by (3.4) and derive a functional integral representation for the heat semigroup associated with it. The main idea is to apply the FKN formula and the FKI formula ([31,Theorem 15.3]).

For an $\mathcal{H}_{-1}$-valued function ρ, $\phi_{\mathcal{F}}(\pi_{-1}(\tilde{\rho}_\mu(x)))$ is a self-adjoint operator for each $x \in \mathbb{R}^d$ as a multiplication operator in $\mathcal{F}$. Then, for each $x, y \in \mathbb{R}^d$, we can define a unitary operator on $\mathcal{F}$ by

$$\begin{aligned} U_\rho(x,y) &\equiv \exp\left\{\frac{1}{2}i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^{d}\pi_{-1}(\tilde{\rho}_\mu(x)+\tilde{\rho}_\mu(y))(x_\mu - y_\mu)\right)\right\} \\ &\equiv \exp(\phi^\rho(x,y)). \end{aligned}$$

Let $p_s(x)$ be the heat kernel function

$$p_s(x) = (2\pi s)^{-\frac{d}{2}}\exp\left(-\frac{1}{2s}|x|^2\right), \quad s>0, x\in\mathbb{R}^d.$$

Then we define a family of the contractive self-adjoint operators $\{Q_{\rho,s}\}_{s\geq 0}$ on $\mathcal{M}$ by

$$\begin{aligned} (Q_{\rho,s}F)(x) &= \int_{\mathbb{R}^d} p_s(x-y)U_\rho(x,y)F(y)dy, \quad s>0, \\ (Q_{\rho,0}F)(x) &= F(x), \end{aligned}$$

where $F(\cdot) \in \mathcal{M}$ and the integral is the $\mathcal{F}$-valued Bochner integral. Actually one can easily see that

$$\begin{aligned} \|Q_{\rho,s}F\|_{\mathcal{M}} &\leq \left\|e^{-s(-\frac{1}{2}\Delta)}\|F(\cdot)\|_{\mathcal{F}}\right\|_{L^2(\mathbb{R}^d)} \\ &\leq \|F\|_{\mathcal{M}}. \end{aligned}$$

Let

$$[C_b^n(\mathbb{R}^d;\mathcal{H}_{-1})] = \left\{\rho(\cdot):\mathbb{R}^d\to\mathcal{H}_{-1}\middle|\,\pi_{-1}(\tilde{\rho}_\mu(\cdot))\in C_b^n(\mathbb{R}^d;[\widetilde{\mathcal{H}}_{-1}]),\mu=1,...,d\right\}.$$

We define a subspace $\mathcal{M}_\rho^\infty$ in $\mathcal{M}$ as follows. For $\rho\in[C_b^1(\mathbb{R}^d;\mathcal{H}_{-1})]$, we say that $F\in\mathcal{M}_\rho^\infty\subset\mathcal{M}$ if and only if the following (i)-(iii) hold
(i) $F(\cdot)\in H^2(\mathbb{R}^d;\mathcal{F})$.
(ii) For each $y\in\mathbb{R}^d$,

$$F(y)\in\mathcal{F}^\infty,\quad \partial_\mu F(y)\in\mathcal{F}^\infty,\quad \mu=1,...,d.$$

(iii) (Integration by parts condition) For all $G\in\mathcal{M}$, $x\in\mathbb{R}^d$ (see Lemma 4.3),

$$\lim_{y\to\infty}\partial_{y_\mu}p_s(x-y)\cdot\langle F(y),U_\rho(x,y)G(x)\rangle_{\mathcal{F}}=0,$$
$$\lim_{y\to\infty}p_s(x-y)\cdot\partial_{y_\mu}\langle F(y),U_\rho(x,y)G(x)\rangle_{\mathcal{F}}=0,\qquad \mu=1,...,d.$$

Lemma 4.1 *Let $\rho\in[C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$, $G\in\mathcal{M}$, and $F\in\mathcal{M}_\rho^\infty$. Then $\langle Q_{\rho,s}F,G\rangle_{\mathcal{M}}$ is differentiable in $s>0$ with*

$$\lim_{s\to0+}\frac{d}{ds}\langle Q_{\rho,s}F,G\rangle_{\mathcal{M}}=\langle-\mathbf{H}_{\rho,0}F,G\rangle_{\mathcal{M}}. \tag{4. 1}$$

□

For the classical cases ([31,Lemma 15.1]), analogue of (4.1) for the Schrödinger Hamiltonian with vector potentials is important for constructions of path integral representations. In the same way as in the classical case, however, (4.1) can not be proven directly. To verify (4.1), we prepare two fundamental lemmas (Lemmas 4.3 and 4.4) as follows. Notice that for $F,G\in\mathcal{M}$,

$$\langle Q_{\rho,s}F,G\rangle_{\mathcal{M}}=\int_{\mathbb{R}^d}dx\int_{\mathbb{R}^d}p_s(x-y)\,\langle U_\rho(x,y)F(y),G(x)\rangle_{\mathcal{F}}\,dy.$$

Fubini's lemma allows one to interchange $\int dx$ and $\int dy$. Moreover, we have

$$\int_{\mathbb{R}^d\times\mathbb{R}^d}dxdy\left|\frac{d}{ds}p_s(x-y)\,\langle U_\rho(x,y)F(y),G(x)\rangle_{\mathcal{F}}\right|$$
$$\le\int_{\mathbb{R}^d}\left(\frac{d}{2s}+\frac{x^2}{2s^2}\right)p_s(x)dx||F||_{\mathcal{M}}||G||_{\mathcal{M}}<\infty,$$

so that we can interchange the differential $\frac{d}{ds}$ and the integral $\int dxdy$. The following proposition is fundamental.

Proposition 4.2 *(I) Let f be a Lebesgue measurable bounded function on $\mathbb{R}^d$ which is continuous at 0. Then*

$$\lim_{s\to0+}\int_{\mathbb{R}^d}p_s(x)f(x)dx=f(0).$$

(II) For any $\alpha>0$

$$\lim_{s\to0+}\int|x|^\alpha p_s(x)dx=0.$$

Proof: Elementary calculations. □

We introduce notations and estimates. For $\rho \in [C_b^r(\mathbb{R}^d;\mathcal{H}_{-1})]$, we set

$$\begin{aligned}
\phi_\mu^{\rho,(n)}(x,y) &\equiv \phi_{\mu+}^{\rho,(n)}(x,y)+\phi_{\mu-}^{\rho,(n)}(x,y),\\
\phi_{\mu+}^{\rho,(n)}(x,y) &\equiv \frac{i}{2}\phi_{\mathcal{F}}\left(\sum_{j=1}^d\left(\partial_\mu^n\pi_{-1}(\tilde\rho_j(y))+\delta_{n,0}\pi_{-1}(\tilde\rho_j(x))\right)(x_j-y_j)\right),\\
\phi_{\mu-}^{\rho,(n)}(x,y) &\equiv \frac{i}{2}\phi_{\mathcal{F}}\left(-n\partial_\mu^{n-1}\pi_{-1}(\tilde\rho_\mu(y))-\delta_{n,1}\pi_{-1}(\tilde\rho_\mu(x))\right),\quad 0\le n\le r.
\end{aligned}$$

Note that $\phi_\mu^{\rho,(0)}(x,y)=\phi^\rho(x,y)$. For $\rho\in[C_b^r(\mathbb{R}^d;\mathcal{H}_{-1})]$, put

$$\begin{aligned}
c_{\mu,n}(\rho) &= \sup_x\sqrt{\sum_{j=1}^d\|\partial_\mu^n\pi_{-1}(\tilde\rho_j(x))\|_{-1}^2},\\
d_{\mu,n}(\rho) &= \sup_x\|\partial_\mu^n\pi_{-1}(\tilde\rho_\mu(x))\|_{-1},\quad 0\le n\le r, 1\le\mu\le d.
\end{aligned}$$

In the case $n=0$, we use notations $c_{\mu,0}(\rho)=c_0(\rho)$ and $d_{\mu,0}(\rho)=d_0(\rho)$. From (3.2) it follows that for $\rho_i\in[C_b^r(\mathbb{R}^d;\mathcal{H}_{-1})], 0\le k_i\le r, 1\le\mu_i\le d, i=1,...,n$, and $\Phi\in\mathcal{F}^\infty$ such that $\mathbf{N}\Phi=N\Phi$,

$$\begin{aligned}
&\left\|\phi_{\mu_1}^{\rho_1,(k_1)}(x,y)\phi_{\mu_2}^{\rho_2,(k_2)}(x,y)...\phi_{\mu_n}^{\rho_n,(k_n)}(x,y)\Phi\right\|_{\mathcal{F}}\\
&\le\frac{\sqrt{2^n}\sqrt{N+1}...\sqrt{N+n}}{2^n}\|\Phi\|_{\mathcal{F}}\\
&\times\Pi_{i=1}^n\left\{(1+\delta_{0,k_i})c_{\mu_i,k_i}(\rho_i)|x-y|+(k_i+\delta_{k_i,1})d_{\mu_i,k_i-1}(\rho_i)\right\}. \qquad (4.\ 2)
\end{aligned}$$

Lemma 4.3 *Let $\rho\in[C_b^r(\mathbb{R}^d;\mathcal{H}_{-1})]$, $G\in\mathcal{F}$ and $F\in C^r(\mathbb{R}^d;\mathcal{F})$ such that $\partial^kF(y)\in\mathcal{F}^\infty$, $k=1,...,r-1$. Then $\langle U_\rho(x,y)F(y),G\rangle_{\mathcal{F}}$ is r-times differentiable in y. In particular,*

$$\begin{aligned}
\partial_{y_\mu}\langle U_\rho(x,y)F(y),G\rangle_{\mathcal{F}} &= \left\langle U_\rho(x,y)\phi_\mu^{\rho,(1)}(x,y)F(y),G\right\rangle_{\mathcal{F}}+\langle U_\rho(x,y)\partial_\mu F(y),G\rangle_{\mathcal{F}},\\
&\qquad \rho\in[C_b^1(\mathbb{R}^d;\mathcal{H}_{-1})], \qquad (4.\ 3)\\
\partial_{y_\mu}^2\langle U_\rho(x,y)F,G\rangle_{\mathcal{F}} &= \left\langle U_\rho(x,y)\partial_\mu^2F(y),G\right\rangle_{\mathcal{F}}+2\left\langle U_\rho(x,y)\phi_\mu^{\rho,(1)}(x,y)\partial_\mu F(y),G\right\rangle_{\mathcal{F}}\\
&+\left\langle U_\rho(x,y)\left\{\left(\phi_\mu^{\rho,(1)}(x,y)\right)^2+\phi_\mu^{\rho,(2)}(x,y)\right\}F(y),G\right\rangle_{\mathcal{F}},\\
&\qquad \rho\in[C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]. \qquad (4.\ 4)
\end{aligned}$$

Proof: Suppose that $H\in\mathcal{F}$ such that $\mathbf{N}H=NH$, and $\rho\in[C_b^1(\mathbb{R}^d;\mathcal{H})]$. For simplicity we put $\phi^\rho(x,y)=\phi(x,y)$. Since, by (3.2), $\mathcal{F}^\infty$ is the set of the analytic vectors ([25,X.6]) of the self-adjoint operator $\phi(x,y)$, the following equation follows

$$\langle U_\rho(x,y)H,G\rangle_{\mathcal{F}}=\sum_{n=0}^\infty\frac{1}{n!}\langle\phi^n(x,y)H,G\rangle_{\mathcal{F}}. \qquad (4.\ 5)$$

One can easily derive from (4.2) that each term on the r.h.s.of (4.5) is differentiable with respect to y_μ with

$$\partial_{y_\mu}\langle \phi^n(x,y)H,G\rangle_{\mathcal{F}} = n\left\langle \phi^{n-1}(x,y)\phi_\mu^{\rho,(1)}(x,y)H,G\right\rangle_{\mathcal{F}},$$

from which and (4.2) it follows that

$$\begin{aligned}&\sum_{n=0}^{k}\frac{1}{n!}\left|\left\langle \phi^n(x,y)\phi_\mu^{\rho,(1)}(x,y)H,G\right\rangle_{\mathcal{F}}\right| \\ &\le \sum_{n=0}^{k}\frac{\sqrt{N+1}\sqrt{N+2}...\sqrt{N+n+1}}{\sqrt{2^n}n!} \\ &\times (c_{\mu,1}(\rho)|x-y|+2d_0(\rho))\,(2c_0(\rho)|x-y|)^n\,||H||_{\mathcal{F}}||G||_{\mathcal{F}}. \qquad (4.\ 6)\end{aligned}$$

Then the left hand side (l.h.s.)of (4.6) converges uniformly in the wider sense with respect to y as $k\to\infty$. Hence the differentiability of $\langle U_\rho(x,y)H,G\rangle_{\mathcal{F}}$ with respect to y_μ follows. From the strong differentiability of F and the fact $\partial F(y)\in\mathcal{F}^\infty$, (4.3) follows. Eq.(4.4) and the remaining statements can be shown similarly. □

Lemma 4.4 *Let* $\rho\in[C_b^r(\mathbb{R}^d;\mathcal{H}_{-1})]$, $G\in\mathcal{M}$, $F\in\mathcal{M}$ *such that* $F(x)\in\mathcal{F}^\infty$, *and* $0\le k_j\le r, j=1,...,n$. *Then*

$$\begin{aligned}&\lim_{X\to 0}\int_{\mathbb{R}^d}\left\langle U_\rho(x,x-X)\phi_{\mu_1-}^{\rho,(k_1)}(x,x-X)...\phi_{\mu_n-}^{\rho,(k_n)}(x,x-X)F(x-X),G(x)\right\rangle_{\mathcal{F}}dx \\ &=\int_{\mathbb{R}^d}\left\langle \phi_{\mu_1-}^{\rho,(k_1)}(x,x)...\phi_{\mu_n-}^{\rho,(k_n)}(x,x)F(x),G(x)\right\rangle_{\mathcal{F}}dx.\end{aligned}$$

Proof: One can easily see that for all $x\in\mathbb{R}^d$

$$s-\lim_{X\to 0}U_\rho(x,x-X)=I_{\mathcal{F}}$$

in $\mathcal{F}$. Let $K\in\mathcal{M}$ be such that $\mathbf{N}K(x)=NK(x)$ for all $x\in\mathbb{R}^d$. Then

$$\begin{aligned}&\int_{\mathbb{R}^d}dx\left|\left\langle\left(\phi_{\mu-}^{\rho,(k)}(x,x-X)-\phi_{\mu-}^{\rho,(k)}(x,x)\right)K(x),G(x)\right\rangle_{\mathcal{F}}\right| \\ &\le\frac{\sqrt{N+1}}{\sqrt{2}}k\tilde{d}_{\mu,k}(\rho)|X|||K||_{\mathcal{M}}||G||_{\mathcal{M}},\end{aligned}$$

where $\tilde{d}_{\mu,k}(\rho)=\sup_x||\sum_{j=1}^d\partial_j\partial_\mu^{k-1}\pi_{-1}(\tilde{\rho}_\mu(x))||_{-1}$. Hence by (4.2) and by

$$\lim_{X\to 0}||F(\cdot-X)-F(\cdot)||_{\mathcal{M}}=0,$$

one can directly derive the lemma. □

Now we can prove Lemma 4.1.
Proof of Lemma 4.1.

Note that

$$\frac{d}{ds}p_s(x-y) = \frac{1}{2}\Delta_y p_s(x-y).$$

Since $F \in \mathcal{M}_\rho^\infty$, one can use the integration by parts formula. Then, by (4.4), we have

$$\begin{aligned}
\frac{d}{ds}\langle Q_{\rho,s}F, G\rangle &= \frac{1}{2}\sum_{\mu=1}^{d}\int_{\mathbb{R}^d\times\mathbb{R}^d} dxdyp_s(x-y)\partial_{y_\mu}^2 \langle U_\rho(x,y)F(y), G(x)\rangle_{\mathcal{F}} \\
&= \frac{1}{2}\sum_{\mu=1}^{d}\int_{\mathbb{R}^d\times\mathbb{R}^d} dxdyp_s(x-y) \\
&\quad \times \Big\{\Big\langle U_\rho(x,y)\partial_\mu^2 F(y), G(x)\Big\rangle_{\mathcal{F}} + 2\Big\langle U_\rho(x,y)\phi_\mu^{\rho,(1)}(x,y)\partial_\mu F(y), G(x)\Big\rangle_{\mathcal{F}} \\
&\quad + \Big\langle U_\rho(x,y)\Big\{\Big(\phi_\mu^{\rho,(1)}(x,y)\Big)^2 + \phi_\mu^{\rho,(2)}(x,y)\Big\} F(y), G(x)\Big\rangle_{\mathcal{F}}\Big\} \\
&\equiv \frac{1}{2}\sum_{\mu=1}^{d}\int_{\mathbb{R}^d\times\mathbb{R}^d} dxdyp_s(x-y)I_\mu(x,y) \\
&= \frac{1}{2}\sum_{\mu=1}^{d}\int_{\mathbb{R}^d} p_s(X)dX\int_{\mathbb{R}^d} dxI_\mu(x,x-X).
\end{aligned}$$

Here, to apply Proposition 4.1, we divide $I_\mu(x,y)$ in two components, $I_\mu = I_{\mu+} + I_{\mu-}$, as follows;

$$\begin{aligned}
I_{\mu+}(x,y) &= 2\Big\langle U_\rho(x,y)\phi_{\mu+}^{\rho,(1)}(x,y)\partial_\mu F(y), G(x)\Big\rangle_{\mathcal{F}} \\
&\quad + \Big\langle U_\rho(x,y)\Big\{\Big(\phi_{\mu+}^{\rho,(1)}(x,y)\Big)^2 + 2\phi_{\mu+}^{\rho,(1)}(x,y)\phi_{\mu-}^{\rho,(1)}(x,y) + \phi_{\mu+}^{\rho,(2)}(x,y)\Big\} F(y), G(x)\Big\rangle_{\mathcal{F}}, \\
I_{\mu-}(x,y) &= \Big\langle U_\rho(x,y)\partial_\mu^2 F(y), G(x)\Big\rangle_{\mathcal{F}} + 2\Big\langle U_\rho(x,y)\phi_{\mu-}^{\rho,(1)}(x,y)\partial_\mu F(y), G(x)\Big\rangle_{\mathcal{F}} \\
&\quad + \Big\langle U_\rho(x,y)\Big\{\Big(\phi_{\mu-}^{\rho,(1)}(x,y)\Big)^2 + \phi_{\mu-}^{\rho,(2)}(x,y)\Big\} F(y), G(x)\Big\rangle_{\mathcal{F}}.
\end{aligned}$$

By (4.2), we have

$$\begin{aligned}
&\left|\int_{\mathbb{R}^d} dxI_{\mu+}(x,x-X)\right| \\
&\le ||G||_{\mathcal{M}}\Big\{2\frac{\sqrt{N+1}}{\sqrt{2}}\left(c_{\mu,1}(\rho)|X|\right)||\partial_\mu F||_{\mathcal{M}} + \frac{\sqrt{N+1}\sqrt{N+2}}{2} \\
&\times \Big\{(c_{\mu,1}(\rho)|X|)^2 + 4d_0(\rho)c_{\mu,1}(\rho)|X|\Big\}||F||_{\mathcal{M}} + \frac{\sqrt{N+1}}{\sqrt{2}}c_{\mu,2}(\rho)|X|||F||_{\mathcal{M}}\Big\} \\
&\equiv \epsilon_1|X| + \epsilon_2|X|^2, \\
&\left|\int_{\mathbb{R}^d} dxI_{\mu-}(x,x-X)\right|
\end{aligned}$$

$$\leq ||G||_{\mathcal{M}}\left\{||\partial_\mu^2 F||_{\mathcal{M}} + 4d_0(\rho)\frac{\sqrt{N+1}}{\sqrt{2}}||\partial_\mu F||_{\mathcal{M}}\right.$$
$$\left. + \left(\frac{\sqrt{N+1}\sqrt{N+2}}{2}4d_0(\rho)^2 + \frac{\sqrt{N+1}}{\sqrt{2}}2d_{\mu,1}(\rho)\right)||F||_{\mathcal{M}}\right\}$$
$$\equiv \epsilon_3. \tag{4. 7}$$

By Proposition 4.1 (II), we have

$$\lim_{s\to 0+}\left|\int dXp_s(X)\int dxI_{\mu+}(x,x-X)\right| \leq \lim_{s\to 0+}\int dXp_s(X)\left(\epsilon_1|X|+\epsilon_2|X|^2\right) = 0.$$

Thus it is enough to analyze the $I_\mu(\cdot,\cdot-X)$ component. By Lemma 4.4, it follows that

$$\lim_{X\to 0}\int_{\mathbb{R}^d} I_{\mu-}(x,x-X)dx = \int_{\mathbb{R}^d} I_{\mu-}(x,x)dx. \tag{4. 8}$$

By (4.7) and (4.8), Proposition 4.1 (I) yields

$$\begin{aligned}
\lim_{s\to 0+}\frac{d}{ds}\langle Q_{\rho,s}F,G\rangle_{\mathcal{M}} &= \lim_{s\to 0+}\sum_{\mu=1}^d \frac{1}{2}\int_{\mathbb{R}^d} dXp_s(X)\int_{\mathbb{R}^d} I_{\mu-}(x,x-X)dx \\
&= \frac{1}{2}\sum_{\mu=1}^d\int_{\mathbb{R}^d} dxI_{\mu-}(x,x) \\
&= -\langle \mathbf{H}_{\rho,0}F,G\rangle_{\mathcal{M}}.
\end{aligned}$$

□

Lemma 4.5 *Let $\rho\in[C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$, $F\in\mathcal{M}_\rho^\infty$, and $G\in\mathcal{M}$. Then $\langle Q_{\rho,s}F,G\rangle_{\mathcal{M}}$ is right side differentiable at $s=0$ with*

$$\frac{d}{ds}\langle Q_{\rho,s}F,G\rangle_{\mathcal{M}}|_{s=0+} = -\langle \mathbf{H}_{\rho,0}F,G\rangle_{\mathcal{M}}. \tag{4. 9}$$

Proof: We have that

$$\begin{aligned}
\langle Q_{\rho,s}F,G\rangle_{\mathcal{M}} &= \int_{\mathbb{R}^d} p_s(X)dX\int_{\mathbb{R}^d}\langle U_\rho(x,x-X)F(x-X),G(x)\rangle_{\mathcal{F}}dx \\
\langle Q_{\rho,0}F,G\rangle_{\mathcal{M}} &= \int_{\mathbb{R}^d}\langle F(x),G(x)\rangle_{\mathcal{F}}dx.
\end{aligned}$$

Hence, similarly to the proof of Lemma 4.1, it follows that $\langle Q_{\rho,s}F,G\rangle_{\mathcal{M}}$ is right continuous in s at $s=0$. Thus by the Taylor expansion we can see that

$$\begin{aligned}
\lim_{\epsilon\to 0+}\frac{\langle Q_{\rho,\epsilon}F,G\rangle_{\mathcal{M}}-\langle F,G\rangle_{\mathcal{M}}}{\epsilon} &= \lim_{s\to 0+}\frac{d}{ds}\langle Q_{\rho,s}F,G\rangle_{\mathcal{M}} \\
&= -\langle \mathbf{H}_{\rho,0}F,G\rangle_{\mathcal{M}}.
\end{aligned}$$

Hence (4.9) follows. □

Following [8,31], we shall construct a contraction C_0-semigroup from $\{Q_{\rho,n}\}_{n\geq 1}$. For simplicity we put $2^n = n*$.

Lemma 4.6 *Let* $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$. *Then, for all* $t \geq 0$, *the strong limit*

$$s - \lim_{n\to\infty} Q^{n*}_{\rho,\frac{t}{n*}} \equiv G_\rho(t)$$

exists. Moreover, $G_\rho(t)$ *has the following functional integral representation with* $F, H \in \mathcal{M}$

$$\begin{aligned}
&\langle F, G_\rho(t) H\rangle_{\mathcal{M}} \\
&= \int_{\mathbb{R}^d\times\Omega} d\mu \int_{Q_{-1}} d\mu_{-1} e^{i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^d \left(\int_0^t \pi_{-1}(\tilde{\rho}_\mu(b(s)+x))db_\mu + \frac{1}{2}\int_0^t \partial_\mu \pi_{-1}(\tilde{\rho}_\mu(b(s)+x))ds\right)\right)} \\
&\quad \times \overline{F}(b(t)+x)H(x).
\end{aligned} \tag{4. 10}$$

Proof: To prove the existence, we show that $\left\{Q^{n*}_{\rho,\frac{t}{n*}}\right\}_{n\geq 1}$ is a Cauchy sequence in $\mathcal{M}$. We see that

$$\begin{aligned}
&\left\|Q^{n*}_{\rho,\frac{t}{n*}}F - Q^{m*}_{\rho,\frac{t}{m*}}F\right\|^2_{\mathcal{M}} \\
&= \left\langle F, Q^{2n*}_{\rho,\frac{t}{n*}}F\right\rangle_{\mathcal{M}} + \left\langle F, Q^{2m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}} - 2\Re\left\langle F, Q^{n*}_{\rho,\frac{t}{n*}}Q^{m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}}.
\end{aligned} \tag{4. 11}$$

The last term on the r.h.s.of (4.11) is

$$\begin{aligned}
&\left\langle F, Q^{n*}_{\rho,\frac{t}{n*}}Q^{m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}} \\
&= \int_{\mathbb{R}^d}\left\langle F(x), \left(Q^{n*}_{\frac{t}{n*}}Q^{m*}_{\frac{t}{m*}}F\right)(x)\right\rangle_{\mathcal{F}} dx \\
&= \int_{\mathbb{R}^d\times\mathbb{R}^{n*d}\times\mathbb{R}^{m*d}} p_{\frac{t}{n*}}(x-x_1)..p_{\frac{t}{n*}}(x_{n*-1}-x_{n*})p_{\frac{t}{m*}}(x_{n*}-y_1)..p_{\frac{t}{m*}}(y_{m*-1}-y_{m*}) \\
&\times \langle F(x), U_\rho(x,x_1)..U_\rho(x_{n*-1},x_{n*})U_\rho(x_{n*},y_1)..U_\rho(y_{m*-1},y_{m*})F(y_{m*})\rangle_{\mathcal{F}}\, dx d\vec{x} d\vec{y} \\
&= \int_{\mathbb{R}^d} dx \left\langle F(b(2t)+x), e^{i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^d \pi_{-1}(\Box_{\mu,m,n}(x))\right)}F(x)\right\rangle_{L^2(\Omega;\mathcal{F})},
\end{aligned}$$

where

$$\begin{aligned}
\Box_{\mu,m,n}(x) &= \sum_{k=1}^{m*}\left\{\tilde{\rho}_\mu\left(b\left(\frac{t}{m*}k\right)+x\right) + \tilde{\rho}_\mu\left(b\left(\frac{t}{m*}(k-1)\right)+x\right)\right\} \\
&\qquad \times\left\{b_\mu\left(\frac{t}{m*}k\right) - b_\mu\left(\frac{t}{m*}(k-1)\right)\right\} \\
&+ \sum_{k=1}^{n*}\left\{\tilde{\rho}_\mu\left(b\left(\frac{t}{n*}k+t\right)+x\right) + \tilde{\rho}_\mu\left(b\left(\frac{t}{n*}(k-1)+t\right)+x\right)\right\} \\
&\qquad \times\left\{b_\mu\left(\frac{t}{n*}k+t\right) - b_\mu\left(\frac{t}{n*}(k-1)+t\right)\right\}.
\end{aligned}$$

From Lemma 2.3 it follows that for each $x \in \mathbb{R}^d$

$$\begin{aligned}
&s - \lim_{m\to\infty}\lim_{n\to\infty} \pi_{-1}(\Box_{\mu,m,n}(x)) \\
&= \int_0^{2t} \pi_{-1}\left(\tilde{\rho}_\mu(x+b(s))\right) db_\mu + \frac{1}{2}\int_0^{2t} \partial_\mu \pi_{-1}\left(\tilde{\rho}_\mu(b(s)+x)\right) ds
\end{aligned}$$

in $L^2(\Omega;[\widetilde{\mathcal{H}}_{-1}])$. One can easily see that the strong convergence of $\pi_{-1}(\square_{\mu,m,n}(x))$ in $L^2(\Omega;[\widetilde{\mathcal{H}}_{-1}])$ implies that for each $x \in \mathbb{R}^d$ and $\Phi \in \mathcal{F}$,

$$s-\lim_{m\to\infty}\lim_{n\to\infty}\exp\left(i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^{d}\pi_{-1}\left(\square_{\mu,m,n}(x)\right)\right)\right)\Phi$$
$$=\exp\left(i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^{d}\int_0^{2t}\pi_{-1}\left(\widetilde{\rho}_\mu(x+b(s))\right)db_\mu+\frac{1}{2}\sum_{\mu=1}^{d}\int_0^{2t}\partial_\mu\pi_{-1}\left(\widetilde{\rho}_\mu(x+b(s))\right)ds\right)\right)\Phi. \tag{4. 12}$$

in $L^2(\Omega;\mathcal{F})$. On the other hand we have

$$\left|\left\langle F(b(2t)+\cdot),e^{i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^{d}\pi_{-1}(\square_{\mu,m,n}(\cdot))\right)}F(\cdot)\right\rangle_{L^2(\Omega,\mathcal{F})}\right|$$
$$\leq ||F(b(2t)+\cdot)||_{L^2(\Omega;\mathcal{F})}||F(\cdot)||_{L^2(\Omega;\mathcal{F})}\in L^1(\mathbb{R}^d).$$

Hence, by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\left\langle F,Q^{n*}_{\rho,\frac{t}{n*}}Q^{m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}}$$
$$=\int_{\mathbb{R}^d}dx\,\langle F(b(2t)+x),$$
$$e^{i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^{d}\left(\int_0^{2t}\pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))db_\mu+\frac{1}{2}\int_0^{2t}\partial_\mu\pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))ds\right)\right)}F(x)\Big\rangle_{L^2(\Omega;\mathcal{F})}. \tag{4. 13}$$

Similarly it can be easily seen that $\left\langle F,Q^{2n*}_{\rho,\frac{t}{n*}}F\right\rangle_{\mathcal{M}}$ and $\left\langle F,Q^{2m*}_{\rho,\frac{t}{m*}}F\right\rangle_{\mathcal{M}}$ converge to the r.h.s. of (4.13) as $n,m\to\infty$, respectively. Then it follows that $\{Q^{n*}_{\rho,\frac{t}{n*}}\}_{n\geq 0}$ is a Cauchy. Eq.(4.10) easily follows from (4.13). □

Lemma 4.7 *Let $\rho\in[C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$. Then the family $\{G_\rho(t)\}_{t\geq 0}$ is a contraction C_0-semigroup on $\mathcal{M}$.*

Proof: By the definition of $G_\rho(t)$ and the proof of Lemma.4.6, it holds that

$$G_\rho(t)G_\rho(s)=G_\rho(t+s). \tag{4. 14}$$

We show the strong right continuity at $t=0$. Because of (4.14), the weak continuity implies the strong continuity. Hence it is enough to show that

$$\lim_{t\to 0+}\langle G_\rho(t)F,H\rangle_{\mathcal{M}}=\langle G_\rho(0)F,H\rangle_{\mathcal{M}},\quad F\in\mathcal{M},H\in\mathcal{M}.$$

From (4.10), we have

$$\begin{aligned}
&\langle F, G_\rho(t)H\rangle_{\mathcal{M}} - \langle F, G_\rho(0)H\rangle_{\mathcal{M}} \\
&= \int_{\mathbb{R}^d} dx\, \Big\langle F(b(t)+x) - F(x), \\
&\qquad e^{i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^d \left(\int_0^t \pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))db_\mu + \frac{1}{2}\int_0^t \partial_\mu \pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))ds\right)\right)} H(x)\Big\rangle_{L^2(\Omega;\mathcal{F})} \\
&+ \int_{\mathbb{R}^d} dx\, \Big\langle F(x), \\
&\qquad \left(e^{i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^d \left(\int_0^t \pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))db_\mu + \frac{1}{2}\int_0^t \partial_\mu \pi_{-1}(\widetilde{\rho}_\mu(b(s)+x)ds)\right)\right)} - I\right) H(x)\Big\rangle_{L^2(\Omega;\mathcal{F})}.
\end{aligned} \tag{4. 15}$$

The first summand on the r.h.s.of (4.15) can be evaluated as follows

$$\begin{aligned}
&\Bigg|\int_{\mathbb{R}^d} dx\, \Big\langle F(b(t)+x) - F(x), \\
&\qquad e^{i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^d \left(\int_0^t \pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))db_\mu + \int_0^t \partial_\mu \pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))ds\right)\right)} H(x)\Big\rangle_{L^2(\Omega;\mathcal{F})}\Bigg| \\
&\le \|H\|_{\mathcal{M}} \left(\int_\Omega \|F(b(t)+\cdot) - F(\cdot)\|_{\mathcal{M}}^2 d\mu\right)^{\frac{1}{2}}.
\end{aligned}$$

Since

$$\lim_{t\to 0} \|F(b(t)+\cdot) - F(\cdot)\|_{\mathcal{M}} = 0, \quad a.s. b \in \Omega,$$

the first summand on the r.h.s.of (4.15) converges to 0. On the second summand on the r.h.s. of (4.15), one can see that by Remark 2.2 (3)

$$s - \lim_{t\to 0}\left(\int_0^t \pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))\, db_\mu + \frac{1}{2}\int_0^t \partial_\mu \pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))\, ds\right) = 0$$

in $L^2(\Omega; [\widetilde{\mathcal{H}}_{-1}])$. As in the case of (4.12), we see that for $\Phi \in \mathcal{F}$

$$s - \lim_{t\to 0} \exp\left(i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^d \left(\int_0^t \pi_{-1}(\widetilde{\rho}_\mu(b(s)+x))\, db_\mu + \frac{1}{2}\int_0^t \partial_\mu \pi_{-1}(\widetilde{\rho}(b(s)+x))\, ds\right)\right)\right)\Phi = \Phi$$

in $L^2(\Omega;\mathcal{F})$.

Hence, by the Lebesgue dominated convergence theorem, we can derive that the second summand converges to 0 as $t \to 0$. The strong continuity in the case $t > 0$ is proven similarly. □

By Lemma 4.7 and Hille-Yoshida's theorem, for each $\rho \in [C_b^2(\mathbb{R}^d; \mathcal{H}_{-1})]$, there exists a unique positive self-adjoint operator $\widetilde{\mathbf{H}}_{\rho,0}$ in $\mathcal{M}$ such that

$$G_\rho(t) = e^{-t\widetilde{\mathbf{H}}_{\rho,0}}.$$

Lemma 4.8 *Let* $\rho \in [C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$. *Then the self-adjoint operator* $\widetilde{\mathbf{H}}_{\rho,0}$ *is a self-adjoint extension of* $\mathbf{H}_{\rho,0}|_{\mathcal{M}_\rho^\infty}$.

Proof: Let $F \in D(\widetilde{\mathbf{H}}_{\rho,0})$ and $G \in \mathcal{M}_\rho^\infty$. Then we have

$$\begin{aligned}
\left\langle \frac{1}{t}\left(e^{-t\widetilde{\mathbf{H}}_{\rho,0}} - I\right) G, F\right\rangle_{\mathcal{M}} &= \lim_{n\to\infty} \left\langle \frac{1}{t}\left(Q^{n*}_{\rho,\frac{t}{n*}} - I\right) G, F\right\rangle_{\mathcal{M}} \\
&= \lim_{n\to\infty} \sum_{j=0}^{n*-1} \frac{1}{n*} \left\langle \frac{n*}{t}\left(Q_{\rho,\frac{t}{n*}} - I\right) G, Q^{n*\frac{j}{n*}}_{\rho,\frac{t}{n*}} F\right\rangle_{\mathcal{M}} \\
&= \lim_{n\to\infty} \int_0^1 \left\langle \frac{n*}{t}\left(Q_{\rho,\frac{t}{n*}} - I\right) G, Q^{[n*s]}_{\rho,\frac{t}{n*}} F\right\rangle_{\mathcal{M}} ds.
\end{aligned}$$

Since, by Lemma 4.5,

$$w - \lim_{n\to 0} \frac{n*}{t}\left(Q_{\frac{t}{n*}} - I\right) G = -\mathbf{H}_{\rho,0} G,$$

the norm $||\frac{n*}{t}(Q_{\frac{t}{n*}} - I)G||_{\mathcal{M}}$ is uniformly bounded in n. By Remark 2.4, we can see that

$$s - \lim_{n\to\infty} Q^{[n*s]}_{\rho,\frac{t*}{n}} = G_\rho(ts).$$

Then we have

$$\left\langle \frac{1}{t}\left(e^{-t\widetilde{\mathbf{H}}_{\rho,0}} - I\right) G, F\right\rangle_{\mathcal{M}} = \int_0^1 ds \left\langle -\mathbf{H}_{\rho,0} G, e^{-ts\widetilde{\mathbf{H}}_{\rho,0}} F\right\rangle_{\mathcal{M}}.$$

As $t \to 0$ on the both sides, we get

$$\left\langle G, \widetilde{\mathbf{H}}_{\rho,0} F\right\rangle_{\mathcal{M}} = \langle \mathbf{H}_{\rho,0} G, F\rangle_{\mathcal{M}},$$

which implies that $G \in D(\widetilde{\mathbf{H}}_{\rho,0})$ and

$$\widetilde{\mathbf{H}}_{\rho,0} G = \mathbf{H}_{\rho,0} G.$$

Thus the proof is complete. □

We denote the extension $\widetilde{\mathbf{H}}_{\rho,0}$ by the same symbol $\mathbf{H}_{\rho,0}$. We give a rigorous definition of $\mathbf{H}_\rho$ in terms of the form sum $\dot{+}$ of $\mathbf{H}_{\rho,0}$ and $I \otimes \mathbf{H}_0$;

$$\mathbf{H}_\rho = \mathbf{H}_{\rho,0} \dot{+} I \otimes \mathbf{H}_0.$$

Next we study functional integral representations concerning $e^{-t\mathbf{H}_\rho}$. We introduce a multiplication operator in $L^2(\mathbb{R}^d;\mathcal{E})$ by

$$\phi^{\rho,s}_{\mathcal{E},\mu} \equiv \int_{\mathbb{R}^d}^{\oplus} \phi_{\mathcal{E}}\left(\pi_{-2}\left(\tilde{j}_s \tilde{\rho}_\mu(x)\right)\right) dx.$$

We define an operator acting in $L^2(\mathbb{R}^d;\mathcal{E})$ by

$$\mathbf{H}_{\rho,0,s}=\frac{1}{2}\sum_{\mu=1}^{d}\left(-iD_\mu\otimes I-\phi_{\mathcal{E},\mu}^{\rho,s}\right)^2,\quad s\geq 0.$$

Since for $\rho\in[C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$, $\pi_{-2}(\widetilde{j}_s\widetilde{\rho}_\mu(\cdot))\in C_b^2(\mathbb{R}^d;[\widetilde{\mathcal{H}}_{-2}])$, we see that $\widetilde{j}_s\widetilde{\rho}_\mu(\cdot)\in[C_b^2(\mathbb{R}^d;\mathcal{H}_{-2})]$. Then one can define a self-adjoint operator $\mathbf{H}_{\rho,0,s}$ in the same way as $\mathbf{H}_{\rho,0}$. Then, the following equation holds for $F,H\in L^2(\mathbb{R}^d;\mathcal{E})$

$$\begin{aligned}&\left\langle F,e^{-t\mathbf{H}_{\rho,0,k}}H\right\rangle_{\mathcal{E}}\\&=\int_{\mathbb{R}^d\times\Omega}d\mu\int_{Q_{-2}}d\mu_{-2}e^{i\phi_{\mathcal{E}}\left(\sum_{\mu=1}^d\left(\int_0^t\pi_{-2}\left(\widetilde{j}_k\widetilde{\rho}_\mu(b(s)+x)\right)db_\mu+\frac{1}{2}\int_0^t\partial_\mu\pi_{-2}\left(\widetilde{j}_k\widetilde{\rho}_\mu(b(s)+x)ds\right)\right)\right)}\\&\quad\times\overline{F}(b(t)+x)H(x).\end{aligned}$$

Lemma 4.9 *Let $\rho\in[C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$. Then the following equation holds on $L^2(\mathbb{R}^d;\mathcal{E})$*

$$J_se^{-t\mathbf{H}_{\rho,0,}}J_s^*=E_se^{-t\mathbf{H}_{\rho,0,s}}E_s,$$

where J_s and E_s are defined in (3.3).

Proof: Note that for any $A\in\widetilde{\mathcal{H}}_{-1}$, $J_se^{i\phi_{\mathcal{F}}(A)}J_s^*=E_se^{i\phi_{\mathcal{E}}(\widetilde{j}_sA)}E_s$, and $(J_s^*F)(x)=J_s^*(F(x))$. From (4.10) it follows that for $F,H\in L^2(\mathbb{R}^d;\mathcal{E})$,

$$\begin{aligned}&\left\langle F,J_se^{-t\mathbf{H}_{\rho,0}}J_s^*H\right\rangle_{L^2(\mathbb{R}^d;\mathcal{E})}\\&=\int_{\mathbb{R}^d\times\Omega}d\mu\,\Big\langle(J_s^*F)(b(t)+x),\\&\qquad e^{i\phi_{\mathcal{F}}\left(\sum_{\mu=1}^d\left(\int_0^t\pi_{-1}\left(\widetilde{\rho}_\mu(b(s')+x)\right)db_\mu+\frac{1}{2}\int_0^t\partial_\mu\pi_{-1}\left(\widetilde{\rho}_\mu(b(s')+x)ds'\right)\right)\right)}(J_s^*H)(x)\Big\rangle_{\mathcal{F}}\\&=\int_{\mathbb{R}^d\times\Omega}d\mu\,\Big\langle F(b(t)+x),\\&\qquad E_se^{i\phi_{\mathcal{E}}\left(\sum_{\mu=1}^d\widetilde{j}_s\left(\int_0^t\pi_{-1}\left(\widetilde{\rho}_\mu(b(s')+x)\right)db_\mu+\frac{1}{2}\int_0^t\partial_\mu\pi_{-1}\left(\widetilde{\rho}_\mu(b(s')+x)ds'\right)\right)\right)}E_sH(x)\Big\rangle_{\mathcal{E}}\\&=\left\langle F,E_se^{-t\mathbf{H}_{\rho,0,s}}E_sH\right\rangle_{L^2(\mathbb{R}^d;\mathcal{E})}.\end{aligned}$$

Since $F,H\in L^2(\mathbb{R}^d;\mathcal{E})$ are arbitrary, the proof is complete. □

Now we are ready to state the main theorem in this section.

Theorem 4.10 *Let $F,G\in\mathcal{M}$, $V\in C_b(\mathbb{R}^d)$ and $\rho\in[C_b^2(\mathbb{R}^d;\mathcal{H}_{-1})]$ such that*

$$\sup_{x\in\mathbb{R}^d}\|[\widetilde{\omega}]\pi_{-1}(\widetilde{\rho}_\mu(x))\|_{-1}=\sup_{x\in\mathbb{R}^d}|\widetilde{\omega}\widetilde{\rho}_\mu(x)|_{-1}<\infty,\quad\mu=1,...,d,\tag{4. 16}$$

$$\left\|\sum_{\mu=1}^d\partial_\mu\pi_{-1}(\widetilde{\rho}_\mu(x))\right\|_{-1}=0,(\textit{the Coulomb gauge}).\tag{4. 17}$$

Then

$$\left\langle F, e^{-t(\mathbf{H}_\rho+V)}G\right\rangle_{\mathcal{M}} = \int_{\mathbb{R}^d\times\Omega} d\mu \int_{Q_{-2}} d\mu_{-2}\exp\left(-\int_0^t V(b(s)+x)ds\right)$$

$$\times\exp\left(i\phi_{\mathcal{E}}\left(\sum_{\mu=1}^{d}\int_0^t [\widetilde{j}_{0\to t}]\pi_{-1}\left(\widetilde{\rho}_\mu(b(s)+x)\right)db_\mu\right)\right) J_t\overline{F}(b(t)+x)J_0G(x), \quad (4.\ 18)$$

where

$$\int_0^t [\widetilde{j}_{0\to t}]\pi_{-1}\left(\widetilde{\rho}_\mu(b(s)+x)\right)db_\mu$$

is the time-ordered $[\widetilde{\mathcal{H}}_{-2}]$-valued stochastic integral associated with the family of isometries $[\widetilde{j}_t]$ from $[\widetilde{\mathcal{H}}_{-1}]$ to $[\widetilde{\mathcal{H}}_{-2}]$.

Proof: By the strong Trotter product formula [18] and Proposition 4.3 (c), we see that

$$\begin{aligned}&\left\langle F, e^{-t(\mathbf{H}_\rho+V)}G\right\rangle_{\mathcal{M}}\\ &= \lim_{n\to\infty}\left\langle F, \left(e^{-\frac{t}{n*}\mathbf{H}_{\rho,0}}e^{-\frac{t}{n*}\mathbf{H}_0}e^{-\frac{t}{n*}V}\right)^{n*}G\right\rangle_{\mathcal{M}}\\ &= \lim_{n\to\infty}\left\langle F, J_t^*\left(J_te^{-\frac{t}{n*}\mathbf{H}_{\rho,0}}J_t^*\right)e^{-\frac{t}{n*}V}\left(J_{t-\frac{t}{n*}}e^{-\frac{t}{n*}\mathbf{H}_{\rho,0}}J^*_{t-\frac{t}{n*}}\right)e^{-\frac{t}{n*}V}\ldots\ldots\right.\\ &\qquad\left.\ldots\ldots\left(J_{\frac{t}{n*}}e^{-\frac{t}{n*}\mathbf{H}_{\rho,0}}J^*_{\frac{t}{n*}}\right)e^{-\frac{t}{n*}V}J_0G\right\rangle_{\mathcal{M}},\end{aligned}$$

from which and Lemma 4.9 it follows that

$$\begin{aligned}&= \lim_{n\to\infty}\left\langle F, J_t^*\left(E_te^{-\frac{t}{n*}\mathbf{H}_{\rho,0,t}}E_t\right)e^{-\frac{t}{n*}V}\left(E_{t-\frac{t}{n*}}e^{-\frac{t}{n}\mathbf{H}_{\rho,0,t-\frac{t}{n*}}}E_{t-\frac{t}{n*}}\right)e^{-\frac{t}{n*}V}\right.\\ &\qquad\left.\ldots e^{-\frac{t}{n*}V}\left(E_{\frac{t}{n*}}e^{-\frac{t}{n*}\mathbf{H}_{\rho,0,\frac{t}{n*}}}E_{\frac{t}{n*}}\right)e^{-\frac{t}{n*}V}J_0G\right\rangle_{\mathcal{M}}\\ &\equiv \lim_{n\to\infty}S_{n*}.\end{aligned}$$

Let $\frac{t}{n*}=s$. From the definition of $\mathbf{H}_{\rho,0,t'}$ and Lemma 4.6 it follows that

$$\begin{aligned}S_{n*} &= \lim_{k\to\infty}\left\langle F, J_t^*\left(E_tQ^{k*}_{\rho,\frac{s}{k*},t}E_t\right)e^{-sV}\left(E_{t-s}Q^{k*}_{\rho,\frac{s}{k*},t-s}E_{t-s}\right)e^{-sV}\ldots\right.\\ &\qquad\left.\ldots e^{-sV}\left(E_sQ^{k*}_{\rho,\frac{s}{k*},s}E_s\right)e^{-sV}J_0G\right\rangle_{\mathcal{M}}\\ &\equiv \lim_{k\to\infty}S_{n*,k*},\end{aligned}$$

where $Q_{\rho,t,t'}$ is defined by operators on $L^2(\mathbb{R}^d;\mathcal{E})$ such that

$$\begin{aligned}(Q_{\rho,t,t'}F)(x) &= \int_{\mathbb{R}^d} p_t(x-y)U_{\rho,t'}(x,y)F(y)dy, \quad t>0,\\ (Q_{\rho,0,t'}F)(x) &= F(x),\end{aligned}$$

and

$$U_{\rho,t'}(x,y) = \exp\left\{\frac{1}{2}i\phi_{\mathcal{E}}\left(\sum_{\mu=1}^{d}[\widetilde{j}_{t'}]\pi_{-1}\left(\widetilde{\rho}_\mu(x)+\widetilde{\rho}_\mu(y)\right)(x_\mu-y_\mu)\right)\right\}.$$

One can see that

$$S_{n*,k*} = \int_{\mathbb{R}^d\times\underbrace{\mathbb{R}^{dk*}\times\ldots\times\mathbb{R}^{dk*}}_{n*}} P_s(\vec{x_1})\ldots P_s(\vec{x}_{n*})e^{-s\sum_{j=1}^{n*}V(x_j^{k*})}$$
$$\times\langle F(x), J_t^*\left(E_tU_{\rho,t}(\vec{x_1})E_t\right)\left(E_{t-s}U_{\rho,t-s}(\vec{x_2})E_{t-s}\right)\ldots$$
$$\ldots\left(E_sU_{\rho,s}(\vec{x_{n*}})E_s\right)J_0G(x_{n*}^k)\rangle_{\mathcal{F}}, \qquad (4.\ 19)$$

where

$$\begin{aligned} P_s(\vec{x_j}) &= p_s(x_{j-1}^{k*}-x_j^1)p_s(x_j^1-x_j^2)\ldots p_s(x_j^{k*-1}-x_j^{k*}),\\ U_{\rho,\alpha}(\vec{x_j}) &= U_{\rho,\alpha}(x_{j-1}^{k*},x_j^1)U_{\rho,\alpha}(x_j^1,x_j^2)\ldots U_{\rho,\alpha}(x_j^{k*-1},x_j^{k*}),\\ &= \exp\left\{\frac{1}{2}i\phi_{\mathcal{E}}\left(\sum_{i=1}^{k*}[\tilde{j}_\alpha]\sum_{\mu=1}^{d}\pi_{-1}\left(\tilde{\rho}_\mu(x_j^{i-1})+\tilde{\rho}_\mu(x_j^i)\right)(x_{j,\mu}^{i-1}-x_{j,\mu}^i)\right)\right\},\\ &\quad x_j^0 = x_{j-1}^{k*},\ x_0^{k*}\equiv x, j=1,\ldots,n*. \end{aligned}$$

By Proposition 3.3 one can neglect E_j in (4.19), so that

$$\begin{aligned} S_{n*,k*} &= \int_{\mathbb{R}^d}dx\Big\langle F(b(t)+x),\\ &\quad J_t^*\exp\left(i\phi_{\mathcal{E}}\left(\sum_{\mu=1}^{d}\sum_{j=0}^{n*-1}[\tilde{j}_{js}]\pi_{-1}\left(\square_{\mu,j,k}(x)\right)\right)-s\sum_{j=1}^{n*}V\left(b(js)+x\right)\right)J_0G(x)\Big\rangle_{L^2(\Omega;\mathcal{F})}, \end{aligned}$$

where

$$\begin{aligned} \square_{\mu,j,k}(x) &= \sum_{m=1}^{k*}\left\{\tilde{\rho}_\mu\left(b\left(\frac{m}{k*}s+js\right)+x\right)+\tilde{\rho}_\mu\left(b\left(\frac{m-1}{k*}s+js\right)+x\right)\right\}\\ &\quad\times\left\{b_\mu\left(\frac{m}{k*}s+js\right)-b_\mu\left(\frac{m-1}{k*}s+js\right)\right\},\quad j=0,\ldots,n*-1. \end{aligned}$$

As in the case of (4.12), by the Coulomb gauge condition (4.17), we see that for each $x\in\mathbb{R}^d$ and $\Phi\in\mathcal{E}$,

$$\begin{aligned} &s-\lim_{k\to\infty}\exp\left(i\phi_{\mathcal{E}}\left(\sum_{\mu=1}^{d}\sum_{j=0}^{n*-1}[\tilde{j}_{js}]\pi_{-1}\left(\square_{\mu,j,k}(x)\right)\right)\right)\Phi\\ &=\exp\left(i\phi_{\mathcal{E}}\left(\sum_{\mu=1}^{d}\sum_{j=0}^{n*-1}\int_{js}^{(j+1)s}[\tilde{j}_{js}]\pi_{-1}\left(\tilde{\rho}_\mu(x+b(s'))\right)db_\mu\right)\right)\Phi \end{aligned}$$

in $L^2(\Omega;\mathcal{E})$. On the other hand, we have

$$\begin{aligned} &\left|\Big\langle J_tF(b(t)+\cdot),\exp\left(i\phi_{\mathcal{E}}\left(\sum_{\mu=1}^{d}\sum_{j=0}^{n*-1}[\tilde{j}_{js}]\pi_{-1}\left(\square_{\mu,j,k}(\cdot)\right)\right)-s\sum_{j=1}^{n*}V\left(b(js)+\cdot\right)\right)J_0G(\cdot)\Big\rangle_{L^2(\Omega;\mathcal{E})}\right|\\ &\le\exp\left(-s\inf_x V(x)\right)||F(b(t)+\cdot)||_{L^2(\Omega;\mathcal{F})}||G(\cdot)||_{L^2(\Omega;\mathcal{F})}\in L^1(\mathbb{R}^d). \end{aligned}$$

Hence, by the Lebesgue dominated convergence theorem, we have

$$\lim_{k\to\infty} S_{n*,k*} = \int_{\mathbb{R}^d} dx \Big\langle J_t F(b(t)+x), \exp\left(i\phi_{\mathcal{E}} \left(\sum_{\mu=1}^{d} \sum_{j=0}^{n*-1} \square_{\mu,j}(x) \right) - s\sum_{j=1}^{n*} V\left(b(js)+x\right) \right) J_0 G(x) \Big\rangle_{L^2(\Omega;\mathcal{E})},$$

where

$$\square_{\mu,j}(x) = \int_{js}^{(j+1)s} [\widetilde{j}_{js}]\pi_{-1}\left(\widetilde{\rho}_\mu(b(s')+x)\right) db_\mu.$$

Note that for sufficiently small $\epsilon > 0$

$$\begin{aligned} \left\| [\widetilde{j}_{t+\epsilon}]^* [\widetilde{j}_t] \pi_{-1}(\widetilde{\rho}_\mu(x)) - \pi_{-1}(\widetilde{\rho}_\mu(x)) \right\|_{-1} &= \left\| e^{-\epsilon[\widetilde{\omega}]} \pi_{-1}(\widetilde{\rho}_\mu(x)) - \pi_{-1}(\widetilde{\rho}_\mu(x)) \right\|_{-1} \\ &\le \epsilon \sup_{x\in\mathbb{R}^d} |\widetilde{\omega}\widetilde{\rho}_\mu(x)|_{-1}. \end{aligned}$$

Hence by Theorem 2.5 and (4.16), we see that for $x \in \mathbb{R}^d$

$$\begin{aligned} s-\lim_{n\to\infty} \sum_{j=0}^{n*-1} \square_{\mu,j}(x) &= s-\lim_{n\to\infty} \sum_{j=0}^{n*-1} \int_{\frac{jt}{n*}}^{\frac{j+1}{n*}t} [\widetilde{j}_{\frac{jt}{n*}}]\pi_{-1}\left(\widetilde{\rho}_\mu(b(s)+x)\right) db_\mu \\ &= \int_0^t [\widetilde{j}_{0\to t}]\pi_{-1}\left(\widetilde{\rho}_\mu(x+b(s))\right) db_\mu, \end{aligned}$$

in $L^2(\Omega; [\widetilde{\mathcal{H}}_{-2}])$. Then again as in the case of (4.12), for each $x \in \mathbb{R}^d$ and $\Phi \in \mathcal{E}$

$$\begin{aligned} &s-\lim_{n\to\infty} \exp\left(i\phi_{\mathcal{E}} \left(\sum_{\mu=1}^{d} \sum_{j=0}^{n*-1} \int_{\frac{jt}{n*}}^{\frac{j+1}{n*}t} [\widetilde{j}_{\frac{jt}{n*}}]\pi_{-1}\left(\widetilde{\rho}_\mu(b(s)+x)\right) db_\mu \right) \right) \Phi \\ &= \exp\left(i\phi_{\mathcal{E}} \left(\sum_{\mu=1}^{d} \int_0^t [\widetilde{j}_{0\to t}]\pi_{-1}\left(\widetilde{\rho}_\mu(x+b(s))\right) db_\mu \right) \right) \Phi \end{aligned} \tag{4. 20}$$

in $L^2(\Omega; \mathcal{E})$. Passing to the subsequences for n (hereafter denoted again by n), (4.20) holds for each $x \in \mathbb{R}^d$ and $a.s.b \in \Omega$ in the strong topology of $\mathcal{E}$. Since $V(b(s)+x)$ is continuous in s for each $x \in \mathbb{R}^d$ and $a.s.b \in \Omega$, we have

$$\lim_{n\to\infty} \frac{t}{n*} \sum_{j=0}^{n*-1} V\left(b\left(\frac{jt}{n*}\right) + x \right) = \int_0^t V(b(s)+x)ds, \quad x \in \mathbb{R}^d \ a.s.b \in \Omega.$$

Furthermore,

$$\begin{aligned} &\left| \exp\left(-s\sum_{j=1}^{n*} V\left(b(js)+\cdot\right) \right) \Big\langle J_t F(b(t)+\cdot), \exp\left(i\phi_{\mathcal{E}} \left(\sum_{\mu=1}^{d} \sum_{j=0}^{n*-1} \square_{\mu,j}(\cdot) \right) \right) J_0 G(\cdot) \Big\rangle_{\mathcal{E}} \right| \\ &\le \exp\left(-s\inf_x V(x) \right) \|F(b(t)+\cdot)\|_{\mathcal{F}} \|F(\cdot)\|_{\mathcal{F}} \in L^1(\mathbb{R}^d \times \Omega; d\mu). \end{aligned}$$

Hence, again by the Lebesgue dominated convergence theorem we get (4.18). □

Similarly to the classical case [31,Theorem 15.5], we have an interest in extending (4.18) to more general potentials. From (4.18) it follows that for $V \in C_b(\mathbb{R}^d)$ and ρ satisfying the conditions stated in Theorem 4.10

$$\left|\left\langle F, e^{-t(\mathbf{H}_\rho+V)}G\right\rangle_{\mathcal{M}}\right| \leq \left\langle \|F\|_{\mathcal{F}}, e^{-t(-\frac{1}{2}\Delta+V)}\|G\|_{\mathcal{F}}\right\rangle_{L^2(\mathbb{R}^d)}. \tag{4. 21}$$

We define for $G \in \mathcal{M}$

$$(signG)(x) \equiv \begin{cases} \frac{G(x)}{\|G(x)\|_{\mathcal{F}}}, & \|G(x)\|_{\mathcal{F}} \neq 0, \\ 0, & \|G(x)\|_{\mathcal{F}} = 0. \end{cases}$$

Lemma 4.11 *Let $|V|$ be a multiplication operator which is $-\frac{1}{2}\Delta$ -form bounded with relative bound ϵ. Then for ρ satisfying the condition in Theorem 4.10, $|V|$ is $\mathbf{H}_\rho$-form bounded with relative bound $\leq \epsilon$.*

Proof: Substituting $V = 0$ and $F = sign\left(e^{-t\mathbf{H}_\rho}G\right) \cdot \psi$, where $\psi \in C_0^\infty(\mathbb{R}^d)$ and $\psi \geq 0$, in (4.21), we have

$$\left\langle \psi, \|e^{-t\mathbf{H}_\rho}G\|_{\mathcal{F}}\right\rangle_{L^2(\mathbb{R}^d)} \leq \left\langle \psi, e^{-t(-\frac{1}{2}\Delta)}\|G\|_{\mathcal{F}}\right\rangle_{L^2(\mathbb{R}^d)}.$$

Hence it follows that

$$\left\|\left(e^{-t\mathbf{H}_\rho}G\right)(x)\right\|_{\mathcal{F}} \leq \left(e^{-t(-\frac{1}{2}\Delta)}\|G(\cdot)\|_{\mathcal{F}}\right)(x), \quad a.e.x \in \mathbb{R}^d.$$

Since

$$\left((\mathbf{H}_\rho + E)^{-\frac{1}{2}}G\right)(x) = \Gamma\left(\frac{1}{2}\right)\int_0^\infty e^{-Et}t^{-\frac{1}{2}}\left(e^{-t\mathbf{H}_\rho}G\right)(x)dt, \quad a.e.x \in \mathbb{R}^d, E > 0,$$

one can see that

$$\left\|\left((\mathbf{H}_\rho + E)^{-\frac{1}{2}}G\right)(x)\right\|_{\mathcal{F}} \leq \left(\left(-\frac{1}{2}\Delta + E\right)^{-\frac{1}{2}}\|G(\cdot)\|_{\mathcal{F}}\right)(x), \quad a.e.x \in \mathbb{R}^d.$$

Then we have

$$\int_{\mathbb{R}^d}\left\{|V(x)|^{\frac{1}{2}}\left\|\left((\mathbf{H}_\rho + E)^{-\frac{1}{2}}G\right)(x)\right\|_{\mathcal{F}}\right\}^2 dx$$
$$\leq \int_{\mathbb{R}^d}\left\{|V(x)|^{\frac{1}{2}}\left(\left(-\frac{1}{2}\Delta + E\right)^{-\frac{1}{2}}\|G(\cdot)\|_{\mathcal{F}}\right)(x)\right\}^2 dx.$$

Thus

$$\frac{\left\||V|^{\frac{1}{2}}(\mathbf{H}_\rho + E)^{-\frac{1}{2}}G\right\|_{\mathcal{M}}}{\|G\|_{\mathcal{M}}} \leq \frac{\left\||V|^{\frac{1}{2}}\left(-\frac{1}{2}\Delta + E\right)^{-\frac{1}{2}}\|G(\cdot)\|_{\mathcal{F}}\right\|_{L^2(\mathbb{R}^d)}}{\|\|G(\cdot)\|_{\mathcal{F}}\|_{L^2(\mathbb{R}^d)}},$$

which implies that the following operator norm estimate holds

$$\left\| |V|^{\frac{1}{2}} (\mathbf{H}_\rho + E)^{-\frac{1}{2}} \right\|_{\mathcal{M}} \leq \left\| |V|^{\frac{1}{2}} \left(-\frac{1}{2}\Delta + E \right)^{-\frac{1}{2}} \right\|_{L^2(\mathbb{R}^d)} . \tag{4. 22}$$

Since

$$\lim_{E\to\infty} \left\| |V|^{\frac{1}{2}} \left(-\frac{1}{2}\Delta + E \right)^{-\frac{1}{2}} \right\|_{L^2(\mathbb{R}^d)} = \epsilon,$$

the lemma follows. □

For a multiplication operator V, we set $V_+ = max\{0, V\}$ and $V_- = max\{0, -V\}$. Let us introduce a class P of potentials.

Definition 4.12 *A potential V is in the set P if and only if $V_+ \in L^1_{loc}(\mathbb{R}^d)$ and V_- is $-\frac{1}{2}\Delta$-form bounded with relative bound < 1.*

□

For $V \in P$, we define a quadratic form t by

$$t(F, F) = \left\langle \mathbf{H}_\rho^{\frac{1}{2}} F, \mathbf{H}_\rho^{\frac{1}{2}} F \right\rangle_{\mathcal{M}} + \left\langle V_+^{\frac{1}{2}} F, V_+^{\frac{1}{2}} F \right\rangle_{\mathcal{M}} - \left\langle V_-^{\frac{1}{2}} F, V_-^{\frac{1}{2}} F \right\rangle_{\mathcal{M}},$$
$$Q(t) = Q(\mathbf{H}_\rho) \cap Q(V_+),$$

where $Q(A)$ denotes the form domain of a positive self-adjoint operator A, i.e. $Q(A) = D(A^{\frac{1}{2}})$. By Lemma 4.11, t is positive closed form on $Q(t)$. We denote the positive self-adjoint operator associated with t by

$$\mathbf{H}_\rho \dot{+} V_+ \dot{-} V_-,$$

so that $Q(\mathbf{H}_\rho \dot{+} V_+ \dot{-} V_-) = Q(t)$.

Theorem 4.13 *Let $V \in P$ and $\rho \in [C_b^2(\mathbb{R}^d; \mathcal{H}_{-1})]$ satisfy the conditions in Theorem 4.10. Then (4.18) holds with $\mathbf{H}_\rho + V$ replaced by $\mathbf{H}_\rho \dot{+} V_+ \dot{-} V_-$.*

Proof: As in [31,Theorem 6.2], one can easily see that by an approximation argument, (4.18) holds for $V \in L^\infty(\mathbb{R}^d)$(the set of bounded functions). Fix $V \in P$. We set

$$V_{+n}(x) = \begin{cases} V_+(x), & V_+(x) < n, \\ n, & V_+(x) \geq n \end{cases}, V_{-m}(x) = \begin{cases} V_-(x), & V_-(x) < m, \\ m, & V_-(x) \geq m \end{cases}.$$

Then we have

$$\left\langle F, e^{-t(\mathbf{H}_\rho + V_{+n} - V_{-m})} G \right\rangle_{\mathcal{M}} = \int_{\mathbb{R}^d \times \Omega} d\mu \exp\left(-\int_0^t (V_{+n} - V_{-m})(b(s) + x) ds \right)$$
$$\times \left\langle J_t F(b(t) + x), \exp\left(i\phi_{\mathcal{E}} \left(\sum_{\mu=1}^d \int_0^t [\tilde{j}_{0\to t}] \pi_{-1} \left(\tilde{\rho}_\mu(b(s) + x) \right) db_\mu \right) \right) J_0 G(x) \right\rangle_{\mathcal{E}} . \tag{4. 23}$$

Define closed quadratic forms by

$$\begin{aligned} t_{n,m}(F,F) &= \left\langle \mathbf{H}_\rho^{\frac12}, \mathbf{H}_\rho^{\frac12} \right\rangle_{\mathcal{M}} + \left\langle V_{+n}^{\frac12} F, V_{+n}^{\frac12} F \right\rangle_{\mathcal{M}} - \left\langle V_{-m}^{\frac12} F, V_{-m}^{\frac12} F \right\rangle_{\mathcal{M}}, \\ & \quad Q(t_{n,m}) = Q(\mathbf{H}_\rho), \\ t_{n,\infty}(F,F) &= \left\langle \mathbf{H}_\rho^{\frac12}, \mathbf{H}_\rho^{\frac12} \right\rangle_{\mathcal{M}} + \left\langle V_{+n}^{\frac12} F, V_{+n}^{\frac12} F \right\rangle_{\mathcal{M}} - \left\langle V_{-}^{\frac12} F, V_{-}^{\frac12} F \right\rangle_{\mathcal{M}}, \\ & \quad Q(t_{n,\infty}) = Q(\mathbf{H}_\rho), \\ t_{\infty,\infty}(F,F) &= \left\langle \mathbf{H}_\rho^{\frac12}, \mathbf{H}_\rho^{\frac12} \right\rangle_{\mathcal{M}} + \left\langle V_{+}^{\frac12} F, V_{+}^{\frac12} F \right\rangle_{\mathcal{M}} - \left\langle V_{-}^{\frac12} F, V_{-}^{\frac12} F \right\rangle_{\mathcal{M}}, \\ & \quad Q(t_{\infty,\infty}) = Q(\mathbf{H}_\rho) \cap Q(V_+). \end{aligned}$$

We denote the self-adjoint operator associated with $t_{n,\infty}$ by $\mathbf{H}_\rho + V_{+n} \dot{-} V_-$. We have

$$t_{n,m} \geq t_{n,m+1} \geq t_{n,m+2} \geq \dots \geq t_{n,\infty}$$

and $t_{n,m} \to t_{n,\infty}$ in the sense of quadratic form on $\cup_m Q(t_{n,m}) = Q(\mathbf{H}_\rho)$. Since $t_{n,\infty}$ is closed on $Q(\mathbf{H}_\rho)$, by the monotone convergence theorem for forms ([17,VIII. Theorem 3.11]), the associated positive self-adjoint operators satisfy

$$\mathbf{H}_\rho + V_{+n} - V_{-m} \to \mathbf{H}_\rho + V_{+n} \dot{-} V_-$$

in the strong resolvent sense, which implies that for all $t \geq 0$,

$$\exp\left(-t\left(\mathbf{H}_\rho + V_{+n} - V_{-m}\right)\right) \to \exp\left(-t\left(\mathbf{H}_\rho + V_{+n} \dot{-} V_-\right)\right)$$

strongly. Similarly, we have

$$t_{n,\infty} \leq t_{n+1,\infty} \leq t_{n+2,\infty} \leq \dots \leq t_{\infty,\infty}$$

and $t_{n,\infty} \to t_{\infty,\infty}$ in the sense of quadratic form on

$$\left\{ F \in \cap_n Q(t_{n,\infty}) \,\middle|\, \sup_n t_{n,\infty}(F,F) < \infty \right\} = Q(\mathbf{H}_\rho) \cap Q(V_+).$$

Hence the monotone convergence theorem for forms ([17,VIII. Theorem 3.13],[30]), we get

$$\exp\left(-t\left(\mathbf{H}_\rho \dot{+} V_{+n} \dot{-} V_-\right)\right) \to \exp\left(-t\left(\mathbf{H}_\rho \dot{+} V_{+} \dot{-} V_-\right)\right), \quad t \geq 0,$$

strongly. On the other hand, the r.h.s.of (4.23) converges to

$$\begin{aligned} &\int_{\mathbb{R}^d \times \Omega} d\mu \exp\left(-\int_0^t \left(V_{+n} - V_-\right)(b(s)+x) ds\right) \\ &\times \left\langle J_t F(b(t)+x), \exp\left(i\phi_{\mathcal{E}}\left(\sum_{\mu=1}^d \int_0^t [\widetilde{j}_{0\to t}] \phi_{-1}\left(\widetilde{\rho}_\mu(b(s)+x)\right) db_\mu\right)\right) J_0 G(x) \right\rangle_{\mathcal{E}} . \end{aligned} \tag{4. 24}$$

as $m \to \infty$ by the monotone convergence theorem for integrals. Also (4.24) converges to the r.h.s of (4.18) as $n \to \infty$ by the Lebesgue dominated convergence theorem. Thus the proof is complete. □

5 INEQUALITIES

In this section we shall derive some inequalities similar to classical models from the functional integral representation constructed in Section IV. Let $\sigma(A)$ be the spectrum of A. For simplicity, we put for $V \in P$

$$\begin{aligned} \mathbf{H}_\rho \dot{+} V_+ \dot{-} V_- &\equiv \mathbf{H}_\rho + V, \\ -\frac{1}{2}\Delta \dot{+} V_+ \dot{-} V_- &\equiv -\frac{1}{2}\Delta + V. \end{aligned}$$

Theorem 5.1 *(Diamagnetic inequality)*
Let $V \in P$ and $\rho \in [C_b^2(\mathbb{R}^d; \mathcal{H}_{-1})]$ satisfy the conditions in Theorem 4.10. Then

$$\inf \sigma\left(-\frac{1}{2}\Delta + V\right) \leq \inf \sigma\left(\mathbf{H}_\rho + V\right). \tag{5. 1}$$

Proof: Similarly to (4.21), we can also see that for $V \in P$

$$\left|\left\langle F, e^{-t(\mathbf{H}_\rho + V)} G\right\rangle_{\mathcal{M}}\right| \leq \left\langle ||F||_{\mathcal{F}}, e^{-t(-\frac{1}{2}\Delta + V)} ||G||_{\mathcal{F}}\right\rangle_{L^2(\mathbb{R}^d)}. \tag{5. 2}$$

Fix $G \in \mathcal{M}$ such that $E_{\mathbf{H}_\rho + V}([E_0, E_0 + \epsilon))G \neq 0$, for all $0 < \epsilon < \epsilon_0$ with some $\epsilon_0 > 0$, where $E_{\mathbf{H}_\rho + V}$ denotes the spectral projection of $\mathbf{H}_\rho + V$ and $E_0 = \inf \sigma(\mathbf{H}_\rho + V)$. Then

$$\begin{aligned} \inf \sigma\left(\mathbf{H}_\rho + V\right) &= \lim_{t \to \infty} -\frac{1}{t} \log \left\langle G, e^{-t(\mathbf{H}_\rho + V)} G\right\rangle_{\mathcal{M}} \\ &\geq \lim_{t \to \infty} -\frac{1}{t} \log \left\langle ||G||_{\mathcal{F}}, e^{-t(-\frac{1}{2}\Delta + V)} ||G||_{\mathcal{F}}\right\rangle_{L^2(\mathbb{R}^d)} \\ &\geq \lim_{t \to \infty} -\frac{1}{t} \log \left(||G||_{\mathcal{F}}^2 e^{-t \inf \sigma(-\frac{1}{2}\Delta + V)}\right) \\ &\geq \inf \sigma\left(-\frac{1}{2}\Delta + V\right). \end{aligned}$$

Thus (5.1) follows. □

Theorem 5.2 *(Abstract Kato's inequality)*
Let $V \in P$ and $\rho \in [C_b^2(\mathbb{R}^d; \mathcal{H}_{-1})]$ satisfy the conditions in Theorem 4.10.
Suppose that $\psi \in D((-\frac{1}{2}\Delta + V)^{\frac{1}{2}})$, $\psi \geq 0$ and $G \in D\left((\mathbf{H}_\rho + V)\right)$.
Then $||G(\cdot)||_{\mathcal{F}} \in D\left(\left(-\frac{1}{2}\Delta + V\right)^{\frac{1}{2}}\right)$ and the following inequality holds

$$\Re \left\langle (sign G)\psi, \left(\mathbf{H}_\rho + V\right) G\right\rangle_{\mathcal{M}} \geq \left\langle \left(-\frac{1}{2}\Delta + V\right)^{\frac{1}{2}} \psi, \left(-\frac{1}{2}\Delta + V\right)^{\frac{1}{2}} ||G||_{\mathcal{F}}\right\rangle_{L^2(\mathbb{R}^d)}. \tag{5. 3}$$

Proof: The proof is a slight modification of that of [15]. By (4.21), we have

$$\left\langle ||F||_{\mathcal{F}}, \frac{I - e^{-t(\frac{1}{2}\Delta + V)}}{t} ||F||_{\mathcal{F}} \right\rangle_{L^2(\mathbb{R}^d)} \leq \left\langle F, \frac{I - e^{-t(\mathbf{H}_\rho + V)}}{t} F \right\rangle_{\mathcal{M}}. \tag{5. 4}$$

Putting

$$s_t(\phi, \phi) \equiv \left\langle \phi, \frac{I - e^{-t(\frac{1}{2}\Delta + V)}}{t} \phi \right\rangle_{L^2(\mathbb{R}^d)}, \quad \phi \in L^2(\mathbb{R}^d),$$

one can see that $\{s_t\}_{t \geq 0}$ is a family of positive closed quadratic forms such that

$$s_t \leq s_k, \quad t \leq k.$$

Thus the monotone convergence theorem for forms, we see that $s_\infty(G, G) \equiv \sup_t s_t(G, G)$ is a closed quadratic form on

$$R_\infty = \{F \in \mathcal{M} |\sup_t (F, F) < \infty\},$$

moreover the corresponding positive self-adjoint operators to s_t converge to $\left(-\frac{1}{2}\Delta + V\right)^{\frac{1}{2}}$ in the strong resolvent sense. Since, by (5.4), $F \in D(\mathbf{H}_\rho + V)$ implies that $||F||_{\mathcal{F}} \in R_\infty$, we have $||F||_{\mathcal{F}} \in D\left(\left(-\frac{1}{2}\Delta + V\right)^{\frac{1}{2}}\right)$. By (5.2), we have

$$\Re \left\langle F, \frac{1}{t}\left(e^{-t(\mathbf{H}_\rho + V)} - I\right) G \right\rangle_{\mathcal{M}} \leq \left\langle ||F||_{\mathcal{F}}, \frac{1}{t} e^{-t(-\frac{1}{2}\Delta + V)} ||G||_{\mathcal{F}} \right\rangle_{L^2(\mathbb{R}^d)} - \frac{1}{t} \Re \langle F, G \rangle_{\mathcal{M}}. \tag{5. 5}$$

Substituting $F = (signG)\psi$ into (5.5), we have

$$\Re \left\langle (signG)\,\psi, \frac{1}{t}\left(e^{-t(\mathbf{H}_\rho + V)} - I\right) G \right\rangle_{\mathcal{M}} \leq \left\langle \psi, \frac{1}{t}\left(e^{-t(-\frac{1}{2}\Delta + V)} - I\right) ||G||_{\mathcal{F}} \right\rangle_{L^2(\mathbb{R}^d)}.$$

Then, taking $t \to 0+$ we see that

$$\lim_{t \to 0+} \Re \left\langle (signG)\,\psi, \frac{1}{t}\left(e^{-t(\mathbf{H}_\rho + V)} - I\right) G \right\rangle_{\mathcal{M}} = -\Re \langle (signG)\psi, (\mathbf{H}_\rho + V)\, G \rangle_{\mathcal{M}},$$

and

$$\lim_{t \to 0+} \left\langle \psi, \frac{1}{t}\left(e^{-t(-\frac{1}{2}\Delta + V)} - I\right) ||G||_{\mathcal{F}} \right\rangle_{L^2(\mathbb{R}^d)}$$
$$= -\left\langle \left(-\frac{1}{2}\Delta + V\right)^{\frac{1}{2}} \psi, \left(-\frac{1}{2}\Delta + V\right)^{\frac{1}{2}} ||G|| \right\rangle_{L^2(\mathbb{R}^d)}.$$

Thus (5.3) follows. □

6 REMARKS

(1) Any concrete core of the Hamiltonian $\mathbf{H}_\rho$ defined in Section IV is not known. In [23], in the case when the coupling cconstant is sufficiently small, the authors proved essentially self-adjointness of $\mathbf{H}_\rho$.
(2) In the FKI formula, the Wiener path measure $d\mu$ is more useful than the Brownian path measure Db. For the Schrödinger Hamiltonian

$$\mathbf{H}_{cl} = \frac{1}{2}\sum_{\mu=1}^{d}(-iD_\mu - A_\mu)^2 + V$$

the FKI formula is, first, established for a magnetic vector potential $A_\mu(\cdot) \in C_0^\infty(\mathbb{R}^d), \mu = 1, ..., d$, and after that it is extended to $A_\mu(\cdot) \in L^2_{loc}(\mathbb{R}^d)$ by a limiting argument([31]).

In the model which we consider, $\phi^\rho_{\mathcal{F},\mu}$ corresponds to the classical magnetic vector potential $A_\mu(\cdot)$. But we have no strategy of limiting argument used in the classical model. Then it is necessary to deal with the Hilbert space-valued stochastic integral for $\phi^\rho_{\mathcal{F},\mu}$ such that $\pi_{-1}(\tilde{\rho}_\mu(\cdot)) \in C_b^2(\mathbb{R}^d; [\widetilde{\mathcal{H}}_{-1}])$ directly (not using limiting arguments as in the classical case). Then it can not be assumed that $\pi_{-1}(\tilde{\rho}_\mu(\cdot)) \in H^2(\mathbb{R}^d; [\widetilde{\mathcal{H}}_{-1}])$ i.e., one can not define (see (2.6))

$$\int_0^t \phi_{\mathcal{F}}(\pi_{-1}(\tilde{\rho}_\mu(\omega(s))))\, d\omega_\mu.$$

Therefore we consider the Hilbert space-valued stochastic integral not on the Wiener path but on the Brownian path. .
(3) The FKI formula holds without the Coulomb gauge condition([31]). However, in our model, if one does not assume the Coulomb gauge condition (4.17), then in the integrand on the r.h.s.of (4.18), the factor

$$\exp\left(-\frac{1}{2}\sum_{k=1}^{2^n}\int_{\frac{k-1}{2^n}t}^{\frac{k}{2^n}t}[\tilde{j}_{\frac{k-1}{2^n}t}]\sum_{\mu=1}^{d}\partial_\mu\pi_{-1}(\tilde{\rho}_\mu(b(s)))\,ds\right) \qquad (6.\ 1)$$

appears. It is not clear how to show the convergence of (6.1) as $n \to \infty$ in the strong topology in $L^2(\Omega; [\widetilde{\mathcal{H}}_{-2}])$.
(4) In scalar field theory ([21,22,26]), the range of the projection $e_{[a,b]}$ (notations follow [26]) can be characterized by some support properties([26,Proposition III.4]), i.e.

$$Ran\left(e_{[a,b]}\right) = \left\{f \in N \,\middle|\, suppf \subset (a,b)\times\mathbb{R}^d\right\}^-.$$

In particular

$$Ran(e_t) = \left\{f \in N \,\middle|\, suppf \subset \{t\}\times\mathbb{R}^d\right\}^-.$$

However, the corresponding projection $[e_{[a,b]}]$, which we introduce in Proposition 3.2, can not be characterized in such a way. For example

$$Ran([e_t]) \neq \left\{\pi_2(f) \in [\widetilde{\mathcal{H}}_{-2}] \,\middle|\, suppf \subset \oplus_{\mu=1}^{d}\{t\}\times\mathbb{R}^d\right\}^-.$$

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