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CONDITIONAL EXPECTATION IN CLASSICAL AND QUANTUM WHITE NOISE CALCULI

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Introduction

The present paper continues the new approach to quantum stochastic processes on Fock space developed in a series of papers [22], [23], [24], [25]. It is the noticeable feature of this approach that the quantum white noise, i.e., the time derivative of quantum Brownian motion, is formulated as a C^{∞}-flow of operators on Fock space. More precisely, the role of the annihilation process $\{A_t\}$ and the creation process $\{A_t^*\}$ in the works of Belavkin [1], Hudson–Parthasarathy [13], Meyer [19] and Parthasarathy [26] is played by their infinitesimal increments:

$$a_t = \frac{d}{dt}A_t, \qquad a_t^* = \frac{d}{dt}A_t^*.$$

It is very common that these operators are understood as *operator-valued distributions* and hence are not defined pointwisely. On the other hand, it is also known (though not widely used in practice) that the creation and annihilation operators are defined pointwisely using a suitable Gelfand triple, see e.g., [3], [6], [14]. In particular, the special choice of Gelfand triple of white noise functions

$$(E) \subset L^2(E^*, \mu) \cong \Gamma(L^2(\mathbb{R})) \subset (E)^*$$

yields such situation; in fact, $a_t \in \mathcal{L}((E),(E))$ and $a_t^* \in \mathcal{L}((E)^*,(E)^*)$. The above Gelfand triple is referred to as the *Hida–Kubo–Takenaka space* [8], [15]. A similar structure called Fock scale is introduced by Belavkin [1] in order to develop a non-adapted Itô theory on Fock space, though the pointwisely defined annihilation and creation operators are not formulated. A big advantage of the Hida–Kubo–Takenaka space is also found in [20] where a general theory of operators in $\mathcal{L}((E),(E)^*)$ is established systematically in terms of pointwisely defined annihilation and creation operators, see also [21] for generalization to vector-valued white noise distributions.

There lives a canonical flow $\{B_t\}_{t\in\mathbb{R}}$ called *Brownian motion* in $L^2(E^*,\mu) \cong \Gamma(L^2(\mathbb{R}))$. Then the conditional expectation E_t relative to the σ-field generated by $\{B_s \;\; s \le t\}$ becomes one of the most fundamental concepts in both classical and quantum stochastic analyses.

In fact, the conditional expectation E_t is an orthogonal projection acting on $L^2(E^*, \mu)$ and therefore, belongs to $\mathcal{L}((E), (E)^*)$. In that sense it can be treated fully within our operator theory; however, in various applications we need to discuss the conditional expectation of a white noise distribution. Unfortunately, the conditional expectation is not defined on the whole space $(E)^*$ of white noise distributions due to the fact that pointwise multiplication of distributions is not defined in general. This would be one of the reasons why the conditional expectation has not been discussed actively along with the Hida–Kubo–Takenaka space. While, being based on a different framework of white noise distributions Hida [9] introduced the conditional expectation and suggested possibility of application to prediction theory[1].

In this paper we propose an idea to overcome the above mentioned difficulty. Namely, we introduce a certain space of test white noise functions, denoted by $(\mathcal{A})$, which is bigger than (E) and obtain by duality a space of white noise distributions, denoted by $(\mathcal{A})^*$. There holds a simple inclusion relation among these spaces:

$$(E) \subset (\mathcal{A}) \subset L^2(E^*, \mu) \subset (\mathcal{A})^* \subset (E)^*.$$

A white noise distribution belonging to $(\mathcal{A})^*$ is called *admissible.* It is shown that the conditional expectation E_t becomes a continuous operator from $(\mathcal{A})^*$ into itself which keeps $(\mathcal{A})$ invariant. Accordingly, in both classical and quantum cases the notion of an *admissible process* is naturally introduced and the conditional expectation of such a process becomes an interesting subject to study. In this paper we study the Hitsuada–Skorokhod integral of an admissible process and observe how the conditional expectation acts on it. Moreover, we derive prototypes of representation of a martingale in terms of stochastic integrals both in classical and quantum cases. In particular, the result in classical case is thought of as a variant of the so-called Clark formula [4] which has been discussed with great interests in various aspects, e.g., in connection with martingale representation, see also [28] for a white noise approach[2].

The paper is organized as follows: Section 1 is devoted to assembling some technical instruments in the operator theory on white noise distributions. In Section 2 we introduce admissible white noise distributions and the conditional expectation. In Section 3 we study the Hitsuda–Skorokhod integral and derive a variant of the Clark formula. In Section 4 we introduce the conditional expectation for operators and the notion of an admissible quantum stochastic process. In Section 5 we discuss quantum stochastic integrals in terms of white noise calculus. In particular, we obtain the conditional expectation of a quantum Hitsuda–Skorokhod integral and discuss representation of a quantum martingale in terms of stochastic integrals.

1 Preliminaries

In the recent development the basic framework of white noise calculus is constructed from an arbitrary topological space T keeping in mind applications to quantum and random fields [15], [20]. This framework is called the *standard setup of white noise calculus* [11]. The present paper being devoted to a study of a stochastic "process," we take $T = \mathbb{R}$

[1] I thank Professor H.-H. Kuo for the information.

[2] I thank Professor H. Watanabe for interesting conversation on this topic.

and regard it as the time axis. Some of the results obtained below remain valid under the standard setup after straightforward modification.

1.1 Triplet of white noise functionals

Let $H = L^2(\mathbb{R}, dt)$ be the Hilbert space of $\mathbb{R}$-valued L^2-functions on $\mathbb{R}$ with norm $|\cdot|_0$ and inner product $\langle\cdot, \cdot\rangle$, and consider the Gelfand triple:

$$E = \mathcal{S}(\mathbb{R}) \subset H = L^2(\mathbb{R}, dt) \subset E^* = \mathcal{S}'(\mathbb{R}). \tag{1.1}$$

It is known that the topology of E is defined by the norms:

$$|\xi|_p = |A^p\xi|_0, \qquad \xi \in E, \qquad p \in \mathbb{R},$$

where

$$A = 1 + t^2 - \frac{d^2}{dt^2}.$$

These norms are linearly ordered in the sense that

$$|\xi|_p \le \rho^q |\xi|_{p+q}, \qquad p \in \mathbb{R}, \quad q \ge 0, \tag{1.2}$$

where

$$\rho = \inf \operatorname{Spec}(A) = \frac{1}{2}.$$

In fact, E is a countable Hilbert nuclear space. The canonical bilinear form on $E^* \times E$, being compatible with the real inner product of H, is denoted also by $\langle\cdot, \cdot\rangle$.

The *Gaussian measure* associated with the Gelfand triple (1.1) is the unique probability measure μ on E^* satisfying

$$\exp\left(-\frac{1}{2}|\xi|_0^2\right) = \int_{E^*} e^{i\langle x, \xi\rangle}\mu(dx), \qquad \xi \in E.$$

The probability space (E^*, μ) is called the *Gaussian space.* Let

$$(L^2) \equiv L^2(E^*, \mu; \mathbb{C})$$

denote the Hilbert space of $\mathbb{C}$-valued L^2-functions on the Gaussian space (E^*, μ). When a probabilistic aspect is emphasized, we also use the symbol

$$\mathbf{E}(\phi) = \int_{E^*} \phi(x)\mu(dx),$$

which is the *mean value (random average)* of a random variable $\phi \in L^1(E^*, \mu)$.

The canonical bilinear form on $(E^{\otimes n})^* \times E^{\otimes n}$ is denoted by $\langle\cdot, \cdot\rangle$ again and its $\mathbb{C}$-bilinear extension to $(E_{\mathbb{C}}^{\otimes n})^* \times E_{\mathbb{C}}^{\otimes n}$ is also denoted by the same symbol[3]. For a non-negative integer n and $x \in E^*$ an element $:x^{\otimes n}: \in (E^{\otimes n})^*_{\mathrm{sym}}$ is uniquely defined by

$$\phi_\xi(x) \equiv \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \frac{\xi^{\otimes n}}{n!} \right\rangle = \exp\left(\langle x, \xi\rangle - \frac{1}{2}\langle \xi, \xi\rangle\right), \qquad \xi \in E_{\mathbb{C}}, \quad x \in E^*, \tag{1.3}$$

[3]Throughout the paper we do not use a specific symbol for the hermitian inner product.

where ϕ_ξ is the so-called *exponential vector*. In particular, ϕ_0 is called the *vacuum*. As is well known, each $\phi \in (L^2)$ is expressed in the following form:

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f_n \right\rangle, \qquad x \in E^*, \quad f_n \in H_{\mathbb{C}}^{\widehat{\otimes} n}, \tag{1.4}$$

where each function $x \mapsto \langle :x^{\otimes n}:, f_n \rangle$ and the convergence of the series are understood in the L^2-sense. Expression (1.4) is called the *Wiener–Itô expansion* of ϕ. In that case,

$$\| \phi \|_0^2 \equiv \int_{E^*} |\phi(x)|^2 \mu(dx) = \sum_{n=0}^{\infty} n! \, | f_n |_0^2 .$$

Thus we have a unitary isomorphism between (L^2) and $\Gamma(H_{\mathbb{C}})$, the Boson Fock space over $H_{\mathbb{C}}$. This is the celebrated *Wiener–Itô–Segal isomorphism.*

For $\phi \in (L^2)$ with Wiener–Itô expansion given as in (1.4) we put

$$\Gamma(A)\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, A^{\otimes n} f_n \right\rangle .$$

Then $\Gamma(A)$ becomes a positive selfadjoint operator on (L^2) with Hilbert–Schmidt inverse, and a complex Gelfand triple[4)] is thereby obtained:

$$(E) \subset (L^2) \equiv L^2(E^*, \mu; \mathbb{C}) \cong \Gamma(H_{\mathbb{C}}) \subset (E)^*. \tag{1.5}$$

Elements in (E) and $(E)^*$ are called a *test (white noise) functional* and a *generalized (white noise) functional*, respectively. We denote by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ the canonical bilinear form on $(E)^* \times (E)$ and by $\| \cdot \|_p$ the norm induced from $\Gamma(A)$, namely,

$$\| \phi \|_p^2 = \| \Gamma(A)^p \phi \|_0^2 = \sum_{n=0}^{\infty} n! | (A^{\otimes n})^p f_n |_0^2 = \sum_{n=0}^{\infty} n! \, | f_n |_p^2 , \qquad p \in \mathbb{R}, \tag{1.6}$$

where ϕ and $(f_n)_{n=0}^{\infty}$ are related through the Wiener–Itô expansion (1.4). It is obvious from (1.6) that $\phi \in (L^2)$ belongs to (E) if and only if $f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}$ for all n and $\sum_{n=0}^{\infty} n! \, | f_n |_p^2 < \infty$ for all $p \geq 0$.

We use a similar (but formal) expression for a generalized white noise functional. For each non-negative integer n let $F_n \in (E_{\mathbb{C}}^{\otimes n})^*_{\mathrm{sym}}$ be given and assume that $\sum_{n=0}^{\infty} n! \, | F_n |_{-p}^2 < \infty$ for some $p \geq 0$. Then there exists a unique $\Phi \in (E)^*$ such that

$$\langle\!\langle \Phi, \phi \rangle\!\rangle = \sum_{n=0}^{\infty} n! \, \langle F_n, f_n \rangle , \qquad \phi \in (E),$$

where ϕ and $(f_n)_{n=0}^{\infty}$ are related as in (1.4). In that case Φ is written in a formal series:

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, F_n \right\rangle . \tag{1.7}$$

[4)] The notation (1.5) is commonly used in the standard setup of white noise calculus, see e.g., [20]. As the white noise triplet discussed here is constructed from the special Gelfand triple (1.1), it is often denoted by $(\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^*$ instead, see e.g., [10] or I. Dôku's paper in this volume. Note also the remark at the beginning of this section.

Conversely, every $\Phi \in (E)^*$ is obtained in this way. Expression (1.7) is called the *Wiener-Itô expansion* of Φ. Note that (1.6) is also true for Φ. Moreover, for $f \in E_{\mathbb{C}}^*$ we define the *exponential vector* $\phi_f \in (E)^*$ through its Wiener–Itô expansion in a similar manner as in (1.3).

1.2 Brownian motion and white noise

Through the Wiener–Itô–Segal isomorphism we define $B_t \in (L^2)$ by

$$B_t(x) = \begin{cases} \left\langle x,\, 1_{[0,t]} \right\rangle, & t \geq 0, \\ -\left\langle x,\, 1_{[t,0]} \right\rangle, & t < 0, \end{cases}$$

where 1_J denotes the indicator function of $J \subset \mathbb{R}$. Note that $:x^{\otimes 1}:= x$ by definition. Since the delta function δ_t belongs to $E^* = \mathcal{S}'(\mathbb{R})$, by construction

$$W_t(x) = \langle x,\, \delta_t \rangle\,, \qquad t \in \mathbb{R},$$

is a white noise distribution, i.e., $W_t \in (E)^*$. As is easily seen,

$$B_0 = 0, \qquad \mathbf{E}(B_t) = 0, \qquad \mathbf{E}(B_s B_t) = s \wedge t \equiv \min\{s, t\}, \quad s, t \geq 0,$$

which means that $\{B_t\}$ is a Brownian motion. It is easily verified that the map $t \mapsto B_t \in L^2(E^*, \mu)$ is continuous. An important consequence of our approach is illustrated in the following

Proposition 1.1 *The map $t \mapsto B_t \in (E)^*$ is a C^∞-map*[5] *and it holds that*

$$\frac{d}{dt}\, B_t = W_t, \qquad t \in \mathbb{R}.$$

Hence $t \mapsto W_t \in (E)^*$ is also a C^∞-map.

Thus the one-parameter family of white noise distributions $\{W_t\}$, which is justifiably called the *white noise*, is a C^∞-flow in $(E)^*$.

1.3 Integral kernel operators, symbols and Fock expansion

Throughout the paper $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are locally convex spaces, denotes the space of continuous linear maps from $\mathfrak{X}$ into $\mathfrak{Y}$. Unless otherwise stated $\mathcal{L}(\mathfrak{X}, \mathfrak{Y})$ carries the topology of uniform convergence on every bounded subset of $\mathfrak{X}$ (the topology of bounded convergence).

We sketch briefly the essence of the operator theory on white noise distributions, see [20] and [21] for the detailed account. For each $y \in E_{\mathbb{C}}^*$ there exists a unique operator $D_y \in \mathcal{L}((E), (E))$ such that

$$D_y \phi_\xi = \langle y,\, \xi \rangle\, \phi_\xi, \qquad \xi \in E_{\mathbb{C}}.$$

[5]Throughout the paper the dual space of a locally convex space carries the strong dual topology.

This is called the *annihilation operator*. In particular,

$$a_t = D_{\delta_t}, \qquad t \in \mathbb{R},$$

is called the *annihilation operator at a point* or *Hida's differential operator*[6]. Then $a_t \in \mathcal{L}((E),(E))$ and $a_t^* \in \mathcal{L}((E)^*,(E)^*)$. The latter is called the *creation operator at a point.* It is emphasized that these operators are *not* operator-valued distributions but continuous operators for themselves.

For each $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$ there exists a unique operator $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E)^*)$ such that

$$\langle\!\langle \Xi_{l,m}(\kappa)\phi,\, \psi\rangle\!\rangle = \langle \kappa,\, \eta_{\phi,\psi}\rangle\,, \qquad \phi,\psi \in (E),$$

where

$$\eta_{\phi,\psi}(s_1,\cdots,s_l,t_1,\cdots,t_m) = \left\langle\!\!\left\langle a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m}\phi,\, \psi\right\rangle\!\!\right\rangle.$$

We use a formal (but descriptive) integral expression:

$$\Xi_{l,m}(\kappa) = \int_{\mathbb{R}^{l+m}} \kappa(s_1,\cdots,s_l,t_1,\cdots,t_m) a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m, \tag{1.8}$$

which is called an *integral kernel operator* with *kernel distribution* κ. It is known that κ is uniquely determined whenever it is taken from the subspace

$$(E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)} = \left\{\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*\,;\; s_{l,m}(\kappa) = \kappa\right\},$$

where $s_{l,m}$ is the symmetrizing operator with respect to the first l and the last m variables independently.

The *symbol* of $\Xi \in \mathcal{L}((E),(E)^*)$ is a $\mathbb{C}$-valued function on $E_{\mathbb{C}} \times E_{\mathbb{C}}$ defined by

$$\widehat{\Xi}(\xi,\eta) = \langle\!\langle \Xi\phi_\xi,\, \phi_\eta\rangle\!\rangle\,, \qquad \xi,\eta \in E_{\mathbb{C}}. \tag{1.9}$$

Since the exponential vectors $\{\phi_\xi;\, \xi \in E_{\mathbb{C}}\}$ span a dense subspace of (E), the symbol determines the operator uniquely.

Theorem 1.2 *For a function $\Theta : E_{\mathbb{C}} \times E_{\mathbb{C}} \to \mathbb{C}$ there exists an operator $\Xi \in \mathcal{L}((E),(E)^*)$ such that $\widehat{\Xi} = \Theta$ if and only if the following two conditions are satisfied:*

(O1) *for any $\xi,\xi_1,\eta,\eta_1 \in E_{\mathbb{C}}$, the function $(z,w) \mapsto \Theta(z\xi+\xi_1, w\eta+\eta_1)$ is entire holomorphic in $z,w \in \mathbb{C}$;*

(O2) *there exist constant numbers $C \ge 0$, $K \ge 0$ and $p \in \mathbb{R}$ such that*

$$|\Theta(\xi,\eta)| \le C \exp K\left(|\,\xi\,|_p^2 + |\,\eta\,|_p^2\right), \qquad \xi,\eta \in E_{\mathbb{C}}.$$

In that case,

$$\|\,\Xi\phi\,\|_{-(p+q+1)} \le CM(K,p,q)\,\|\,\phi\,\|_{p+q+1}\,, \qquad \phi \in (E),$$

where $M(K,p,q) \ge 0$ is a constant number depending on $K \ge 0$, $p \ge 0$, $q > q_0(K,p)$; and $q_0(K,p) > 0$ is also a constant number depending on $K \ge 0$, $p \ge 0$.

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[6] In most literature of white noise calculus the annihilation operator at a point is denoted by ∂_t. However, respecting the common notation among a wide audience, we use in this paper a_t instead.

Theorem 1.3 *For any $\Xi \in \mathcal{L}((E),(E)^*)$ there exists a unique family of kernel distributions $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes(l+m)})^*_{\mathrm{sym}(l,m)}$ such that*

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}), \tag{1.10}$$

where the right hand side converges in $\mathcal{L}((E),(E)^*)$.

Expression (1.10) is called the *expansion of Ξ in terms of integral kernel operators* or the *Fock expansion.* It seems that such expression of a Fock space operator in terms of normal ordered products of annihilation and creation operators is common among theoretical physicists[7)]. The idea traces certainly back to Haag [7] and has been developed in various contexts in quantum field theory, see e.g., [2], [3]. It is the strong point of our theory that a wide class of Fock space operators is determined to be discussed with mathematical rigor using distribution theory. Thus our contribution here is purely mathematical.

Here are some of parallel results for an operator in $\mathcal{L}((E),(E))$ which is a subspace of $\mathcal{L}((E),(E)^*)$.

Lemma 1.4 *Let $\kappa \in (E_{\mathbb{C}}^{\otimes(l+m)})^*$. Then $\Xi_{l,m}(\kappa) \in \mathcal{L}((E),(E))$ if and only if $\kappa \in (E_{\mathbb{C}}^{\otimes l}) \otimes (E_{\mathbb{C}}^{\otimes m})^*$. In particular, $\Xi_{0,m}(\kappa) \in \mathcal{L}((E),(E))$ for any $\kappa \in (E_{\mathbb{C}}^{\otimes m})^*$.*

Theorem 1.5 *For a function $\Theta : E_{\mathbb{C}} \times E_{\mathbb{C}} \to \mathbb{C}$ there exists an operator $\Xi \in \mathcal{L}((E),(E))$ such that $\widehat{\Xi} = \Theta$ if and only if* (O1) *in Theorem 1.2 and the next condition are satisfied:*

(O2$'$) *for any $p \ge 0$ and $\epsilon > 0$ there exist $C \ge 0$ and $q \ge 0$ such that*

$$|\Theta(\xi,\eta)| \le C \exp \epsilon \left(|\xi|_{p+q}^2 + |\eta|_{-p}^2 \right), \qquad \xi,\eta \in E_{\mathbb{C}}.$$

In that case,

$$\| \Xi\phi \|_{p-1} \le CM(\epsilon,q,r) \, |\phi|_{p+q+r+1}, \qquad \phi \in (E),$$

where $M(\epsilon,q,r) \ge 0$ is a constant number depending on ϵ with $0 < \epsilon < (2e^3\delta^2)^{-1}$, $q \ge 0$, $r \ge r_0(q)$; and $r_0(q) \ge 0$ is also a constant number depending on $q \ge 0$.

Theorem 1.6 *For $\Xi \in \mathcal{L}((E),(E))$ let the Fock expansion be given as in (1.10). Then $\kappa_{l,m} \in (E_{\mathbb{C}}^{\otimes l}) \otimes (E_{\mathbb{C}}^{\otimes m})^*$ for all $l,m = 0,1,2,\cdots$, and the right hand side of (1.10) converges in $\mathcal{L}((E),(E))$.*

1.4 How to define an operator on white noise functions – An example

The operator symbol provides a useful criterion for checking whether or not an operator formally defined in Fock space falls into a continuous operator on the white noise functions (Theorems 1.2 and 1.5). Here is a simple illustration.

Recall first that $E_{\mathbb{C}}$ is closed under the pointwise multiplication; in fact, it yields a continuous bilinear map from $E_{\mathbb{C}} \times E_{\mathbb{C}}$ into $E_{\mathbb{C}}$. Therefore multiplication of $\xi \in E_{\mathbb{C}}$ and $f \in E_{\mathbb{C}}^*$, denoted by $f\xi = \xi f \in E_{\mathbb{C}}^*$, is defined as

$$\langle f\xi, \eta \rangle = \langle f, \xi\eta \rangle, \qquad \eta \in E_{\mathbb{C}}.$$

[7)] I thank Professor I. Ojima for this comment.

Proposition 1.7 *For any $f \in E_{\mathbb{C}}^*$ there exists a unique operator $\Xi \in \mathcal{L}((E),(E)^*)$ such that $\Xi\phi_\xi = \phi_{f\xi}$, $\xi \in E_{\mathbb{C}}$.*

PROOF. Since the exponential vectors are linearly independent, the correspondence $\phi_\xi \mapsto \phi_{f\xi}$, $\xi \in E_{\mathbb{C}}$, is extended to a linear operator from the linear space spanned by the exponential vectors into $(E)^*$. We put

$$\Theta(\xi,\eta) = \langle\!\langle \phi_{f\xi}, \phi_\eta \rangle\!\rangle = e^{\langle f\xi, \eta\rangle}, \qquad \xi,\eta \in E_{\mathbb{C}}. \tag{1.11}$$

It should be checked that Θ satisfies conditions (O1) and (O2) in Theorem 1.2. Since (O1) is obvious, we shall prove (O2). We choose $p \geq 0$ such that $| f |_{-p} < \infty$. Then,

$$| \langle f\xi, \eta\rangle | = | \langle f, \xi\eta\rangle | \leq | f |_{-p} | \xi\eta |_p .$$

By the continuity of pointwise multiplication of $E_{\mathbb{C}}$ we choose $q \geq 0$ and $C \geq 0$ such that

$$| \xi\eta |_p \leq C | \xi |_{p+q} | \eta |_{p+q}, \qquad \xi,\eta \in E_{\mathbb{C}},$$

and hence

$$| \langle f\xi, \eta\rangle | \leq C | f |_{-p} | \xi |_{p+q} | \eta |_{p+q} \leq \frac{C}{2} | f |_{-p} \left(| \xi |_{p+q}^2 + | \eta |_{p+q}^2 \right).$$

Thus (1.11) is estimated as

$$|\Theta(\xi,\eta)| \leq \exp\left\{ \frac{C}{2} | f |_{-p} \left(| \xi |_{p+q}^2 + | \eta |_{p+q}^2 \right) \right\},$$

which proves (O2). It then follows from Theorem 1.2 that there exists $\Xi \in \mathcal{L}((E),(E)^*)$ such that $\widehat{\Xi} = \Theta$. In other words,

$$\langle\!\langle \Xi\phi_\xi, \phi_\eta \rangle\!\rangle = \langle\!\langle \phi_{f\xi}, \phi_\eta \rangle\!\rangle, \qquad \xi,\eta \in E_{\mathbb{C}},$$

namely, $\Xi\phi_\xi = \phi_{f\xi}$ for any $\xi \in E_{\mathbb{C}}$. qed

REMARK. (1) The explicit action of Ξ in Proposition 1.7 is obtained easily. For $\phi \in (E)$ of which Wiener–Itô expansion is given as

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f_n \right\rangle, \qquad f_n \in E_{\mathbb{C}}^{\otimes n},$$

it holds that

$$\Xi\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f^{\otimes n} \cdot f_n \right\rangle, \tag{1.12}$$

where $f^{\otimes n} \cdot f_n$ is pointwise multiplication. In fact, for an exponential vector $\phi = \phi_\xi$ identity (1.12) is obvious. On the other hand, it is easy to see that the operator Ξ' defined by the right hand side of (1.12) belongs to $\mathcal{L}((E),(E)^*)$. Therefore Ξ in Proposition 1.7 coincides with Ξ' on the exponential vectors $\{\phi_\xi\}$, hence on the whole space (E).

(2) If $f \in L^\infty(\mathbb{R})$, then $\Xi \in \mathcal{L}((E)_q, (L^2))$ for some $q \geq 0$. In fact, in view of (1.12) we have

$$\| \Xi\phi \|_0^2 = \sum_{n=0}^{\infty} n! \, | f^{\otimes n} \cdot f_n |_0^2 \leq \sum_{n=0}^{\infty} n! \, \| f^{\otimes n} \|_\infty^2 \, | f_n |_0^2 .$$

Choose $q \geq 0$ such that $\rho^q \| f \|_\infty \leq 1$. Then

$$\| \Xi\phi \|_0^2 \leq \sum_{n=0}^{\infty} n! \, \| f \|_\infty^{2n} \rho^{2nq} \, | f_n |_q^2 \leq \sum_{n=0}^{\infty} n! \, | f_n |_q^2 = \| \phi \|_q^2 .$$

Namely, $\Xi \in \mathcal{L}((E)_q, (L^2))$. In particular, if $\| f \|_\infty \leq 1$, we see that $\Xi \in \mathcal{L}((L^2), (L^2))$. A typical example is the conditional expectation discussed below.

2 Conditional expectation for white noise distributions

2.1 Slowly increasing functions

For a $\mathbb{C}$-valued measurable function f on $\mathbb{R}$ we put

$$\| f \|_r^2 = \int_{-\infty}^{+\infty} |f(t)|^2 (1+t^2)^r dt, \qquad r \in \mathbb{R}.$$

Note the obvious inequality:

$$\| f \|_r \leq \| f \|_{r+r'}, \qquad r \in \mathbb{R}, \quad r' \geq 0.$$

Then $\mathcal{A}_r = \{f \, ; \, \| f \|_r < \infty\}$ becomes a Hilbert space with norm $\| \cdot \|_r$ (modulo null-functions) and forms an increasing chain of Hilbert spaces:

$$\cdots \mathcal{A}_2 \subset \mathcal{A}_1 \subset \mathcal{A}_0 = H_{\mathbb{C}} = L^2(\mathbb{R}, dt; \mathbb{C}) \subset \mathcal{A}_{-1} \subset \mathcal{A}_{-2} \subset \cdots . \tag{2.1}$$

Then

$$\mathcal{A} = \operatorname*{proj\,lim}_{r \to \infty} \mathcal{A}_r = \bigcap_{r \geq 0} \mathcal{A}_r$$

becomes a countable Hilbert space, and by general theory we have

$$\mathcal{A}^* = \operatorname*{ind\,lim}_{r \to \infty} \mathcal{A}_{-r} = \bigcup_{r \geq 0} \mathcal{A}_{-r},$$

where $\mathcal{A}^*$ is equipped with the strong dual topology as we have agreed. We say that $\mathcal{A}^*$ consists of *slowly increasing* functions.

Lemma 2.1 *For any $r \geq 0$ there exists $p \geq 0$ such that the natural injection $E_p \to \mathcal{A}_r$ is well defined and continuous. Therefore the natural injection $E_{\mathbb{C}} \to \mathcal{A}$ is continuous and has a dense image.*

PROOF. It is obvious that $\| \cdot \|_r$ is a continuous norm on $E_{\mathbb{C}}$. Since the defining norms $| \cdot |_p$ is linearly ordered (see (1.2)), given $r \geq 0$ there exist $p \geq 0$ and $C \geq 0$ such that

$$\| \xi \|_r \leq C \, | \xi |_p , \qquad \xi \in E_{\mathbb{C}}. \tag{2.2}$$

Hence the natural injection $E_p \to \mathcal{A}_r$ is well defined and continuous. Therefore the natural injection $E_{\mathbb{C}} \to \mathcal{A}$ is continuous. That $E_{\mathbb{C}}$ is a dense subspace of $\mathcal{A}_r$ is proved with a standard argument. qed

Lemma 2.2 *For any $r \geq 0$ there exists $p \geq 0$ such that the natural injection $\mathcal{A}_{-r} \to E_{-p}$ is well defined and continuous. In particular, $\mathcal{A}^* \to E^*_{\mathbb{C}}$ is a continuous injection.*

PROOF. Given $r \geq 0$ we choose $p \geq 0$ and $C \geq 0$ satisfying (2.2). Then for $f \in \mathcal{A}_{-r}$ we have

$$|\langle f, \xi \rangle| \leq \| f \|_{-r} \| \xi \|_r \leq C \| f \|_{-r} |\xi|_p .$$

Therefore $f \in E_{-p}$ and

$$|f|_{-p} \leq C \| f \|_{-r}, \qquad f \in \mathcal{A}_{-r}.$$

This completes the proof.

qed

REMARK. We shall prove that $\mathcal{A}$ is not a nuclear space. Let $\{e_n\}$ be a complete orthonormal basis of $L^2(\mathbb{R}, dt)$. Then

$$f_n(t) = e_n(t)(1+t^2)^{-(r+r')/2}$$

forms a complete orthonormal basis of $\mathcal{A}_{r+r'}$. We note that

$$\| f_n \|_r^2 = \int_{-\infty}^{+\infty} |f_n(t)|^2 (1+t^2)^r dt = \int_{-\infty}^{+\infty} |e_n(t)|^2 (1+t^2)^{-r'} dt.$$

Let T be the multiplication operator by $(1+t^2)^{-r'/2}$. Then

$$\| f_n \|_r^2 = \langle Te_n, Te_n \rangle = |Te_n|_0^2 .$$

Thus the natural injection $\mathcal{A}_{r+r'} \to \mathcal{A}_r$ is of Hilbert–Schmidt type if and only if so is the operator T on $L^2(\mathbb{R}, dt)$. If so T should be compact. But this never occurs because there is no non-zero multiplication operator on $L^2(\mathbb{R}, dt)$ which is compact.

2.2 Cut-off operator

For $t \in \mathbb{R}$ we put

$$\chi_t(s) = 1_{(-\infty,t]}(s) = \begin{cases} 1 & s \leq t \\ 0 & s > t \end{cases}$$

The multiplication operator induced by χ_t is denoted by the same symbol. Obviously we have

Lemma 2.3 *$\chi_t \in \mathcal{L}(\mathcal{A}_r, \mathcal{A}_r)$ and is an orthogonal projection for any $r \in \mathbb{R}$.*

Lemma 2.4 *For each $r \geq 0$ there exist $p \geq 0$ and $C \geq 0$ such that*

$$|\chi_t f|_{-p} \leq C \| f \|_{-r},$$
$$|(\chi_t - \chi_s) f|_{-p} \leq C |t-s|^{1/2} \| f \|_{-r},$$

where $s, t \in \mathbb{R}$ and $f \in \mathcal{A}_{-r}$. In particular, $t \mapsto \chi_t \in \mathcal{L}(\mathcal{A}_{-r}, E_{-p})$ is continuous.

PROOF. Let $f \in \mathcal{A}_{-r}$. Then for $\xi \in E_{\mathbb{C}}$ we have by the Schwartz inequality

$$|\langle \chi_t f, \xi \rangle| \le \| \chi_t f \|_{-r} \| \xi \|_r \le \| f \|_{-r} \| \xi \|_r . \tag{2.3}$$

In view of Lemma 2.1 we take $p_1 \ge 0$ and $C_1 \ge 0$ such that

$$\| \xi \|_r \le C_1 | \xi |_{p_1} , \qquad \xi \in E_{\mathbb{C}}.$$

Then (2.3) becomes

$$|\langle \chi_t f, \xi \rangle| \le C_1 \| f \|_{-r} | \xi |_{p_1} ,$$

and therefore

$$| \chi_t f |_{-p_1} \le C_1 \| f \|_{-r} , \qquad t \in \mathbb{R}, \quad f \in \mathcal{A}_{-r}. \tag{2.4}$$

Suppose next that $s \le t$. Since

$$\begin{aligned} |\langle (\chi_t - \chi_s) f, \xi \rangle|^2 &= \left| \int_s^t f(u) \xi(u) \, du \right|^2 \\ &\le \int_s^t |f(u)|^2 (1+u^2)^{-r} du \int_s^t |\xi(u)|^2 (1+u^2)^r du, \end{aligned}$$

we have

$$|\langle (\chi_t - \chi_s) f, \xi \rangle| \le \| f \|_{-r} (t-s)^{1/2} \max_{u \in \mathbb{R}} |\xi(u)| (1+u^2)^{r/2}, \qquad f \in \mathcal{A}_{-r}, \quad \xi \in E_{\mathbb{C}}.$$

Note that $\xi \mapsto \max_{u \in \mathbb{R}} |\xi(u)|(1+u^2)^{r/2}$ is a continuous norm on $E_{\mathbb{C}}$, one may find $p_2 \ge 0$ and $C_2 \ge 0$ such that

$$\max_{u \in \mathbb{R}} |\xi(u)|(1+u^2)^{r/2} \le C_2 | \xi |_{p_2} , \qquad \xi \in E_{\mathbb{C}}.$$

Then we see that

$$|\langle (\chi_t - \chi_s) f, \xi \rangle| \le C_2 (t-s)^{1/2} \| f \|_{-r} | \xi |_{p_2} ,$$

and therefore

$$| (\chi_t - \chi_s) f |_{-p_2} \le C_2 (t-s)^{1/2} \| f \|_{-r} , \qquad s \le t, \quad f \in \mathcal{A}_{-r}. \tag{2.5}$$

Finally we take $p = \max\{p_1, p_2\}$ and $C = \max\{C_1, C_2\}$. Then in view of (2.4) we have

$$| \chi_t f |_{-p} = | \chi_t f |_{-p_1 - (p - p_1)} \le \rho^{p - p_1} | \chi_t f |_{-p_1} \le \rho^{p-p_1} C_1 \| f \|_{-r} \le C \| f \|_{-r} ,$$

which proves the first inequality. The second one follows similarly from (2.5). qed

Lemma 2.5 *For each $r \ge 0$ there exist $p \ge 0$ and $C \ge 0$ such that*

$$\begin{aligned} &\| \chi_t \xi \|_r \le C | \xi |_p , \\ &\| (\chi_t - \chi_s) \xi \|_r \le C |t-s|^{1/2} | \xi |_p , \end{aligned}$$

where $s, t \in \mathbb{R}$ and $\xi \in E_p$. In particular, $t \mapsto \chi_t \in \mathcal{L}(E_p, \mathcal{A}_r)$ is continuous.

PROOF. This is the dual result of Lemma 2.4. qed

REMARK. It follows from Lemma 2.3 and the chain (2.1) that $\chi_t \in \mathcal{L}(\mathcal{A}_{r+r'}, \mathcal{A}_r)$ for any $r \in \mathbb{R}$ and $r' \geq 0$. But $t \mapsto \chi_t \in \mathcal{L}(\mathcal{A}_{r+r'}, \mathcal{A}_r)$ is not continuous whatever $r \in \mathbb{R}$ and $r' \geq 0$. In fact, suppose that $t \mapsto \chi_t \in \mathcal{L}(\mathcal{A}_{r+r'}, \mathcal{A}_r)$ is continuous at $t \in \mathbb{R}$ for $r \in \mathbb{R}$ and $r' \geq 0$. We further assume that $t \geq 0$; the case of $t \leq 0$ is proved in a similar manner. Then we have

$$\lim_{s \downarrow t} \sup_{\| f \|_{r+r'} \leq 1} \| (\chi_s - \chi_t) f \|_r^2 = 0. \tag{2.6}$$

On the other hand, if $t < s$ there exists a measurable function f such that $\operatorname{supp} f \subset (t, s)$ and

$$\| f \|_{r+r'}^2 = \int_t^s |f(u)|^2 (1+u^2)^{r+r'} du = 1.$$

Then

$$\begin{aligned} \| (\chi_s - \chi_t) f \|_r^2 &= \int_t^s |f(u)|^2 (1+u^2)^r du \\ &= \int_t^s |f(u)|^2 (1+u^2)^{r+r'} (1+u^2)^{-r'} du \\ &\geq (1+s^2)^{-r'}. \end{aligned}$$

Therefore

$$\sup_{\| f \|_{r+r'} \leq 1} \| (\chi_s - \chi_t) f \|_r^2 \geq (1+s^2)^{-r'},$$

and hence

$$\liminf_{s \downarrow t} \sup_{\| f \|_{r+r'} \leq 1} \| (\chi_s - \chi_t) f \|_r^2 \geq (1+t^2)^{-r'} > 0.$$

This contradicts (2.6).

2.3 Admissible white noise distributions

We introduce a new family of norms on white noise functions. For $\phi \in (E)$ with Wiener–Itô expansion

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \, f_n \right\rangle, \qquad f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n},$$

we put

$$\| \phi \|_{r,\beta}^2 = \sum_{n=0}^{\infty} n! e^{2\beta n} \| f_n \|_r^2, \qquad r, \beta \in \mathbb{R}. \tag{2.7}$$

Suppose $r \geq 0$ and $\beta \in \mathbb{R}$ are fixed. According to Lemma 2.1 we choose $p \geq 0$ and $C \geq 0$ such that

$$\| \xi \|_r \leq C \, | \xi |_p, \qquad \xi \in E_{\mathbb{C}}.$$

Then we have

$$\| f_n \|_r \leq C^n \, | f_n |_p, \qquad f_n \in E_{\mathbb{C}}^{\otimes n}. \tag{2.8}$$

Combining (2.7) and (2.8), we obtain

$$\begin{aligned}\| \phi \|_{r,\beta}^2 &\leq \sum_{n=0}^{\infty} n! e^{2\beta n} C^{2n} | f_n |_p^2 \\ &\leq \sum_{n=0}^{\infty} n! e^{2\beta n} C^{2n} \rho^{2qn} | f_n |_{p+q}^2 \\ &\leq \sum_{n=0}^{\infty} n! \left(C e^{\beta} \rho^q \right)^{2n} | f_n |_{p+q}^2 .\end{aligned}$$

Take $q \geq 0$ sufficiently large to have $Ce^{\beta}\rho^q \leq 1$. Then

$$\| \phi \|_{r,\beta}^2 \leq \sum_{n=0}^{\infty} n! \, | f_n |_{p+q}^2 = \| \phi \|_{p+q}^2 , \qquad \phi \in (E).$$

Let $(\mathcal{A})_{r,\beta}$ be the completion of (E) with respect to the norm $\| \cdot \|_{r,\beta}$. What we have proved above is summarized in the following

Lemma 2.6 *For any $r \geq 0$ and $\beta \in \mathbb{R}$ there exists $p \geq 0$ such that*

$$\| \phi \|_{r,\beta} \leq \| \phi \|_p , \qquad \phi \in (E). \tag{2.9}$$

In particular, the natural injection $(E)_p \to (\mathcal{A})_{r,\beta}$ is well defined and continuous.

In an obvious manner $\{(\mathcal{A})_{r,\beta}\}_{r,\beta \geq 0}$ forms a projective system of Hilbert spaces. Then

$$(\mathcal{A}) = \operatorname*{proj\,lim}_{r,\beta \to \infty} (\mathcal{A})_{r,\beta}$$

becomes a countable Hilbert space. On the other hand, $\{(\mathcal{A})_{-r,-\beta}\}_{r,\beta \geq 0}$ being an inductive system of Hilbert spaces, we have

$$(\mathcal{A})^* = \operatorname*{ind\,lim}_{r,\beta \to \infty} (\mathcal{A})_{-r,-\beta}.$$

In view of Lemma 2.6 we obtain an inclusion relation:

$$(E) \subset (\mathcal{A}) \subset (\mathcal{A})_{0,0} = (L^2) \subset (\mathcal{A})^* \subset (E)^*,$$

where the injections are all continuous. A white noise distribution belonging to $(\mathcal{A})^*$ is called *admissible*. Suppose $\Phi \in (E)^*$ is given with Wiener–Itô expansion

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, F_n \right\rangle .$$

Then Φ is admissible, i.e., $\Phi \in (\mathcal{A})^*$ if and only if there exist $r \geq 0$ and $\beta \geq 0$ such that $F_n \in \mathcal{A}_{-r}^{\otimes n}$ for all n and

$$\| \Phi \|_{-r,-\beta}^2 = \sum_{n=0}^{\infty} n! e^{-2\beta n} \| F_n \|_{-r}^2 < \infty.$$

2.4 Conditional expectation on admissible white noise distributions

For an admissible white noise distribution $\Phi \in (E)^*$ with Wiener–Itô expansion

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, F_n \right\rangle,$$

we put

$$E_t\Phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \chi_t^{\otimes n} \cdot F_n \right\rangle, \qquad t \in \mathbb{R}. \tag{2.10}$$

Lemma 2.7 *$E_t \in \mathcal{L}((\mathcal{A})_{r,\beta}, (\mathcal{A})_{r,\beta})$ and is an orthogonal projection for any $r, \beta \in \mathbb{R}$.*

In particular, $E_t \in \mathcal{L}((\mathcal{A}), (\mathcal{A}))$ and hence $E_t^* \in \mathcal{L}((\mathcal{A})^*, (\mathcal{A})^*)$. On the other hand, E_t^* being the unique continuous extension of E_t, we write $E_t^* = E_t$ for simplicity. The operator $E_t \in \mathcal{L}((\mathcal{A})^*, (\mathcal{A})^*)$ is called the *conditional expectation* (on admissible white noise distributions). Thus the conditional expectation E_t belongs to any of the spaces: $\mathcal{L}((\mathcal{A})_{r,\beta}, (\mathcal{A})_{r,\beta})$, $\mathcal{L}((\mathcal{A}), (\mathcal{A}))$, $\mathcal{L}((\mathcal{A})^*, (\mathcal{A})^*)$, $\mathcal{L}((E), (\mathcal{A}))$, $\mathcal{L}((\mathcal{A})^*, (E)^*)$, and $\mathcal{L}((E), (E)^*)$.

Theorem 2.8 *Both $t \mapsto E_t \in \mathcal{L}((E), (\mathcal{A}))$ and $t \mapsto E_t \in \mathcal{L}((\mathcal{A})^*, (E)^*)$ are continuous.*

PROOF. For $\phi \in (E)$ with Wiener–Itô expansion

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f_n \right\rangle, \qquad f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n},$$

we have by definition

$$\| (E_s - E_t)\phi \|_{r,\beta}^2 = \sum_{n=0}^{\infty} n! e^{2\beta n} \left\| (\chi_s^{\otimes n} - \chi_t^{\otimes n}) f_n \right\|_r^2, \qquad s,t \in \mathbb{R}, \quad r, \beta \in \mathbb{R}. \tag{2.11}$$

Since

$$\chi_s^{\otimes n} - \chi_t^{\otimes n} = \sum_{k=1}^{n} \chi_s^{\otimes n-k} \otimes (\chi_s - \chi_t) \otimes \chi_t^{\otimes k-1},$$

we have

$$\left\| (\chi_s^{\otimes n} - \chi_t^{\otimes n}) f_n \right\|_r \le \sum_{k=1}^{n} \left\| (\chi_s^{\otimes n-k} \otimes (\chi_s - \chi_t) \otimes \chi_t^{\otimes k-1}) f_n \right\|_r, \qquad f_n \in E_{\mathbb{C}}^{\widehat{\otimes} n}. \tag{2.12}$$

Take $p \ge 0$ and $C \ge 0$ exactly as in Lemma 2.5. Then (2.12) becomes

$$\left\| (\chi_s^{\otimes n} - \chi_t^{\otimes n}) f_n \right\|_r \le n C^n |t-s|^{1/2} \, | f_n |_p .$$

Inserting this into (2.11), we obtain

$$\begin{aligned} \| (E_s - E_t)\phi \|_{r,\beta}^2 &\le \sum_{n=1}^{\infty} n! e^{2\beta n} n^2 C^{2n} |s-t| \, | f_n |_p^2 \\ &\le \sum_{n=1}^{\infty} n! n^2 \left(C e^{\beta} \rho^q \right)^{2n} |s-t| \, | f_n |_{p+q}^2 \\ &\le |s-t| \sup_{n \ge 1} n^2 \left(C e^{\beta} \rho^q \right)^{2n} \| \phi \|_{p+q}^2 . \end{aligned}$$

Take $q \geq 0$ large enough to have $Ce^{\beta}\rho^{q} < 1$. Then

$$M \equiv \sup_{n\geq 1} n \left(Ce^{\beta}\rho^{q}\right)^{n} < \infty$$

and

$$\| (E_s - E_t)\phi \|_{r,\beta} \leq M|s-t|^{1/2} \| \phi \|_{p+q}, \qquad \phi \in (E).$$

This proves that $t \mapsto E_t \in \mathcal{L}((E), (\mathcal{A}))$ is continuous. The second half of the statement follows immediately by taking the adjoint. qed

2.5 Fock expansion of the conditional expectation

It has been already noted that $E_t \in \mathcal{L}((E), (E)^*)$. Here we record the Fock expansion.

Lemma 2.9 *The Fock expansion of the conditional expectation E_t is given by*

$$E_t = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_t^{+\infty} \cdots \int_t^{+\infty} a_{s_1}^* \cdots a_{s_n}^* a_{s_1} \cdots a_{s_n} ds_1 \cdots ds_n.$$

PROOF. By definition (2.10) we have

$$E_t \phi_\xi = \phi_{\chi_t \xi}, \qquad \xi \in E_{\mathbb{C}}.$$

(In fact, the above relation characterizes the conditional expectation.) Then

$$\langle\langle E_t \phi_\xi, \phi_\eta \rangle\rangle = \langle\langle \phi_{\chi_t \xi}, \phi_\eta \rangle\rangle = \exp \langle \chi_t \xi, \eta \rangle = \exp \int_{-\infty}^{t} \xi(s)\eta(s)\, ds.$$

Hence we have

$$e^{-\langle \xi, \eta \rangle} \langle\langle E_t \phi_\xi, \phi_\eta \rangle\rangle = \exp\left(- \int_t^{+\infty} \xi(s)\eta(s)\, ds \right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_t^{+\infty} \xi(s)\eta(s)\, ds \right)^n,$$

which completes the proof. qed

3 Adapted processes and the Hitsuda–Skorokhod integral

3.1 Adapted processes, admissible processes and martingales

The *support* of a distribution $F \in (E_{\mathbb{C}}^{\otimes n})^*$, denoted by supp F, is the smallest closed subset $K \subset \mathbb{R}^n$ such that F vanishes in $\mathbb{R}^n - K$. The next definition is essentially due to Hida [9].

Definition 3.1 Let $t \mapsto \Phi_t \in (E)^*$ be a continuous map defined on an interval and

$$\Phi_t(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, F_n^{(t)} \right\rangle$$

be the Wiener–Itô expansion. Then $\{\Phi_t\}$ is called an *adapted process* if supp $F_n^{(t)} \subset (-\infty, t]^n$ for all $n \geq 1$ and t.

Definition 3.2 A continuous map $t \mapsto \Phi_t \in (E)^*$, is called an *admissible process* if Φ_t is admissible for each t, i.e., $\Phi_t \in (\mathcal{A})^*$ for all t.

In the above definition we do not require that $t \mapsto \Phi_t \in (\mathcal{A})^*$ is continuous with respect to the topology of $(\mathcal{A})^*$. Our condition above is weaker than this.

Proposition 3.3 *Let $\{\Phi_t\}$ be an admissible process. Then it is adapted if and only if $E_t\Phi_t = \Phi_t$ for all t, hence if and only if $E_s\Phi_t = \Phi_t$ for $s \ge t$.*

PROOF. The assertion follows immediately from the fact that supp $F_n^{(t)} \subset (-\infty, t]^n$ if and only if $\chi_t^{\otimes n} \cdot F_n^{(t)} = F_n^{(t)}$. qed

Definition 3.4 An admissible process $\{\Phi_t\}$ is called a *martingale* if $E_s\Phi_t = \Phi_s$ for $s \le t$.

By definition a martingale is an adapted admissible process. The next assertion contains a typical example.

Proposition 3.5 *Let $\Phi \in (\mathcal{A})^*$ be an admissible white noise distribution. Then $\{E_t\Phi\}_{t\in\mathbb{R}}$ is a martingale.*

PROOF. It follows from Theorem 2.8 that $t \mapsto E_t\Phi \in (E)^*$ is continuous. Obviously $E_t\Phi \in (\mathcal{A})^*$ for any t, which means that $\{E_t\Phi\}$ is an admissible process. Since $E_s(E_t\Phi) = E_s\Phi$ for $s \le t$ obviously, $\{E_t\Phi\}$ is a martingale. qed

The Brownian motion $\{B_t\}_{t\ge 0}$ is expressed as $B_t = E_t\Phi$, where $\Phi(x) = \left\langle x,\, 1_{[0,+\infty)}\right\rangle$. In particular, the Brownian motion is a martingale.

Proposition 3.6 *Any martingale $\{\Phi_t\}$ admits an expression of the form:*

$$\Phi_t(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:,\, \chi_t^{\otimes n} \cdot F_n \right\rangle, \tag{3.1}$$

where F_n is a $\mathbb{C}$-valued measurable function on $\mathbb{R}^n$.

PROOF. Let

$$\Phi_t(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:,\, F_n^{(t)} \right\rangle$$

be the Wiener–Itô expansion of Φ_t, where $F_n^{(t)}$ is a slowly increasing function on $\mathbb{R}^n$. Since $E_s\Phi_t = \Phi_s$ for $s \le t$ by assumption, we have

$$\chi_s^{\otimes n} \cdot F_n^{(t)} = F_n^{(s)}, \qquad s \le t, \quad n \ge 1.$$

Therefore, we can define a measurable funtion F_n on $\mathbb{R}^n$ by

$$F_n(u_1, \cdots, u_n) = F_n^{(t)}(u_1, \cdots, u_n), \qquad t \ge u_1, \cdots, u_n.$$

Then $F_n^{(t)} = \chi_t^{\otimes n} \cdot F_n$ and we obtain (3.1). qed

REMARK. In Proposition 3.6 one might consider a formal series:

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:,\, F_n \right\rangle .$$

However, Φ is not necessarily a white noise distribution because there is no guarantee that F_n is slowly increasing. Neverthless, one can write $E_t\Phi = \Phi_t$ at a formal level.

3.2 Hitsuda–Skorokhod integral

We first introduce the integral of an $(E)^*$-valued function.

Lemma 3.7 [10, Proposition 8.1] *Let $t \mapsto \Phi_t \in (E)^*$ be a map defined on a (finite or infinite) interval I. Assume that for any $\phi \in (E)$ the function $t \mapsto \langle\!\langle \Phi_t, \phi \rangle\!\rangle$ belongs to $L^1(I, dt)$. Then there exists a unique $\Psi \in (E)^*$ such that*

$$\langle\!\langle \Psi, \phi \rangle\!\rangle = \int_I \langle\!\langle \Phi_t, \phi \rangle\!\rangle \, dt, \qquad \phi \in (E).$$

In that case we write

$$\Psi = \int_I \Phi_t dt.$$

For example, if I is a finite closed interval and $t \mapsto \Phi_t \in (E)^*$ is continuous, the above integral exists.

Again suppose we are given a map $t \mapsto \Phi_t \in (E)^*$, where t runs over an interval I. Note that $a_t^* \Phi_t \in (E)^*$ is defined because $a_t^* \in \mathcal{L}((E)^*, (E)^*)$. If in addition $t \mapsto \langle\!\langle a_t^* \Phi_t, \phi \rangle\!\rangle = \langle\!\langle \Phi_t, a_t \phi \rangle\!\rangle$ belongs to $L^1(I, dt)$ for any $\phi \in (E)$, then

$$\Psi = \int_I a_t^* \Phi_t \, dt \in (E)^*$$

is defined according to Lemma 3.7. This is called the *Hitsuda–Skorokhod integral* of $\{\Phi_t\}$.

If I is a finite closed interval and $t \mapsto \Phi_t \in (E)^*$ is continuous, the Hitsuda–Skorokhod integral exists. In fact, $t \mapsto \langle\!\langle a_t^* \Phi_t, \phi \rangle\!\rangle$ is a continuous function on the interval I for any $\phi \in (E)$.

3.3 Conditional expectation of Hitsuda–Skorokhod integral

Lemma 3.8 *Let $t \mapsto \Phi_t \in (\mathcal{A})^*$ be a map defined on a closed finite interval $[a, b]$ such that*

$$\sup_{a \le t \le b} \| \Phi_t \|_{-r,-\beta} < \infty \tag{3.2}$$

for some $r, \beta \ge 0$. Assume that $t \mapsto \langle\!\langle a_t^* \Phi_t, \phi \rangle\!\rangle$ belongs to $L^1(a, b)$ for any $\phi \in (E)$ and that the Hitsuda–Skorokhod integral

$$\int_a^b a_s^* \Phi_s \, ds$$

belongs to $(\mathcal{A})^*$, i.e., is an admissible white noise distribution. Then

$$E_t \left(\int_a^b a_s^* \Phi_s \, ds \right) = \begin{cases} \displaystyle\int_a^{t \wedge b} a_s^* E_t \Phi_s \, ds, & a \le t, \\ 0 & t \le a. \end{cases} \tag{3.3}$$

PROOF. Suppose $t \in \mathbb{R}$ is fixed throughout. First note that the map $s \mapsto a_s^* E_t \Phi_s \in (E)^*$ is well defined. We shall show that $s \mapsto \langle\langle a_s^* E_t \Phi_s, \phi \rangle\rangle$ belongs to $L^1(a,b)$ for any $\phi \in (E)$. In fact,

$$|\langle\langle a_s^* E_t \Phi_s, \phi \rangle\rangle| = |\langle\langle E_t \Phi_s, a_s \phi \rangle\rangle| \le \| E_t \Phi_s \|_{-r,-\beta} \| a_s \phi \|_{r,\beta} \le \| \Phi_s \|_{-r,-\beta} \| a_s \phi \|_{r,\beta}.$$

In view of Lemma 2.6 we may find $p \ge 0$ such that

$$\| a_s \phi \|_{r,\beta} \le \| a_s \phi \|_p .$$

On the other hand, by [20, Theorem 4.1.1] there exist $q > 0$ and $C \ge 0$ such that

$$\| a_s \phi \|_p \le C \, | \delta_s |_{-(p+q)} \| \phi \|_{p+q} .$$

Thus we obtain

$$|\langle\langle a_s^* E_t \Phi_s, \phi \rangle\rangle| \le C \| \Phi_s \|_{-r,-\beta} | \delta_s |_{-(p+q)} \| \phi \|_{p+q} .$$

Since $s \mapsto \delta_s$ is continuous, taking $q > 0$ large enough we see that

$$\sup_{a \le s \le b} | \delta_t |_{-(p+q)} < \infty.$$

Combining this with (3.2), we see that $s \mapsto |\langle\langle a_s^* E_t \Phi_s, \phi \rangle\rangle|$ is bounded on $[a,b]$ and hence integrable. Then by Lemma 3.7 the Hitsuda–Skorokhod integral exists:

$$\int_a^{t \wedge b} a_s^* E_t \Phi_s \, ds \in (E)^*.$$

For simplicity we put

$$\Psi = \int_a^b a_s^* \Phi_s \, ds.$$

For the assertion it is sufficient to prove that

$$E_t \Psi = \int_a^{t \wedge b} a_s^* E_t \Phi_s \, ds, \quad t > a; \qquad E_t \Psi = 0, \quad t \le a.$$

Since both sides in the above identities are white noise distributions, it is sufficient to prove

$$\langle\langle E_t \Psi, \phi_\eta \rangle\rangle = \int_a^{t \wedge b} \langle\langle a_s^* E_t \Phi_s, \phi_\eta \rangle\rangle \, ds, \quad t > a, \tag{3.4}$$

$$\langle\langle E_t \Psi, \phi_\eta \rangle\rangle = 0, \quad t \le a, \tag{3.5}$$

for any $\eta \in E_{\mathbb{C}}$. We shall prove (3.4) for (3.5) is verified in a similar manner.

Suppose t is fixed as $t > a$. Note first that

$$\langle\langle E_t \Psi, \phi_\eta \rangle\rangle = \langle\langle \Psi, E_t \phi_\eta \rangle\rangle = \langle\langle \Psi, \phi_{\chi_t \eta} \rangle\rangle . \tag{3.6}$$

We take an approximate sequence $\eta_n \in E_{\mathbb{C}}$ with supp $\eta_n \subset (-\infty, t]$ such that $\eta_n \to \chi_t \eta$ in $\mathcal{A}$. Since $\Psi \in (\mathcal{A})^*$ by assumption,

$$\langle\langle \Psi, \phi_{\chi_t \eta} \rangle\rangle = \lim_{n \to \infty} \langle\langle \Psi, \phi_{\eta_n} \rangle\rangle . \tag{3.7}$$

Now we see that

$$\begin{aligned}
\langle\!\langle \Psi, \phi_{\eta_n} \rangle\!\rangle &= \int_a^b \langle\!\langle a_s^* \Phi_s, \phi_{\eta_n} \rangle\!\rangle \, ds \\
&= \int_a^b \langle\!\langle \Phi_s, a_s \phi_{\eta_n} \rangle\!\rangle \, ds \\
&= \int_a^b \langle\!\langle \Phi_s, \phi_{\eta_n} \rangle\!\rangle \, \eta_n(s) \, ds \\
&= \int_a^{t\wedge b} \langle\!\langle \Phi_s, \phi_{\eta_n} \rangle\!\rangle \, \eta_n(s) \, ds.
\end{aligned}$$

Then in view of (3.6) and (3.7) we obtain

$$\langle\!\langle E_t\Psi, \phi_\eta \rangle\!\rangle = \lim_{n\to\infty} \langle\!\langle \Psi, \phi_{\eta_n} \rangle\!\rangle = \lim_{n\to\infty} \int_a^{t\wedge b} \langle\!\langle \Phi_s, \phi_{\eta_n} \rangle\!\rangle \, \eta_n(s) \, ds = \int_a^{t\wedge b} \langle\!\langle \Phi_s, \phi_{\chi_t\eta} \rangle\!\rangle \, \eta(s) \, ds.$$

Therefore, viewing

$$\langle\!\langle \Phi_s, \phi_{\chi_t\eta} \rangle\!\rangle \, \eta(s) = \langle\!\langle a_s^* E_t \Phi_s, \phi_\eta \rangle\!\rangle$$

we come to (3.4). qed

Proposition 3.9 *Let $\{\Phi_t\}$ be an adapted admissible process, where t runs over a closed finite interval $[a, b]$. Assume that*

$$\sup_{a\le t\le b} \| \Phi_t \|_{-r,-\beta} < \infty$$

for some $r, \beta \ge 0$, that $t \mapsto \langle\!\langle a_t^* \Phi_t, \phi \rangle\!\rangle$ belongs to $L^1(a, b)$ and that the Hitsuda–Skorokhod integral

$$\int_a^b a_s^* \Phi_s \, ds$$

belongs to $(\mathcal{A})^*$, i.e., is an admissible white noise distribution. Then

$$E_t \left(\int_a^b a_s^* \Phi_s \, ds \right) = \begin{cases} \displaystyle\int_a^{t\wedge b} a_s^* \Phi_s \, ds, & a \le t, \\ 0 & t < a. \end{cases}$$

PROOF. By the assumption of adaptedness we have $E_t \Phi_s = \Phi_s$ for $t \ge s$. It then follows from Lemma 3.8 that

$$E_t \left(\int_a^b a_s^* \Phi_s \, ds \right) = \int_a^{t\wedge b} a_s^* E_t \Phi_s \, ds = \int_a^{t\wedge b} a_s^* \Phi_s \, ds, \qquad a \le t,$$

which completes the proof. qed

Theorem 3.10 *Let $\Phi \in (\mathcal{A})^*$ be an admissible white noise distribution with Wiener–Itô expansion*

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle : x^{\otimes n} :, F_n \right\rangle.$$

Assume that every F_n is a continuous function. Then there exists an adapted admissible process $\{\Psi_t\}$ such that

$$E_t\Phi = \mathbf{E}(\Phi)\phi_0 + \int_{-\infty}^{t} a_s^*\Psi_s ds, \tag{3.8}$$

where $\mathbf{E}(\Phi) = \langle\langle \Phi,\, \phi_0\rangle\rangle = F_0$ *is the vacuum expectation of* Φ.

PROOF. Since $E_t\phi_0 = \phi_0$ it is sufficient to prove (3.8) under the assumption that $\mathbf{E}(\Phi) = 0$, i.e., $F_0 = 0$. By assumption there exists $r \geq 0$ such that $F_n \in \mathcal{A}_{-r}^{\otimes n}$ for all $n \geq 1$. Now for $n \geq 0$ we put

$$G_n^{(s)}(u_1, \cdots, u_n) = (n+1)F_{n+1}(s, u_1, \cdots, u_n)\chi_s(u_1)\cdots\chi_s(u_n), \qquad s, u_1, \cdots, u_n \in \mathbb{R}.$$

Obviously, $G_n^{(s)} \in \mathcal{A}_{-r}^{\otimes n}$. We put

$$\Psi_s(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:,\, G_n^{(s)} \right\rangle.$$

Then $\{\Psi_s\}$ is an adapted admissible process. To prove (3.8) it is sufficient to see that

$$\int_{-\infty}^{t} \langle\langle a_s^*\Psi_s,\, \phi_\xi\rangle\rangle\, ds = \langle\langle E_t\Phi,\, \phi_\xi\rangle\rangle, \qquad \xi \in E_{\mathbb{C}}.$$

We first observe that

$$\int_{-\infty}^{t} \langle\langle a_s^*\Psi_s,\, \phi_\xi\rangle\rangle\, ds = \int_{-\infty}^{t} \langle\langle \Psi_s,\, a_s\phi_\xi\rangle\rangle\, ds = \int_{-\infty}^{t} \xi(s)\, \langle\langle \Psi_s,\, \phi_\xi\rangle\rangle\, ds.$$

Since by definition

$$\begin{aligned}
\langle\langle \Psi_s,\, \phi_\xi\rangle\rangle &= \sum_{n=0}^{\infty} n! \left\langle G_n^{(s)},\, \frac{\xi^{\otimes n}}{n!} \right\rangle \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} G_n^{(s)}(u_1, \cdots, u_n)\xi(u_1)\cdots\xi(u_n)\, du_1\cdots du_n \\
&= \sum_{n=0}^{\infty} (n+1) \int_{-\infty}^{s} \cdots \int_{-\infty}^{s} F_{n+1}(s, u_1, \cdots, u_n)\xi(u_1)\cdots\xi(u_n)\, du_1\cdots du_n,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\int_{-\infty}^{t} \langle\langle a_s^*\Psi_s,\, \phi_\xi\rangle\rangle\, ds = \\
&\quad = \sum_{n=0}^{\infty} (n+1) \int_{-\infty}^{t} \xi(s)ds \int_{-\infty}^{s} \cdots \int_{-\infty}^{s} F_{n+1}(s, u_1, \cdots, u_n)\xi(u_1)\cdots\xi(u_n)\, du_1\cdots du_n.
\end{aligned}$$

On the other hand, by symmetry we have

$$\begin{aligned}
&\int_{-\infty}^{s} \cdots \int_{-\infty}^{s} F_{n+1}(s, u_1, \cdots, u_n)\xi(u_1)\cdots\xi(u_n)\, du_1\cdots du_n = \\
&\quad = n! \int_{-\infty}^{s} du_1 \int_{-\infty}^{u_1} du_2 \cdots \int_{-\infty}^{u_{n-1}} du_n F_{n+1}(s, u_1, \cdots, u_n)\xi(u_1)\cdots\xi(u_n).
\end{aligned}$$

Finally we come to

$$\begin{aligned}
&\int_{-\infty}^{t} \langle\langle a_s^* \Psi_s,\, \phi_\xi \rangle\rangle \, ds = \\
&= \sum_{n=0}^{\infty} (n+1)! \int_{-\infty}^{t} \xi(s) ds \int_{-\infty}^{s} du_1 \cdots \int_{-\infty}^{u_{n-1}} du_n F_{n+1}(s, u_1, \cdots, u_n) \xi(u_1) \cdots \xi(u_n) \\
&= \sum_{n=0}^{\infty} \int_{-\infty}^{t} \cdots \int_{-\infty}^{t} F_{n+1}(s, u_1, \cdots, u_n) \xi(s) \xi(u_1) \cdots \xi(u_n) \, ds du_1 \cdots du_n \\
&= \sum_{n=0}^{\infty} \left\langle \chi_t^{\otimes(n+1)} F_{n+1},\, \xi^{\otimes(n+1)} \right\rangle \\
&= \sum_{n=1}^{\infty} \left\langle \chi_t^{\otimes n} F_n,\, \xi^{\otimes n} \right\rangle \\
&= \langle\langle E_t \Phi,\, \phi_\xi \rangle\rangle .
\end{aligned}$$

This completes the proof. qed

We have shown in Proposition 3.5 that $\{E_t \Phi\}_{t \in \mathbb{R}}$ is a martingale for $\Phi \in (\mathcal{A})^*$. The above result is a prototype of representation of a martingale by means of the Hitsuda–Skorokhod integral.

3.4 Clark formula

Since $E_t \in \mathcal{L}((E),(E)^*)$, the composition $a_t^* E_t a_t \in \mathcal{L}((E),(E)^*)$ is defined. We shall consider

$$M_t \equiv \int_{-\infty}^{t} a_s^* E_s a_s \, ds, \qquad -\infty < t \le +\infty.$$

In fact, M_t is defined in the following

Lemma 3.11 *There exists a unique $M_t \in \mathcal{L}((E),(E)^*)$, $-\infty < t \le +\infty$, such that*

$$\langle\langle M_t \phi_\xi,\, \phi_\eta \rangle\rangle = \int_{-\infty}^{t} \langle\langle a_s^* E_s a_s \phi_\xi,\, \phi_\eta \rangle\rangle \, ds, \qquad \xi, \eta \in E_{\mathbb{C}}. \tag{3.9}$$

Moreover, $M_t^* = M_t$.

PROOF. Note first that

$$\begin{aligned}
\langle\langle a_s^* E_s a_s \phi_\xi,\, \phi_\eta \rangle\rangle &= \xi(s)\eta(s) \langle\langle E_s \phi_\xi,\, \phi_\eta \rangle\rangle \\
&= \xi(s)\eta(s) \exp \int_{-\infty}^{s} \xi(u)\eta(u) \, du \\
&= \frac{d}{ds} \exp \int_{-\infty}^{s} \xi(u)\eta(u) \, du.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{-\infty}^{t} \langle\langle a_s^* E_s a_s \phi_\xi,\, \phi_\eta \rangle\rangle \, ds &= \exp \int_{-\infty}^{s} \xi(u)\eta(u) \, du \Big|_{s=-\infty}^{s=t} \\
&= \exp \int_{-\infty}^{t} \xi(u)\eta(u) \, du - 1.
\end{aligned} \tag{3.10}$$

Consequently,

$$\left|\int_{-\infty}^{t} \langle\langle a_s^* E_s a_s \phi_\xi,\, \phi_\eta \rangle\rangle\, ds\right| \le \exp \int_{-\infty}^{t} |\xi(u)\eta(u)|\, du + 1 \le \exp(|\,\xi\,|_0\,|\,\eta\,|_0) + 1.$$

Hence

$$\left|\int_{-\infty}^{t} \langle\langle a_s^* E_s a_s \phi_\xi,\, \phi_\eta \rangle\rangle\, ds\right| \le \exp \frac{1}{2}(|\,\xi\,|_0^2 + |\,\eta\,|_0^2) + 1 \le 2 \exp \frac{1}{2}(|\,\xi\,|_0^2 + |\,\eta\,|_0^2).$$

It follows from Theorem 1.2 that the right hand side of (3.9) is the symbol of an operator in $\mathcal{L}((E),(E)^*)$, which we denote by M_t. That $M_t^* = M_t$ is obvious by definition. qed

One may prove by a slightly modified argument that $M_\infty \in \mathcal{L}((E),(E))$. Hence $M_\infty^* \in \mathcal{L}((E)^*,(E)^*)$ and is the unique continuous extension of M_∞.

Lemma 3.12 *It holds that*

$$E_t\phi = \mathbf{E}(\phi)\phi_0 + M_t\phi, \qquad \phi \in (E). \tag{3.11}$$

PROOF. In (3.10) we have already established

$$\langle\langle M_t\phi_\xi,\, \phi_\eta \rangle\rangle = \exp \int_{-\infty}^{t} \xi(u)\eta(u)\, du - 1 = \exp \langle \chi_t\xi,\, \eta \rangle - 1, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

In other words,

$$\langle\langle M_t\phi_\xi,\, \phi_\eta \rangle\rangle = \langle\langle \phi_{\chi_t\xi},\, \phi_\eta \rangle\rangle - \langle\langle \phi_0,\, \phi_\eta \rangle\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}.$$

Hence

$$M_t\phi_\xi = \phi_{\chi_t\xi} - \phi_0 = \phi_{\chi_t\xi} - \langle\langle \phi_\xi,\, \phi_0 \rangle\rangle\, \phi_0 = E_t\phi_\xi - \mathbf{E}(\phi_\xi)\phi_0.$$

Then by continuity we obtain (3.11). qed

The map $\phi \mapsto \mathbf{E}(\phi)\phi_0$ is called the *vacuum projection* and, obviously, is extended to a continuous linear operator from $(E)^*$ into (E) by putting $\mathbf{E}(\Phi) = \langle\langle \Phi,\, \phi_0 \rangle\rangle$. It is known (§2.4) that E_t belongs to $\mathcal{L}((\mathcal{A})^*,(\mathcal{A})^*)$, while from the above consideration so does the vacuum projection. Therefore from Lemma 3.12 we see that M_t is extended to a continuous operator in $\mathcal{L}((\mathcal{A})^*,(\mathcal{A})^*)$. In that sense we obtain a variant of the Clark formula.

Theorem 3.13 *For* $-\infty < t < +\infty$ *it holds that*

$$E_t\Phi = \mathbf{E}(\Phi)\phi_0 + \left(\int_{-\infty}^{t} a_s^* E_s a_s\, ds\right)\Phi, \qquad \Phi \in (\mathcal{A})^*. \tag{3.12}$$

For $t = +\infty$ *we have*

$$\Phi = \mathbf{E}(\Phi)\phi_0 + \left(\int_{-\infty}^{+\infty} a_s^* E_s a_s\, ds\right)\Phi, \qquad \Phi \in (E)^*.$$

The above result is closely related to representation of a martingale (Theorem 3.10). For instance we have the following

Theorem 3.14 *Let $\Phi \in (\mathcal{A})^*$ be an admissible white noise distribution with Wiener–Itô expansion*

$$\Phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, F_n \right\rangle .$$

Assume that every F_n is a continuous function. Then

$$E_t\Phi = \mathbf{E}(\Phi)\phi_0 + \int_{-\infty}^{t} a_s^* E_s a_s \Phi \, ds, \qquad -\infty < t \leq +\infty. \tag{3.13}$$

REMARK. Since F_n is a continuous function, $a_s\Phi$ is defined by

$$a_s\Phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \delta_s \otimes_1 F_n \right\rangle .$$

PROOF. It is easily verified that Ψ_s defined in Theorem 3.10 coincides with $E_s a_s \Phi$.

qed

REMARK. Note the difference between (3.12) and (3.13). The latter is a more direct generalization of the so-called Clark formula, see [28] for a white noise approach.

4 Quantum Stochastic Processes

4.1 Definition and basic processes

Definition 4.1 [24] A one-parameter family of operators $\{\Xi_t\} \subset \mathcal{L}((E), (E)^*)$ is called a *quantum stochastic process* if $t \mapsto \Xi_t \in \mathcal{L}((E), (E)^*)$ is continuous, where t runs over an interval. A continous linear map $\Xi : E_{\mathbb{C}} \to \mathcal{L}((E), (E)^*)$ is called a *generalized quantum stochastic process.* A generalized quantum stochastic process Ξ is called *regular* if it is extended to a continuous linear map from $E_{\mathbb{C}}^*$ into $\mathcal{L}((E), (E)^*)$.

Since $t \mapsto \delta_t \in E_{\mathbb{C}}^*$ is continuous, for a regular generalized quantum stochastic process Ξ one obtains a quantum stochastic process by putting

$$\Xi_t = \Xi(\delta_t), \qquad t \in \mathbb{R}.$$

A quantum stochastic process obtained in this way is also called *regular.* Note that not every quantum stochastic process is regular.

Proposition 4.2 *The families of annihilation operators $\{a_t\}_{t\in\mathbb{R}}$ and creation operators $\{a_t^*\}_{t\in\mathbb{R}}$ are both regular quantum stochastic processes. Moreover, both $t \mapsto a_t \in \mathcal{L}((E), (E))$ and $t \mapsto a_t^* \in \mathcal{L}((E)^*, (E)^*)$ are C^∞-maps.*

PROOF. Consider an integral kernel operator:

$$\Xi_{0,1}(f) = \int_{\mathbb{R}} f(t) a_t \, dt, \qquad f \in E_{\mathbb{C}}^*.$$

It is proved [20, Theorem 4.1.1 and Proposition 4.3.10] that $\Xi_{0,1} : E^*_{\mathbb{C}} \to \mathcal{L}((E),(E))$ is a continuous map. Since the natural injection $\mathcal{L}((E),(E)) \to \mathcal{L}((E),(E)^*)$ is continuous, $\{a_t = \Xi_{0,1}(\delta_t)\}$ forms a regular quantum stochastic process. It follows by a direct verification that $t \mapsto a_t$ is infinitely many times differentiable in $\mathcal{L}((E),(E))$. In fact,

$$\frac{d^n}{dt^n}\, a_t = (-1)^n \Xi_{0,1}(\delta_t^{(n)}).$$

By taking adjoint one may prove the assertion for a_t^* easily. qed

In a similar manner one obtains

Proposition 4.3 *Put*

$$A_t = \Xi_{0,1}(1_{[0,t]}), \qquad A_t^* = \Xi_{1,0}(1_{[0,t]}), \qquad t \geq 0. \tag{4.1}$$

Then $\{A_t\}_{t\geq 0}$ and $\{A_t^*\}_{t\geq 0}$ are quantum stochastic processes. Moreover, it holds that

$$a_t = \frac{d}{dt}\, A_t, \qquad a_t^* = \frac{d}{dt}\, A_t^*,$$

with respect to the topologies of $\mathcal{L}((E),(E))$ and $\mathcal{L}((E)^*,(E)^*)$, respectively. In particular, $t \mapsto A_t \in \mathcal{L}((E),(E))$ and $t \mapsto A_t^* \in \mathcal{L}((E)^*,(E)^*)$ are C^∞-maps.

Definition 4.4 The quantum stochastic processes $\{A_t\}$ and $\{A_t^*\}$ defined in (4.1) are called the *annihilation process* and the *creation process*, respectively.

The correspondence between classical and quantum stochastic processes is stated in the following

Proposition 4.5 *If $t \mapsto \Phi_t \in (E)^*$ is continuous, regarded as multiplication operators $\{\Phi_t\}$ becomes a quantum stochastic process.*

PROOF. Since the pointwise multiplication of white noise functions yields a continuous bilinear map $(E) \times (E) \to (E)$, multiplication of $\phi \in (E)$ and $\Phi \in (E)^*$, denoted by $\Phi\phi = \phi\Phi$, is defined by

$$\langle\langle \Phi\phi,\, \psi\rangle\rangle = \langle\langle \Phi,\, \phi\psi\rangle\rangle\,, \qquad \phi, \psi \in (E), \quad \Phi \in (E)^*.$$

It is then easily verified that $\phi \mapsto \Phi\phi$, $\phi \in (E)$, is continuous and linear; namely, each Φ gives rise to an operator in $\mathcal{L}((E),(E)^*)$. Moreover, as is easily seen, thus obtained natural injection $(E)^* \to \mathcal{L}((E),(E)^*)$ is continuous. This completes the proof. qed

The *quantum Brownian motion* and the *quantum white noise* are quantum stochastic processes respectively corresponding to the classical Brownian motion $\{B_t\}$ and the classical white noise $\{W_t\}$, for the definitions see §1.2, in such a way as described in Proposition 4.5. The quantum Brownian motion, again denoted by B_t, is decomposed into the sum of the annihilation and creation processes:

$$B_t = A_t + A_t^*, \qquad t \geq 0.$$

Similarly, for the quantum white noise we have

$$W_t = a_t + a_t^*, \qquad t \in \mathbb{R}.$$

It is also noteworthy that the conditional expectations $\{E_t\}_{t\in\mathbb{R}}$ form a quantum stochastic process (see Theorem 2.8).

4.2 Conditional expectation for admissible operators

Definition 4.6 An operator $\Xi \in \mathcal{L}((E),(E)^*)$ is called *admissible* if there exists a continuous operator in $\mathcal{L}((\mathcal{A}),(\mathcal{A})^*)$ of which restriction to (E) coincides with Ξ. For an admissible operator $\Xi \in \mathcal{L}((\mathcal{A}),(\mathcal{A})^*)$ the conditional expectation is defined as $E_t \Xi E_t \in \mathcal{L}((\mathcal{A}),(\mathcal{A})^*)$.

Lemma 4.7 *Let κ be a slowly increasing function on $\mathbb{R}^{l+m}$, i.e., a $\mathbb{C}$-valued measurable function with $\| \kappa \|_{-r} < \infty$ for some $r \geq 0$. Then for any $\beta > 0$*

$$\| \Xi_{l,m}(\kappa)\phi \|_{-r,-\beta} \leq C \, \| \kappa \|_{-r} \, \| \phi \|_{r,\beta} \, , \qquad \phi \in (\mathcal{A}),$$

where

$$C = \sup_{n\geq 0} \left\{ \frac{(l+n)!}{n!} \frac{(m+n)!}{n!} \right\}^{1/2} e^{-(2n+m+l)\beta}. \tag{4.2}$$

In particular, $\Xi_{l,m}(\kappa) \in \mathcal{L}((\mathcal{A}),(\mathcal{A})^*)$.

PROOF. The action of an integral kernel operator $\Xi_{l,m}(\kappa)$ is given explicitly as follows: Let $\phi \in (E)$ be given with Wiener–Itô expansion

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, \, f_n \right\rangle .$$

Then

$$\Xi_{l,m}(\kappa)\phi(x) = \sum_{n=0}^{\infty} \frac{(m+n)!}{n!} \left\langle :x^{\otimes(l+n)}:, \, \kappa \otimes_m f_{m+n} \right\rangle .$$

By definition

$$\begin{aligned} \| \Xi_{l,m}(\kappa)\phi \|_{-r,-\beta}^2 &= \sum_{n=0}^{\infty} (l+n)! e^{-2(l+n)\beta} \left\{ \frac{(m+n)!}{n!} \right\}^2 \| \kappa \otimes_m f_{m+n} \|_{-r}^2 \\ &= \sum_{n=0}^{\infty} \frac{(l+n)!}{n!} \frac{(m+n)!}{n!} e^{-2(l+n)\beta} (m+n)! \, \| \kappa \otimes_m f_{m+n} \|_{-r}^2 . \end{aligned}$$

Using the inequality

$$\| \kappa \otimes_m f_{m+n} \|_{-r} \leq \| \kappa \|_{-r} \, \| f_{m+n} \|_{r} \, , \qquad r \geq 0, \tag{4.3}$$

which is verified easily with the Schwartz inequality, we come to

$$\begin{aligned} \| \Xi_{l,m}(\kappa)\phi \|_{-r,-\beta}^2 &\leq \sum_{n=0}^{\infty} \frac{(l+n)!}{n!} \frac{(m+n)!}{n!} e^{-2(l+n)\beta} (m+n)! \, \| \kappa \|_{-r}^2 \, \| f_{m+n} \|_r^2 \\ &\leq \sum_{n=0}^{\infty} C^2 e^{2(n+m)\beta} (m+n)! \, \| \kappa \|_{-r}^2 \, \| f_{m+n} \|_r^2 \\ &\leq C^2 \, \| \kappa \|_{-r}^2 \, \| \phi \|_{r,\beta}^2 \, , \end{aligned}$$

where C is defined as in (4.2). qed

Moreover, we can give a sufficient condition for an integral kernel operator to belong to $\mathcal{L}((\mathcal{A}),(\mathcal{A}))$. For a measurable function κ on $\mathbb{R}^{l+m}$ we put

$$\begin{aligned}\| \kappa \|_{l,m;r,s}^2 &= \int_{\mathbb{R}^{l+m}} |\kappa(s_1,\cdots,s_l,t_1,\cdots,t_m)|^2(1+s_1^2)^r\cdots(1+s_l^2)^r \times \\ &\qquad (1+t_1^2)^s\cdots(1+t_m^2)^s ds_1\cdots ds_l dt_1\cdots dt_m.\end{aligned}$$

Obviously, $\| \kappa \|_{l,m;r,r} = \| \kappa \|_r$.

Lemma 4.8 *Let κ be a $\mathbb{C}$-valued measurable function on $\mathbb{R}^{l+m}$. If there exists $r_0 \ge 0$ such that $\| \kappa \|_{l,m;r,-r} < \infty$ for all $r \ge r_0$, then for any $\beta > 0$ and $\epsilon > 0$ we have*

$$\| \Xi_{l,m}(\kappa)\phi \|_{r,\beta} \le C \| \kappa \|_{l,m;r,-r} \| \phi \|_{r,\beta+\epsilon}, \qquad \phi \in (\mathcal{A}),$$

where

$$C = \sup_{n\ge 0}\left\{ \frac{(l+n)!}{n!}\frac{(m+n)!}{n!}\right\}^{1/2} e^{-\epsilon n-(\beta+\epsilon)m+\beta l}.$$

In particular, $\Xi_{l,m}(\kappa) \in \mathcal{L}((\mathcal{A}),(\mathcal{A}))$.

PROOF. We need only to modify the proof of Lemma 4.7 using

$$\| \kappa \otimes_m f_{m+n} \|_r \le \| \kappa \|_{l,m;r,-r} \| f_{m+n} \|_r, \qquad r \ge 0,$$

instead of (4.3). qed

REMARK. The converse assertions of Lemma 4.7 and 4.8 are not true. In fact, there exists an admissible integral kernel operator of which kernel distribution is not slowly increasing, see e.g., Proposition 4.12. On the other hand, we have a partial result for characterizing an admissible operator in terms of Fock expansion. Let $\Xi \in \mathcal{L}((\mathcal{A}),(\mathcal{A})^*)$ be given with the Fock expansion

$$\Xi = \sum_{l,m=0}^{\infty} \Xi_{l,m}(\kappa_{l,m}).$$

By general theory of countable Hilbert spaces (see e.g., [5], [20]) there exist $r \ge 0$, $\beta > 0$ and $C \ge 0$ such that

$$\| \Xi\phi \|_{-r,-\beta} \le C \| \phi \|_{r,\beta}, \qquad \phi \in (\mathcal{A}).$$

Then

$$|\langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle| \le C \| \phi_\xi \|_{r,\beta} \| \phi_\eta \|_{r,\beta} = C \exp\frac{e^{2\beta}}{2}(\| \xi \|_r^2 + \| \eta \|_r^2),$$

and hence

$$|\langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle e^{-\langle \xi,\eta\rangle}| \le C \exp\frac{e^{2\beta}+1}{2}(\| \xi \|_r^2 + \| \eta \|_r^2).$$

Then, applying the Cauchy estimate to

$$\langle\langle \Xi\phi_\xi, \phi_\eta\rangle\rangle e^{-\langle \xi,\eta\rangle} = \sum_{l,m=0}^{\infty} \left\langle \kappa_{l,m}, \eta^{\otimes l}\otimes\xi^{\otimes m}\right\rangle,$$

we obtain

$$\left|\left\langle \kappa_{l,m}, \eta^{\otimes l} \otimes \xi^{\otimes m}\right\rangle\right| \leq C\{e(e^{2\beta}+1)\}^{(l+m)/2} l^{-l/2} m^{-m/2} \| \eta \|_r^l \| \xi \|_r^m , \tag{4.4}$$

where the calculation is modelled after [20, Lemma 4.4.8]. However, it does not follow from (4.4) that $\kappa_{l,m}$ is slowly increasing. This is a typical difference between $\mathcal{L}((\mathcal{A}),(\mathcal{A})^*)$ and $\mathcal{L}((E),(E)^*)$; the former is based on $\mathcal{A}$ which is *not* nuclear, while the latter is based on the nuclear space $E_{\mathbb{C}}$.

Lemma 4.9 *Let $\kappa_{l,m} \in \mathcal{A}_{-r}^{\otimes(l+m)}$, $r \geq 0$. Then*

$$\begin{aligned} &E_t \Xi_{l,m}(\kappa_{l,m}) E_t = \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^t ds_1 \cdots \int_{-\infty}^t ds_l \int_{-\infty}^t dt_1 \cdots \int_{-\infty}^t dt_m \int_t^{+\infty} du_1 \cdots \int_t^{+\infty} du_n \times \\ &\qquad \kappa_{l,m}(s_1,\cdots,s_l,t_1,\cdots,t_m) a_{u_1}^* \cdots a_{u_n}^* a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} a_{u_1} \cdots a_{u_n}. \end{aligned}$$

PROOF. By a direct computation modelled after Lemma 2.9. qed

4.3 Admissible processes

Definition 4.10 A quantum stochastic process $\{\Xi_t\} \subset \mathcal{L}((E),(E)^*)$ is called *admissible* if $\Xi_t \in \mathcal{L}((\mathcal{A}),(\mathcal{A})^*)$ for each t.

Here are typical examples.

Proposition 4.11 *The annihilation process $\{A_t\}$ and the creation process $\{A_t^*\}$ are both admissible. Moreover, $A_t \in \mathcal{L}((\mathcal{A}),(\mathcal{A}))$ and $A_t^* \in \mathcal{L}((\mathcal{A})^*,(\mathcal{A})^*)$*

PROOF. It is proved in Proposition 4.3 that $\{A_t = \Xi_{0,1}(1_{[0,t]})\}_{t\geq 0}$ is a quantum stochastic process. Since

$$\left\| 1_{[0,t]} \right\|_{0,1;r,-r}^2 = \int_0^t (1+s^2)^{-r} ds < \infty, \qquad r \geq 0,$$

the assertion follows immediately from Lemma 4.8. qed

The *number process (gauge process)* is defined as

$$\Lambda_t = \int_0^t a_s^* a_s \, ds, \qquad t \geq 0. \tag{4.5}$$

Proposition 4.12 *The number process is admissible. Moreover, $\Lambda_t \in \mathcal{L}((\mathcal{A}),(\mathcal{A}))$.*

PROOF. For $\phi \in (E)$ with Wiener–Itô expansion

$$\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, f_n \right\rangle,$$

we put

$$\Lambda_t\phi(x) = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, g_n \right\rangle .$$

Then by a direct computation we obtain

$$g_n(u_1, \cdots, u_n) = n f_n(u_1, \cdots, u_n) 1_{[0,t]}(u_1).$$

Hence for an arbitrary $\epsilon > 0$ we have

$$\begin{aligned} \| \Lambda_t\phi \|_{r,\beta}^2 &= \sum_{n=0}^{\infty} n! e^{2\beta n} \| g_n \|_r^2 \\ &\le \sum_{n=0}^{\infty} n! e^{2\beta n} n^2 \| f_n \|_r^2 \\ &= \sum_{n=0}^{\infty} n^2 e^{-2\epsilon n} n! e^{2(\beta+\epsilon)n} \| f_n \|_r^2 \\ &\le \left(\sup_{n\ge 0} n e^{-\epsilon n} \right)^2 \| \phi \|_{r,\beta+\epsilon}^2 , \end{aligned}$$

which proves that $\Lambda_t \in \mathcal{L}((\mathcal{A}), (\mathcal{A}))$. qed

REMARK. As is stated in Proposition 4.5, any continuous map $t \mapsto \Phi_t \in (E)^*$ gives rise to a quantum stochastic process by multiplication. It can be proved with a similar argument as in [20, §3.5] that the pointwise multiplication yields a continuous bilinear map from $(\mathcal{A}) \times (\mathcal{A})$ into $(\mathcal{A})$. Therefore a (classical) admissible process is always considered as an admissible quantum stochastic process by multiplication.

5 Qunatum stochastic integrals

5.1 Integrals of quantum stochastic processes

Let $\{L_t\} \subset \mathcal{L}((E), (E)^*)$ be a quantum stochastic process defined on an interval I and fix $a \in I$ as a time origin. Then by general theory of topological vector spaces there exists a unique operator $\Xi_t \in \mathcal{L}((E), (E)^*)$ such that

$$\langle\langle \Xi_t\phi, \psi \rangle\rangle = \int_a^t \langle\langle L_s\phi, \psi \rangle\rangle \, ds, \qquad \phi, \psi \in (E), \qquad t \in I.$$

Moreover, it is proved that $\{\Xi_t\}$ is again a quantum stochastic process. We write

$$\Xi_t = \int_a^t L_s \, ds$$

and call it an integral of $\{L_s\}$ against time. It is also proved that $\{\Xi_t\}$ is differentiable with respect to the topology of $\mathcal{L}((E), (E)^*)$ and

$$\frac{d}{dt} \Xi_t = L_t. \tag{5.1}$$

If $\{L_t\} \subset \mathcal{L}((E),(E)^*)$ is a regular quantum stochastic process, by definition there exists a continuous linear map $L : E_{\mathbb{C}}^* \to \mathcal{L}((E),(E)^*)$ such that $L_t = L(\delta_t)$. It then holds that

$$L(1_{[a,t]}) = \int_a^t L_s\, ds.$$

In particular, it holds that

$$A_t = \int_0^t a_s\, ds, \qquad A_t^* = \int_0^t a_s^*\, ds, \qquad t \geq 0.$$

Again we obtain

$$\frac{d}{dt} A_t = a_t, \qquad \frac{d}{dt} A_t^* = a_t^*, \tag{5.2}$$

with respect to the topologies of $\mathcal{L}((E),(E))$ and $\mathcal{L}((E)^*,(E)^*)$, respectively.

REMARK. Let $\{L_t\}$ be a regular quantum stochastic process. Then we have

$$\lim_{a\to-\infty} L(1_{[a,t]}) = L(1_{(-\infty,t]})$$

in $\mathcal{L}((E),(E)^*)$, since $\lim_{a\to-\infty} 1_{[a,t]} = 1_{(-\infty,t]}$ in $E_{\mathbb{C}}^*$. Hence

$$\int_{-\infty}^t L_s\, ds \equiv \lim_{a\to-\infty} \int_a^t L_s\, ds = L(1_{(-\infty,t]})$$

is well defined.

5.2 Quantum stochastic integrals

It is known [24] that if $\{L_t\} \subset \mathcal{L}((E),(E)^*)$ is a quantum stochastic process, so are $\{L_t a_t\}$ and $\{a_t^* L_t\}$. Then one may define quantum stochastic processes by

$$\int_a^t L_s a_s\, ds, \qquad \int_a^t a_s^* L_s\, ds, \tag{5.3}$$

as in the previous section. The number process (4.5) is understood in this sense as well.

Definition 5.1 The former in (5.3) is called the *quantum stochastic integral against the annihilation process*, and the latter the *quantum stochastic integral against the creation process* or the *quantum Hitsuda–Skorokhod integral.*

These generalize "quantum stochastic integrals" so far discussed by many authors in various contexts. For example, in view of (5.2) we may write

$$\int_a^t L_s a_s\, ds = \int_a^t L_s dA_s. \tag{5.4}$$

If a_s^* and L_s commute[8)], then

$$\int_a^t a_s^* L_s\, ds = \int_a^t L_s a_s^*\, ds = \int_a^t L_s dA_s^*. \tag{5.5}$$

[8)]For the well-definedness of $[a_s^*, L_s]$ we need further assumption, e.g., $L_s \in \mathcal{L}((E)^*,(E)^*)$. On the other hand, this condition relates deeply to the formulation of the adaptedness of $\{L_s\}$.

The right hand sides of (5.4) and (5.5) are introduced by Hudson–Parthasarathy [13] using Riemann–Stieltjes integral of Itô type. To develop a non-adapted quantum stochastic integration a quantum Hitsuda–Skorokhod integral has been discussed in a different context by Belavkin [1], Lindsay [18], and others.

If $\{L_t\}$ is a quantum stochastic process, so is $\{L_t^*\}$. It is then easily verified that

$$\left(\int_a^t L_s a_s \, ds\right)^* = \int_a^t a_s^* L_s^* \, ds. \tag{5.6}$$

Thus in many cases it is sufficient to discuss only quantum Hitsuda–Skorokhod integrals.

We shall observe how the above quantum Hitsuda–Skorokhod integral generalizes a classical one (§3.2). Let $t \mapsto \Phi_t \in (E)^*$ be a continuous map defined on an interval I. Fix $a \in I$ and consider the (classical) Hitsuda–Skorokhod integral:

$$\Psi_t = \int_a^t a_s^* \Phi_s \, ds \in (E)^*.$$

On the other hand, by multiplication $\{\Phi_t\}$ becomes a quantum stochastic process which we denote by $\{\widetilde{\Phi}_t\}$ for clarity. Then we have the quantum Hitsuda–Skorokhod integral:

$$\Xi_t = \int_a^t a_s^* \widetilde{\Phi}_s \, ds \in \mathcal{L}((E), (E)^*).$$

Proposition 5.2 *Notations and assumptions being as above, it holds that*

$$\Psi_t = \Xi_t \phi_0, \qquad t \geq a,$$

where ϕ_0 is the vacuum.

The proof is straightforward. Note that Ξ_t is no longer a multiplication operator. Let $\widetilde{\Psi}_t$ denote the multiplication operator by Ψ_t, i.e., $\{\widetilde{\Psi}_t\}$ is the quantum stochastic process corresponding to $\{\Psi_t\}$. Then we have

$$\widetilde{\Psi}_t = \int_a^t (a_s^* \widetilde{\Phi}_s + \widetilde{\Phi}_s a_s) ds. \tag{5.7}$$

In classical case, in addition to the Hitsuda–Skorokhod integral another approach to a stochastic Itô integral with non-adapted integrand has been discussed by Kuo–Russek [17], see also Kuo [16]. Our observation (5.7) would be a key to study a quantum analogue of their results.

5.3 Generalized integral kernel operators

It is possible to replace the kernel distribution κ in an integral kernel operator (1.8) with an operator-valued distribution, for generalities for such distributions see [21]. With each $L \in \mathcal{L}(E_{\mathbb{C}}^{\otimes(l+m)}, \mathcal{L}((E), (E)^*))$ we may associate an operator $\Xi \in \mathcal{L}((E), (E)^*)$ by the formula:

$$\langle\langle \Xi \phi_\xi, \phi_\eta \rangle\rangle = \left\langle\left\langle L(\eta^{\otimes l} \otimes \xi^{\otimes m}) \phi_\xi, \phi_\eta \right\rangle\right\rangle, \qquad \xi, \eta \in E_{\mathbb{C}}. \tag{5.8}$$

That Ξ is well defined is due to the characterization theorem of symbols (see Theorem 1.2). It is reasonable to write

$$\Xi = \int_{\mathbb{R}^{l+m}} a_{s_1}^* \cdots a_{s_l}^* L(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{t_1} \cdots a_{t_m} ds_1 \cdots ds_l dt_1 \cdots dt_m. \tag{5.9}$$

In fact, if L is a scalar-operator-valued distribution (5.9) is reduced to an integral kernel operator as in (1.8).

We observe that quantum stochastic integrals (5.3) are special cases of (5.9). Let $\{L_s\}$ be a quantum stochastic process and consider

$$\widetilde{L}_t(\eta) = \widetilde{L}(t, \eta) = \int_a^t \eta(s) L_s \, ds, \qquad \eta \in E_{\mathbb{C}},$$

where a is fixed. It is then easily verified that $\widetilde{L}_t \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)^*))$. Hence by the above consideration we can define

$$\Xi_t = \int_{\mathbb{R}} a_s^* \widetilde{L}_t(s) \, ds$$

as a generalized integral kernel operator, the symbol of which is

$$\langle\langle \Xi_t \phi_\xi, \phi_\eta \rangle\rangle = \langle\langle \widetilde{L}_t(\eta) \phi_\xi, \phi_\eta \rangle\rangle = \int_a^t \eta(s) \langle\langle L_s \phi_\xi, \phi_\eta \rangle\rangle \, ds = \int_a^t \langle\langle a_s^* L_s \phi_\xi, \phi_\eta \rangle\rangle \, ds.$$

Therefore

$$\int_{\mathbb{R}} a_s^* \widetilde{L}_t(s) \, ds = \int_a^t a_s^* L_s \, ds.$$

In other words, the Hitsuda–Skorokhod integral of a quantum stochastic process is a special case of a generalized integral kernel operator. The situation for a quantum stochastic integral against the annihilation process is similar.

5.4 Adapted processes

Consider the annihilation and creation operators associated with $\eta \in E_{\mathbb{C}}$:

$$D_\eta = \Xi_{0,1}(\eta) = \int_{\mathbb{R}} \eta(t) a_t \, dt, \qquad D_\eta^* = \Xi_{1,0}(\eta) = \int_{\mathbb{R}} \eta(t) a_t^* \, dt.$$

It follows from general theory [20] that $D_\eta^* \in \mathcal{L}((E), (E))$ and D_η is continuously extended to an operator from $(E)^*$ into itself, i.e., $D_\eta \in \mathcal{L}((E)^*, (E)^*)$. Thus for $\Xi \in \mathcal{L}((E), (E)^*)$ the commutators $[D_\eta, \Xi]$ and $[D_\eta^*, \Xi]$ are meaningful.

Definition 5.3 A generalized quantum stochastic process $\Xi \in \mathcal{L}(E_{\mathbb{C}}, \mathcal{L}((E), (E)^*))$ is called *adapted* if

$$[D_\eta, \Xi(\xi)] = [D_\eta^*, \Xi(\xi)] = 0$$

for any choice of $t \in \mathbb{R}$, $\xi, \eta \in E_{\mathbb{C}}$ such that supp $\xi \subset (-\infty, t)$ and supp $\eta \in (t, +\infty)$.

Lemma 5.4 [24] *Assume that a generalized quantum stochastic process Ξ is regular. Then it is adapted if and only if $[D_\eta, \Xi_t] = [D_\eta^*, \Xi_t] = 0$ for any $t \in \mathbb{R}$ and $\eta \in E_{\mathbb{C}}$ with* supp $\eta \subset (t, +\infty)$.

Definition 5.5 A quantum stochastic process $\{\Xi_t\}$ is called *adapted* if

$$[D_\eta, \Xi_t] = [D_\eta^*, \Xi_t] = 0$$

for any t and $\eta \in E_{\mathbb{C}}$ with supp $\eta \subset (t, +\infty)$.

Adaptedness of a quantum stochastic process was introduced by Hudson–Parthasarathy [13] and has been discussed by Huang [12] along with white noise calculus. Thier definitions are compatible with ours.

By definition, if $\{\Xi_t\}$ is adapted, so is $\{\Xi_t^*\}$. Moreover, if $\{\Xi_t\}$ is adapted and differentiable in $\mathcal{L}((E),(E)^*)$, then $\{d\Xi_t/dt\}$ is also adapted.

Theorem 5.6 [24] *Let $\{\Xi_t\}$ be a quantum stochastic process and let*

$$\Xi_t = \sum_{l+m\geq 1} \Xi_{l,m}(\kappa_{l,m}(t)) + c_t I, \qquad t \in \mathbb{R},$$

be the Fock expansion. Then $\{\Xi_t\}$ is adapted if and only if supp $\kappa_{l,m}(t) \subset (-\infty, t]^{l+m}$ *for any $t \in \mathbb{R}$ and $l + m \geq 1$.*

For example, the annihilation process $\{A_t\}$, the creation process $\{A_t^*\}$ and the number process $\{\Lambda_t\}$ are adapted. It is also obvious that $\{a_t\}$, $\{a_t^*\}$ and $\{a_t^* a_t\}$ are adapted.

The notion of adaptedness for both classical and quantum stochastic processes (Definitions 3.1 and 5.5) are compatible.

Proposition 5.7 *Let $t \mapsto \Phi_t \in (E)^*$ be a continuous map defined on an interval. Then it is adapted (in the classical sense) if and only if it is an adapted quantum stochastic process as multiplication operators.*

PROOF. Let

$$\Phi_t = \sum_{n=0}^{\infty} \left\langle :x^{\otimes n}:, F_n^{(t)} \right\rangle$$

be the Wiener–Itô expansion. Then, as multiplication operator the Fock expansion of Φ_t is given as

$$\Phi_t = \sum_{l,m=0}^{\infty} \frac{(l+m)!}{l!m!} \Xi_{l,m}(F_{l+m}^{(t)}),$$

see [20, Proposition 4.6.4]. Hence, in view of Theorem 5.6 it is adapted (in the quantum sense) if and only if supp $F_n^{(t)} \subset (-\infty, t]^n$ for all t and $n \geq 1$, i.e., $\{\Phi_t\} \subset (E)^*$ is adapted.

qed

5.5 Quantum stochastic integrals of an adapted process

Proposition 5.8 *Let $\{L_t\} \subset \mathcal{L}((E),(E)^*)$ be an adapted quantum stochastic process where t runs over an interval I. Then both*

$$\int_a^t L_s a_s \, ds, \qquad \int_a^t a_s^* L_s \, ds, \qquad t \geq a, \quad t \in I,$$

are adapted.

PROOF. It is sufficient to prove the assertion for the qunatum Hitsuda–Skorokhod integral

$$\Xi_t = \int_a^t a_s^* L_s \, ds, \qquad t \geq a, \quad t \in I.$$

For that purpose we shall show that

$$[D_\eta, \Xi_t] = [D_\eta^*, \Xi_t] = 0$$

whenever $t \geq a$ and $\eta \in E_{\mathbb{C}}$ with supp $\eta \subset (t, \infty)$. For $\phi, \psi \in (E)$ we have by definition

$$\langle\langle \Xi_t D_\eta \phi, \psi \rangle\rangle = \int_a^t \langle\langle a_s^* L_s D_\eta \phi, \psi \rangle\rangle \, ds.$$

Since both $\{a_s^*\}$ and $\{L_s\}$ are adapted, we have

$$\langle\langle a_s^* L_s D_\eta \phi, \psi \rangle\rangle = \langle\langle D_\eta a_s^* L_s \phi, \psi \rangle\rangle = \langle\langle a_s^* L_s \phi, D_\eta^* \psi \rangle\rangle.$$

Therefore

$$\langle\langle \Xi_t D_\eta \phi, \psi \rangle\rangle = \int_a^t \langle\langle a_s^* L_s \phi, D_\eta^* \psi \rangle\rangle ds = \langle\langle \Xi_t \phi, D_\eta^* \psi \rangle\rangle = \langle\langle D_\eta \Xi_t \phi, \psi \rangle\rangle,$$

which proves $[D_\eta, \Xi_t] = 0$. Similarly, $[D_\eta^*, \Xi_t] = 0$ is proved. qed

5.6 Conditional expectation of a quantum Hitsuda–Skorokhod integral

Lemma 5.9 *Let $\{L_t\}$ be an admissible process such that*

$$M \equiv \sup_{a \leq t \leq b} \sup_{\|\!| \phi \|\!|_{r,\beta} \leq 1} \|\!| L_t \phi \|\!|_{-r,-\beta} < \infty$$

for some $r, \beta \geq 0$. Assume that the quantum Hitsuda–Skorokhod integral

$$\Xi = \int_a^b a_s^* L_s \, ds$$

is admissible, i.e., belongs to $\mathcal{L}((\mathcal{A}), (\mathcal{A})^*)$. Then

$$E_t \Xi E_t = \begin{cases} \displaystyle\int_a^{t \wedge b} a_s^* E_t L_s E_t \, ds, & t \geq a, \\ 0, & t < a. \end{cases} \tag{5.10}$$

PROOF. Note that the composition $a_s^* E_t L_s E_t \in \mathcal{L}((E), (E)^*)$ is well defined. Moreover, for $\phi, \psi \in (E)$ we have

$$|\langle\langle a_s^* E_t L_s E_t \phi, \psi \rangle\rangle| \leq |\langle\langle E_t L_s E_t \phi, a_s \psi \rangle\rangle| \leq \|\!| E_t L_s E_t \phi \|\!|_{-r,-\beta} \|\!| a_s \psi \|\!|_{r,\beta}.$$

According to Lemma 2.6 we take $p \geq 0$ such that

$$\|\!| a_s \psi \|\!|_{r,\beta} \leq \| a_s \psi \|_p.$$

Noting also that E_t is an orthogonal projection, we obtain

$$|\langle\langle a_s^* E_t L_s E_t \phi,\, \psi\rangle\rangle| \le \| L_s E_t \phi \|_{-r,-\beta} \| a_s \psi \|_p .$$

By assumption we have

$$|\langle\langle a_s^* E_t L_s E_t \phi,\, \psi\rangle\rangle| \le M \| E_t \phi \|_{r,\beta} \| a_s \psi \|_p \le M \| \phi \|_{r,\beta} \| a_s \psi \|_p \le M \| \phi \|_p \| a_s \psi \|_p .$$

As for $\| a_s \psi \|_p$ it follows from a similar argument as in the proof of Lemma 3.8 there exist $M' \ge 0$ and $q > 0$ such that

$$\| a_s \phi \|_p \le M' \| \phi \|_{p+q}, \qquad \phi \in (E), \quad a \le s \le b.$$

Thus we come to

$$|\langle\langle a_s^* E_t L_s E_t \phi,\, \psi\rangle\rangle| \le MM' \| \phi \|_p \| \psi \|_{p+q}, \qquad a \le s \le b, \quad \phi, \psi \in (E),$$

and hence the integral

$$\int_a^{t\wedge b} a_s^* E_t L_s E_t \, ds, \quad t \ge a,$$

is well defined in $\mathcal{L}((E),(E)^*)$.

To see (5.10) we compute the operator symbol.

$$\begin{aligned}
\langle\langle E_t \Xi E_t \phi_\xi,\, \phi_\eta\rangle\rangle &= \langle\langle \Xi \phi_{\chi_t\xi},\, \phi_{\chi_t\eta}\rangle\rangle \\
&= \int_a^b \langle\langle a_s^* L_s \phi_{\chi_t\xi},\, \phi_{\chi_t\eta}\rangle\rangle \, ds \\
&= \int_a^b \langle\langle L_s \phi_{\chi_t\xi},\, a_s \phi_{\chi_t\eta}\rangle\rangle \, ds \\
&= \int_a^b \langle\langle L_s \phi_{\chi_t\xi},\, \phi_{\chi_t\eta}\rangle\rangle \, \chi_t(s)\eta(s) \, ds.
\end{aligned}$$

In fact, we need a suitable argument of approximation which is similar to the proof of Lemma 3.8. Hence for $t \le a$ we have $E_t \Xi E_t = 0$. Suppose that $t > a$. Then, the last integral being taken over the interval $[a, t\wedge b]$, we come to

$$\langle\langle E_t \Xi E_t \phi_\xi,\, \phi_\eta\rangle\rangle = \int_a^{t\wedge b} \langle\langle L_s \phi_{\chi_t\xi},\, \phi_{\chi_t\eta}\rangle\rangle \, \eta(s) \, ds.$$

Finally we compute the integrand.

$$\begin{aligned}
\langle\langle L_s \phi_{\chi_t\xi},\, \phi_{\chi_t\eta}\rangle\rangle \, \eta(s) &= \langle\langle L_s E_t \phi_\xi,\, E_t \phi_\eta\rangle\rangle \, \eta(s) = \langle\langle E_t L_s E_t \phi_\xi,\, \phi_\eta\rangle\rangle \, \eta(s) \\
&= \langle\langle E_t L_s E_t \phi_\xi,\, a_s \phi_\eta\rangle\rangle = \langle\langle a_s^* E_t L_s E_t \phi_\xi,\, \phi_\eta\rangle\rangle ,
\end{aligned}$$

which completes the proof. qed

Proposition 5.10 *Let $\{L_t\}$ be an admissible process such that*

$$\sup_{a \le t \le b} \sup_{\| \phi \|_{r,\beta} \le 1} \| L_t \phi \|_{-r,-\beta} < \infty$$

for some $r, \beta \geq 0$. Assume that the quantum Hitsuda–Skorokhod integral

$$\Xi_t = \int_a^t a_s^* L_s \, ds, \qquad t \geq a,$$

is admissible, i.e., belongs to $\mathcal{L}((\mathcal{A}), (\mathcal{A})^*)$. If $E_t L_t E_t = L_t$ for all t, then we have

$$E_s \Xi_t E_s = \Xi_s, \qquad a \leq s \leq t.$$

PROOF. Suppose $a \leq s \leq t$. It follows from Lemma 5.9 that

$$E_s \Xi_t E_s = \int_a^s a_u^* E_s L_u E_s \, du.$$

By assumption we have $E_s L_u E_s = E_u L_u E_u = L_u$ for $a \leq u \leq s$. Hence

$$E_s \Xi_t E_s = \int_a^s a_u^* L_u \, du = \Xi_s.$$

qed

5.7 Quantum martingales

Definition 5.11 An adapted process $\{\Xi_t\}$ is called a *quantum martingale* if it is admissible and

$$E_s \Xi_t E_s = E_s \Xi_s E_s, \qquad s \leq t. \tag{5.11}$$

This is a straightforward extension of Parthasarathy–Sinha's one [27]. By definition the adjoint process of a quantum martingale is again a quantum martingale.

Theorem 5.12 *Let $\{L_t\}$ be an adapted admissible process such that*

$$\sup_{a \leq t \leq b} \sup_{\| \phi \|_{r,\beta} \leq 1} \| L_t \phi \|_{-r,-\beta} < \infty$$

for some $r, \beta \geq 0$. Assume that the quantum Hitsuda–Skorokhod integral

$$\Xi_t = \int_a^t a_s^* L_s \, ds, \qquad t \geq a,$$

is admissible. Then $\{\Xi_t\}_{t \geq a}$ is a quantum martingale.

PROOF. That $\{\Xi_t\}_{t \geq a}$ is adapted follows from Proposition 5.8. Condition (5.11) is easily verified with Lemma 5.9. qed

Proposition 5.13 *The annihilation process $\{A_t\}$, the creation process $\{A_t^*\}$ and the number process $\{\Lambda_t\}$ are quantum martingales.*

PROOF. It follows from Theroem 5.12 that $\{A_t\}$ is a quantum martingale; hence so is $\{A_t^*\}$. It has been already shown that the number process is an adapted admissble process. Condition (5.11) is checked easily. qed

Finally we prove a quantum analogue of Proposition 3.6.

Proposition 5.14 *Let $\{\kappa_{l,m}^{(t)}\}$ be a one-parameter family of slowly increasing functions. If $\Xi_t = \Xi_{l,m}(\kappa_{l,m}^{(t)})$ is a quantum martingale, then there exists a $\mathbb{C}$-valued measurable function on $\mathbb{R}^{l+m}$ such that*

$$\Xi_t = \Xi_{l,m}(\chi_t^{\otimes(l+m)} \cdot \kappa_{l,m}).$$

PROOF. Since $\{\Xi_t\}$ is adapted we see from Theorem 5.6 that supp $\kappa_{l,m}^{(t)} \subset (-\infty, t]^{l+m}$. Now, in view of Lemma 4.9 we see that $E_s \Xi_t E_s = E_s \Xi_s E_s$ for $s \le t$ if and only if

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^{s} ds_1 \cdots \int_{-\infty}^{s} ds_l \int_{-\infty}^{s} dt_1 \cdots \int_{-\infty}^{s} dt_m \int_{s}^{+\infty} du_1 \cdots \int_{s}^{+\infty} du_n \times$$
$$\kappa_{l,m}^{(t)}(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{u_1}^* \cdots a_{u_n}^* a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} a_{u_1} \cdots a_{u_n}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{-\infty}^{s} ds_1 \cdots \int_{-\infty}^{s} ds_l \int_{-\infty}^{s} dt_1 \cdots \int_{-\infty}^{s} dt_m \int_{s}^{+\infty} du_1 \cdots \int_{s}^{+\infty} du_n \times$$
$$\kappa_{l,m}^{(s)}(s_1, \cdots, s_l, t_1, \cdots, t_m) a_{u_1}^* \cdots a_{u_n}^* a_{s_1}^* \cdots a_{s_l}^* a_{t_1} \cdots a_{t_m} a_{u_1} \cdots a_{u_n}.$$

Since the Fock expansion is unique, we obtain

$$\kappa_{l,m}^{(t)}(s_1, \cdots, s_l, t_1, \cdots, t_m) = \kappa_{l,m}^{(s)}(s_1, \cdots, s_l, t_1, \cdots, t_m)$$

for $s_1, \cdots, s_l, t_1, \cdots, t_m \le s$. Therefore there exists a $\mathbb{C}$-valued measurable function $\kappa_{l,m}$ on $\mathbb{R}^{l+m}$ such that

$$\kappa_{l,m}^{(t)} = \chi_t^{\otimes(l+m)} \cdot \kappa_{l,m},$$

which completes the proof. qed

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