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ON FUNCTIONAL EQUATIONS OF PREHOMOGENEOUS ZETA DISTRIBUTIONS OVER A LOCAL FIELD OF CHARACTERISTIC P

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ABSTRACT. For a local field of characteristic 0, the functional equations of zeta distributions of prehomogeneous vector spaces are obtained by M.Sato, T.Shintani, J.Igusa and F.Sato (See [17], [9], [13], [15]). In this paper, we shall consider the case of local fields of characteristic p > 0.

§1. K-regular P.V.'s

We fix a local field K of characteristic $p > 0$. Let G be a connected linear algebraic group, ρ its rational representation of G on a finite-dimensional vector space V, all defined over an algebraic closure $\bar{K}$ of K. We call a triplet (G,ρ,V) a prehomogeneous vector space (abbrev. P.V.) if V has a Zariski-dense G-orbit Y.

Any point of Y is called a generic point and the isotropy subgroup $G_y = \{g \in G; \rho(g)y = y\}$ of a generic point y is called a generic isotropy subgroup. Note that we have $dim\ G_y = dim\ G - dim\ V$ if and only if $y \in Y$. A non-zero rational function $f(x)$ on V is called a relative invariant of (G,ρ,V) if $f(\rho(g)x) = \chi(g)f(x)$ holds for any $g \in G$ and $x \in Y$ where $\chi : G \to GL_1$ is a rational character of G.

The complement S of Y is a Zariski-closed set which is called the singular set of the P.V. (G,ρ,V). Now we assume that (G,ρ,V) is defined over K, i.e., G,ρ,V are all defined over K. Let $S_i = \{x \in V; f_i(x) = 0\}$ $(i = 1,....,l)$ be the K-irreducible component of the K-rational points S_K of S of codimension one defined by a K-irreducible (not necessarily absolutely irreducible) polynomial $f_i(x)$ $(i = 1,\cdots,l)$. Then $f_1(x),.......,f_l(x)$ are algebraically independent relative invariants and any relative invariant $f(x)$ in $K(V)$ is of the form $f(x) = c\cdot f_1(x)^{m_1}\cdots\cdot f_l(x)^{m_l}(c \in K^\times,(m_1,.....,m_l) \in \mathbf{Z}^l)$. We call $f_1(x),\cdots,f_l(x)$ the basic K-relative invariants of (G,ρ,V). Let χ_i be the rational character of G corresponding to $f_i (i = 1,...,l)$. Let $X(G)_K$ be the group of K-rational characters of G, $X_1(G)_K$ its subgroup corresponding to K-relative invariants. Then $X_1(G)_K$ is a free abelian group of rank l generated by $\chi_1,.....,\chi_l$.

Let G_1 be a subgroup of G generated by the commutator subgroup $[G,G]$ and a generic isotropy subgroup. This does not depend on a choice of a generic point. For $\chi \in X(G)_K$, it is in $X_1(G)_K$ if and only if $\chi|_{G_1} = 1$. For a relative invariant $f(x)$ of (G,ρ,V), we can define a rational map $\varphi_f : Y \to V^*$ by

$$\varphi_f(x) = {}^t\left(\frac{1}{f(x)}\cdot\frac{\partial f}{\partial x_1}(x), \ldots\ldots, \frac{1}{f(x)}\cdot\frac{\partial f}{\partial x_n}(x)\right)$$

where V^* is the dual vector space of V. We sometimes denote $\varphi_f(x)$ by *grad log* $f(x)$. By a direct calculation, we have

(1) $\varphi_f(\rho(g)x) = \rho^*(g)\varphi_f(x)$ for $g \in G$ and $x \in Y$ where ρ^* denotes the contragradient representation of ρ,

and

(2) $\langle d\rho(A)x, \varphi_f(x)\rangle = \delta\chi(A)$ for $x \in Y$ and $A \in Lie(G)$ where $d\rho$ (resp. $\delta\chi$) is the infinitesimal representation of ρ (resp. the infinitesimal character of χ) of the Lie algebra $Lie(G)$ of G.

A relative invariant $f(x)$ is called non-degenerate if $\varphi_f : Y \to V^*$ is dominant and the Hessian $H_f(x) = \det(\frac{\partial^2 f}{\partial x_i \partial x_j})$ is not identically zero. In this case, a rational function $F(x) = \frac{f(x)^n}{H_f(x)}$ ($n = dim\ V$) is a relative invariant corresponding to the character $\chi_0(g) = \det\rho(g)^2$.

If there exists a non-degenerate relative invariant $f(x)$ in $K(V)$, we say that (G,ρ,V) is a K-regular P.V. Then we have $\det\rho(g)^2 \in X_1(G)_K$. In general, we denote by Y_K, S_K, etc. K-rational points of $Y, S, etc.$ We write $X_1^*(G)_K$ (resp. $X^*(G)_K, Y^*, S^*$, etc.) for (G,ρ^*,V^*) which corresponds to $X_1(G)_K$ (resp. $X(G)_K, Y, S$, etc.) for (G,ρ,V).

Proposition 1.1

Assume that (G,ρ,V) and (G,ρ^*,V^*) are K-regular P.V.'s . Then we have the following assertion.

(1) $X_1(G)_K = X_1^*(G)_K$.

(2) For a non-degenerate K-relative invariant f, the map $\varphi = grad\ log\ f : Y \to Y^*$ is bijective.

[Proof]

Since $\varphi(Y)$ is a Zariski-dense G-orbit in V^*, we have $\varphi(Y) = Y^*$, i.e., $\varphi =$ surjective. Since $\rho^*(g)\varphi(x) = \varphi(\rho(g)x)$, we have $G_x \subset G_{\varphi(x)}$ for $x \in Y$. Now let f^* be a non-degenerate relative invariant in $K(V^*)$, and put $\varphi^* = grad\ log\ f^* : Y^* \to Y$. Similarly we have $G_y \subset G_{x'}$ for $y = \varphi(x)$ and $x' = \varphi^*(y)$, and hence $G_x \subset G_y \subset G_{x'}$. Since $x' = \rho(g_0)x$ for some $g_0 \in G$, we have $G_{x'} = g_0 G_x g_0^{-1} \supset G_x$. Since $dim\ G_{x'} = dim\ G_x$, the algebraic group $G_{x'}$ and G_x have the same connected component H of the identity. Since $G_{x'}$ is isomorphic to G_x, the numbers of their connected components coincide, i.e., $[G_{x'} : H] = [G_x : H]$ with $G_{x'} \supset G_x$. This implies $G_{x'} = G_x$, and hence $G_x = G_y$ with $y = \varphi(x)$.

Thus we have $G_1 = G_1^*$ and hence $X_1(G)_K = X_1^*(G)_K$. Note that $X_1(G)_K = \{\chi \in X(G)_K; \chi|_{G_1} = 1\}$. Now assume that $\varphi(x_1) = \varphi(x_2)$ with $x_2 = \rho(g)x_1$ for some $g \in G$. Then we have $\varphi(x_1) = \varphi(x_2) = \varphi(\rho(g)x_1) = \rho^*(g)\varphi(x_1)$ and hence $g \in G_{\varphi(x_1)} = G_{x_1}, i.e., x_2 = \rho(g)x_1 = x_1$. Thus φ is injective.□

Now assume that (G, ρ, V) is a K-regular P.V. Then, as we have seen above, the dual triplet (G, ρ^*, V^*) is a P.V. For a generic point $y \in Y^*$, a dominant morphism $\psi : G \to V^*$ defined by $\psi(g) = \rho^*(g)y$ is called an open orbit morphism.

Proposition 1.2

Assume that (G, ρ, V) is a K-regular P.V. and an open orbit morphism $\psi : G \to V^*$ is a separable morphism. Then there exists a K-relative invariant f^* such that grad log $f^* : Y^* \to V$ is dominant.

[Proof]

Let f be a non-degenerate relative invariant in $K(V)$ and put $\varphi = grad\ log\ f : Y \to Y^*$. First we show that φ is injective. Assume that $\varphi(x) = \varphi(x')$. Since $\delta\chi(A) = \langle d\rho(A)x, \varphi(x)\rangle = -\langle x, d\rho^*(A)\varphi(x)\rangle$, we have $\langle x - x', d\rho^*(A)\varphi(x)\rangle = 0$ for all $A \in Lie(G)$. Since $\psi : G \to V^*$ with $\psi(g) = \rho^*(g)\varphi(x)$ is separable, we have $\{d\rho^*(A)\varphi(x); A \in Lie(G)\} = V^*$, and hence $x - x' = 0, i.e., x = x'$. For any $g \in G_{\varphi(x)}(\supset G_x)$, we have $\varphi(\rho(g)x) = \rho^*(g)\varphi(x) = \varphi(x)$. As φ is injective, we have $\rho(g)x = x, i.e., g \in G_x$. This implies that $G_x = G_{\varphi(x)}$ and hence $X_1(G)_K = X_1^*(G)_K$. A rational character χ corresponding to f is in $X_1(G)_K$ and hence $\chi^{-1} \in X_1^*(G)_K$. This implies that there exists a relative invariant f^* in $K(V^*)$ satisfying $f^*(\rho^*(g)y) = \chi(g)^{-1}f^*(y)$ for $g \in G$ and $y \in Y^*$.

Put $\varphi^* = grad\ log\ f^*$. Then we have $\langle\varphi^*(y), d\rho^*(A)y\rangle = -\delta\chi(A)$. Since $\delta\chi(A) = \langle d\rho(A)x, \varphi(x)\rangle = -\langle x, d\rho^*(A)\varphi(x)\rangle$, we have

$\langle x - \varphi^*(y), d\rho^*(A)y\rangle = 0$ for $y = \varphi(x)$ and all $A \in Lie(G)$.

Since the open orbit morphism ψ is separable , we have

$\{d\rho^*(A)y; A \in Lie(G)\} = V^*$,

and hence $\varphi^*(y) = x \in Y, i.e., \varphi^*(Y^*) = Y$. □

Note that in the case of $ch(K) = 0$, the proof of Proposition 1.2 gives the equivalence between K-regularity of (G, ρ, V) and that of (G, ρ^*, V^*).

Proposition 1.3

Assume that (G, ρ, V) and (G, ρ^*, V^*) are K-regular P.V.'s. Then we have $\sharp\ \rho(G)_K\backslash Y_K = \sharp\ \rho^*(G)_K\backslash Y_K^*$.

[Proof]

Let f be a non-degenerate relative invariant in $K(V)$ and put $\varphi = grad\ log\ f$. Then for any $x \in Y_K$, we have

$\varphi(\rho(G)_K \cdot x) = \rho^*(G)_K \cdot \varphi(x) \subset Y_K^*$, i.e., φ maps an orbit in Y_K to an orbit in Y_K^*.

By Proposition1.1, this map φ is injective, and hence $\sharp\ \rho(G)_K\backslash Y_K \leqq \sharp\ \rho^*(G)_K\backslash Y_K^*$. Similarly we have $\sharp\ \rho^*(G)_K\backslash Y_K^* \leqq \sharp\ \rho(G)_K\backslash Y_K$. □

Now we shall consider a sufficient condition that $\sharp\ \rho(G)_K\backslash Y_K$ is finite.

Professor J.P.Serre kindly let us know the following theorem with the proof which was explained by Tits to him.

Theorem 1.4

Let K be a local field of characteristic $p > 0$ (or more generally let K be a field complete with respect to a discrete valuation, and with the residue field k of type (F) in the sense of Serre [18]. Let G be a connected smooth reductive group over K. Then $H^1(K,G)$ is finite.

[Proof](after Serre's letter on 9th. September 1992.)

Let K' be the maximal unramified extension of K. The field K' is known to be of *dim.* ≤ 1 (in the sense of CG, II, §3). By a theorem of Steinberg (for K' perfect) and of Borel-Springer (for K' imperfect - see Borel Col. Papers II, p.761) we have $H^1(K',G) = 0$. Hence the Galois cohomology of G over K is killed by K', i.e., it is equal to $H^1(K'/K,G)$. We may now apply a theorem of Bruhat-Tits (J.Fac.Sci.Tokyo, 34 (1987), p.693, th.3.12); this says that $H^1(K'/K,G)$ is contained in a finite union of cohomology sets $H^1(k,G_i)$, where the G_i's are algebraic linear groups (non neccessarily connected) over k. Since k is type (F), each $H^1(k,G_i)$ is finite (see e.g. Borel, Col.Papers II, p.404 , th.6.2, or Coh. Gal. III-30, th.4). Hence $H^1(K,G)$ is finite.

□

Proposition 1.5

Let (G,ρ,V) be a P.V. defined over K with a reductive generic isotropy subgroup. Then $\sharp\rho(G)_K\backslash Y_K$ is finite.

[Proof]

Let H be a generic isotropy subgroup of a point in Y_K. Then there exists a bijection between $\rho(G)_K\backslash Y_K$ and $Ker(H^1(K,H) \to H^1(K,G))$ (see Serre [18]) . By Theorem 1.4, $H^1(K,H)$ is finite , and hence $\rho(G)_K\backslash Y_K$ is a finite set.

□

Example 1.6

Let G be the subgroup of GL_n consisting of all lower triangular matrices. Let V be the totality of symmetric $n \times n$ matrices and define ρ by

$\rho(g)x = gx^tg\ for\ all\ g \in G$

and $x \in V$. Since dim $G =$ dim V, a generic isotropy subgroup is a finite subgroup and hence we have $\sharp\ \rho(G)_K \backslash Y_K = \nu < +\infty$ by Proposition 1.5.

Moreover $\det x$ is a non-degenerate K-relative invariant. By $tr(xy)$ $(x, y \in V)$, we identify V with its dual V^*.

Then (G, ρ, V) and (G, ρ^*, V^*) are K-regular P.V.'s. Hence, by Proposition 1.3, we have $\sharp\ \rho^*(G)_K \backslash Y_K^* = \nu < +\infty$.

Proposition 1.7

Let (G, ρ, V) be an irreducible regular P.V. defined over K. Then we have $\sharp \rho(G)_K \backslash Y_K < +\infty$.

[Proof]

By a classification of irreducible P.V.'s (see Z.Chen [4]), we know that a generic isotropy subgroup is reductive.

□

§2. Zeta distributions

Let K be a local field of characteristic $p > 0$. Assume that (G, ρ, V) and its dual (G, ρ^*, V^*) are K-regular P.V.'s . Moreover we shall assume that $Y_K = Y_1 \cup \cdots \cup Y_\nu$ decomposes into a finite union of $\rho(G)_K$-orbits Y_i $(1 \leqq i \leqq \nu)$, i.e., $\sharp\ \rho(G)_K \backslash Y_K = \nu < +\infty$. Then by Proposition 1.3 , we have $Y_K^* = Y_1^* \cup \cdots \cup Y_\nu^*$.

Let $f_1(x), \cdots, f_l(x)$(resp. $f_1^*(y), \cdots, f_l^*(y)$) be basic K-relative invariants of (G, ρ, V) (resp. (G, ρ^*, V^*)). Let χ_i(resp. χ_i^*) be the corresponding character of f_i (resp. f_i^*). Then we have

$X_1(G)_K = \langle \chi_1, \cdots, \chi_l \rangle$ and $X_1^*(G)_K = \langle \chi_1^*, \cdots, \chi_l^* \rangle$.

By Proposition 1.1, we have $X_1(G)_K = X_1^*(G)_K$ so that there exists uniquely a matrix

$U = (u_{ij}) \in GL_l(\mathbf{Z})$

satisfying $\chi_i = \prod_{j=1}^{l} \chi_j^{*u_{ij}}$. Since $\det \rho(g)^2 \in X_1(G)_K$, we have $\det \rho(g)^2 = \chi_1^{2\lambda_1} \cdots \chi_l^{2\lambda_l}$ for some $\lambda = (\lambda_1, \cdots, \lambda_l) \in (\frac{1}{2}\mathbf{Z})^l$ and $\det \rho^*(g)^2 = \chi_1^{*2\lambda_1^*} \cdots \chi_l^{*2\lambda_l^*}$ for some $\lambda^* = (\lambda_1^*, \cdots, \lambda_l^*) \in (\frac{1}{2}\mathbf{Z})^l$. Since $\det \rho^*(g) = \det \rho(g)^{-1}$, we have $\lambda^* = -\lambda U$.

Example 2.1

For simplicity, we deal with the case $n = 2$ in **Example 1.6.** *Then we have*

$$G = \{g = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} ; ab \neq 0\}$$

and

$$V = \{X = \begin{pmatrix} x & y \\ y & z \end{pmatrix}\}.$$

The basic K-relative invariants of (G, ρ, V) (resp. (G, ρ^*, V^*)) are $f_1(X) = x$ and $f_2(X) = \det X$ (resp. $f_1^*(X) = z$, $f_2^*(X) = \det X$) corresponding to $\chi_1(g) = a^2$, $\chi_2(g) = a^2b^2$ (resp. $\chi_1^*(g) = b^{-2}$, $\chi_2^*(g) = a^{-2}b^{-2}$) for

$$g = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix}$$

in G.
Hence $\chi_1 = \chi_1^*\chi_2^{*-1}$ and $\chi_2 = \chi_2^{*-1}$ so that we have

$$U = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}.$$

Since

$$\det \rho(\begin{pmatrix} a & 0 \\ c & b \end{pmatrix}) = a^3b^3,$$

we have $\lambda = \lambda^* = (0, \frac{3}{2})$.

Let $\{\varepsilon_1, \cdots, \varepsilon_\nu\}$ be the complete representatives of $K^\times/K^{\times 2}$ in $K^\times$. Then we have $Y_K = Y_1 \cup \cdots \cup Y_\nu$ with
$Y_i = \{y \in Y_K;\ f_2(y) \equiv \varepsilon_i \bmod K^{\times 2}\}$ $(i = 1, \cdots, \nu)$.

Let $\omega^{(i)} : K^\times \to \mathbf{C}^\times$ $(i = 1, \cdots, l)$ be a quasicharacter, i.e., a continuous homomorphism.

For $\omega = (\omega^{(1)}, \cdots, \omega^{(l)})$ and the basic K-relative invariants $f(x) = (f_1(x), \cdots, f_l(x))$, we write $\omega(f(x))$ instead of $\Pi_{i=1}^{l}\omega^{(i)}(f_i(x))$ for simplicity of notations.

Let $|\ |$ be the absolute value of K normalized by $|\pi| = q^{-1}$ for a prime element π where q is the module of K. For $s = (s_1, \cdots, s_l)$, we write $\omega_s = (|\ |^{s_1}, \cdots, |\ |^{s_l})$ so that $\omega_s(f(x)) = \Pi_{i=1}^{l}|f_i(x)|^{s_i}$.

Let dx be the Haar measure on $V_K = K^n$ normalized by $\int_{R^n} dx = 1$ where R is the maximal compact subring of K. Since $d(\rho(g)x) = |\det \rho(g)|dx$ and $\omega_\lambda(f(\rho(g)x)) = |\det \rho(g)|\omega_\lambda(f(x))$, the measure $d_Y(x) = \frac{dx}{\omega_\lambda(f(x))}$ is a G-invariant measure on Y.

For $\Phi \in \mathfrak{S}(V_K)$ where $\mathfrak{S}(V_K)$ denotes the Schwartz-Bruhat space of V_K, we define an integral

$$Z_i(\omega, \Phi) = \int_{Y_i} \omega(f(x))\Phi(x)d_Y(x)\ (i = 1, \cdots, \nu).$$

Now any quasi-character $\omega^{(i)} : K^\times \to \mathbf{C}^\times = \{z \in \mathbf{C}; z \neq 0\}$ can be written uniquely as $\omega^{(i)} = |\ |^{s_i} \cdot \phi_i$ for some $s_i \in \mathbf{C}$ and $\phi_i : R^\times \to \mathbf{C}_1^\times = \{z \in \mathbf{C}; |z| = 1\}$ where $R^\times$ is the units of R. Put $Re\ \omega^{(i)} = Re\ s_i$ $(i = 1, \cdots, l)$. The following lemma is easy to prove and we omit the proof (cf. F.Sato [15]).

Lemma 2.2

If $Re\ \omega^{(i)} > \lambda_i$ $(i = 1, \cdots, l)$, *the integral* $Z_i(\omega, \Phi)$ *is absolutely convergent and holomorphic with respect to* $s = (s_1, \cdots, s_l) \in (\mathbf{C}/(\frac{2\pi i}{\log q}\mathbf{Z}))^l \cong \mathbf{C}^{\times l}$ *for* $\omega = (|\ |^{s_1} \cdot \phi_1, \cdots, |\ |^{s_l} \cdot \phi_l)$.

Let $\mathfrak{S}'(V_K) = \{z : \mathfrak{S}(V_K) \to \mathbf{C},\ \mathbf{C} - \textit{linear mapping}\}$ be the space of distributions on V_K. By Lemma 2.2, the mapping $\Phi \mapsto Z_i(\omega, \Phi)$ defines a distribution on V_K when $Re\ \omega^{(i)} > \lambda_i$ $(i = 1, ..., l)$.

For (G, ρ^*, V^*), we can define similar distribution $Z_j^*(\omega)$ $(j = 1, \cdots, \nu)$ given by

$$Z_j^*(\omega, \Phi^*) = \int_{Y_j^*} \omega(f^*(y))\Phi^*(y)d_{Y^*}(y).$$

Now we fix a non-trivial additive character $\psi : K \to \mathbf{C}_1^\times$ and define the Fourier transformation $\mathfrak{S}(V_K^*) \ni \Phi^* \mapsto \hat{\Phi}^* \in \mathfrak{S}(V_K)$ by

$\hat{\Phi}^*(x) = \int_{V_K^*} \Phi^*(y)\psi(\langle x, y\rangle)dy$

where dy is a Haar measure on V_K^* dual to a fixed Haar measure on V_K.

For $\omega = (\omega^{(1)}, \cdots, \omega^{(l)})$, put $\omega^* = \omega^U = (\Pi_{i=1}^l \omega^{(i)u_{i1}}, \cdots, \Pi_{i=1}^l \omega^{(i)u_{il}})$.

Our purpose is to show that $Z_i(\omega)$ and $Z_j^*(\omega)$ are continued analytically to all ω and satisfy the functional equation:

$$\hat{Z}_i(\omega) = \sum_{j=1}^{\nu} \Gamma_{ij}(\omega) Z_j^*(\omega^* \omega_{\lambda^*})\ (i = 1, \cdots, \nu) \tag{2.1}$$

under some additional conditions where

$\hat{Z}_i(\omega)(\Phi^*) = Z_i(\omega, \hat{\Phi}^*)$. Recall that $\omega_{\lambda^*} = (|\ |_{\lambda_1^*}, ..., |\ |_{\lambda_l^*})$ with $\det \rho^*(g)^2 = \chi_1^{2\lambda_1^*} \cdots \chi_l^{2\lambda_l^*}$.

Actually when K is a local field of $ch(K) = 0$, then (2.1) is obtained under some conditions and it is called " the fundamental theorem of P.V. over K ".

§3. Rationality for almost all p

For a rational prime p, let K_p denotes the local field with the constant field $\mathbf{F}_p$. For $f \in \mathbf{Z}[x_1, \cdots, x_n]$, we denote $f\ \textit{mod}\ p \in \mathbf{F}_p[x_1, \cdots, x_n]$ by f_p. Then we have the following theorem which is suggested by Professor M.Kashiwara.

Theorem 3.1
For almost all p, the integral

$$Z_p(s,\Phi_p) = \int_{K_p^n} |f_p(x)|_{K_p}^s \Phi_p(x) d_p x$$

is a rational function of $t = p^{-s}$ *where* $\Phi_p \in \mathfrak{S}(K_p^n)$ *and* $d_p x$ *is a Haar measure on* K_p^n.

[Proof]
Let $K = \mathbf{Q}((t))$ be a field of formal power series over $\mathbf{Q}$, $X = \Omega^n$ the affine space and $X_K = K^n$. Let f denote the morphism $X \to \Omega$ defined by $f(x)$; then there exists a nonsingular algebraic variety Y and a projective morphism $h : Y \to X$ both defined over K with the following property : let b denote an arbitrary point of Y_K, $\mathfrak{O}_K$ the local ring of Y at b relative to K (consisting of " functions " defined over K), and $\mathfrak{M}_K$ the ideal of non-units of $\mathfrak{O}_K$; then there exists an ideal basis $(y_1, \cdots, y_n)$ of $\mathfrak{M}_K$, elements u, v of $\mathfrak{O}_K - \mathfrak{M}_K$, and integers $N_i \geqq 0, \nu_i \geqq 1$ for $1 \leqq i \leqq n$ such that

$$f \circ h = u \cdot \prod_{i=1}^{n} y_i^{N_i}, \quad h^*(dx) = v \cdot \prod_{i=1}^{n} y_i^{\nu_i - 1} dy.$$

The existence of such a pair (Y, h) is guaranteed by Hironaka's theorem [5] p.109 -p.326]. Then for almost all p, the reduction modulo p is well-defined and we have similar results for $K_p, f_p, \cdots$ *etc.* Then by just similar argument as in Appendix of Igusa [11], we obtain our result. □

Remark 3.2
Let K be a number field. For $f \in \mathfrak{O}_K[x_1, \cdots, x_n]$, we have a similar result as Theorem 3.1 for almost all prime ideals $\mathfrak{P}$ of $\mathfrak{O}_K$.

§4. Functional equations

Lemma 4.1
Let G denote a locally compact totally disconnected group, H a closed subgroup of G, $X = H \backslash G$, *and* $\omega : G \to \mathbf{C}^\times$ *a quasicharacter. Put*
$\xi_X(\omega) = \{T \in \mathfrak{S}(X)'; gT = \omega(g)^{-1}T \text{ for all } g \in G\}$.
Then we have $\dim_{\mathbf{C}} \xi_X(\omega) \leqq 1$. *Moreover* $\dim_{\mathbf{C}} \xi_X(\omega) = 1$ *if and only if* $\Delta_G \cdot \omega|_H = \Delta_H$ *where* Δ_G, Δ_H *denotes the module of G, H respectively.*

[Proof]
See Igusa [9] p.1015 .□

Let (G,ρ,V) and its dual (G,ρ^*,V^*) be K-regular P.V.'s with
$\sharp\ \rho(G)_K\backslash Y_K = \nu < +\infty$
where K is a local field of characteristic p. Then, by Proposition 1.3, we have
$Y_K = Y_1 \cup \cdots \cup Y_\nu$ and $Y_K^* = Y_1^* \cup \cdots \cup Y_\nu^*$. i.e., $\sharp\rho^*(G)_K\backslash Y_K^* = \nu$.
As in §2, we can define the zeta distribution $Z_i(\omega,\Phi)$ (resp. $Z_i^*(\omega,\Phi^*)$) which is convergent when $Re\ \omega^{(j)} > \lambda_j$ (resp. $Re\omega^{(j)} > \lambda_j^*$) ($1 \leq i \leq \nu$, $1 \leq j \leq l$).
We denote by $Z_i(\omega)$ the distribution defined by $\Phi \mapsto Z_i(\omega,\Phi)$ etc.

Proposition 4.2
We have
(1) $Z_j^*(\omega^*\omega_{\lambda^*}) \in \xi_{Y_j^*}(\omega^*\omega_{\lambda^*})$
and
(2) $\hat{Z}_i(\omega) \in \xi_{Y_j^*}(\omega^*\omega_{\lambda^*})$.
$(i,j = 1,....,\nu\ )$

[Proof]
By a direct calculation, we obtain our results. □

Proposition 4.3
Let K be a local field of characteristic $p > 0$ with the module q. For $\omega = (\omega^{(1)},\cdots,\omega^{(l)})$ with $\omega^{(i)} = \omega_{s_i}\cdot\phi_i$ ($\phi_i(\pi) = 1$ for a prime element π), assume that $Z_i(\omega,\Phi)$ and $Z_j^*(\omega,\Phi^*)$ are rational functions of $q^{-s_1},\cdots,q^{-s_l}$. Then for all $\Phi^* \in \mathfrak{S}(Y_K^*)$, we have

$$Z_i(\omega,\hat{\Phi^*}) = \sum_j \Gamma_{ij}(\omega)Z_j^*(\omega^*\omega_{\lambda^*},\Phi^*)$$

for $i,j = 1,\cdots,\nu$.

[Proof]
Since $Z_i(\omega,\Phi)$ and $Z_j(\omega,\Phi^*)$ are rational functions, it is defined for all ω except poles and hence by Lemma 4.1 and Proposition 4.2, we have our result. □

Theorem 4.4
Let (G,ρ,V) be a K-regular P.V. satisfying the following conditions:
(C1) its dual (G,ρ^*,V^*) is a K-regular P.V. such that
$\sharp\ \rho^*(G)_K\backslash V_K^* < +\infty$,
(C2) for $x \in S_K^*$, there exists $\chi \in X_1(G)_K$ satisfying $\chi(G_{x,K}) \not\subseteq R^\times$ where $R^\times$ is the units of the maximal compact subring R of K
and
(C3) $Z_j(\omega,\Phi)$ is a rational function of $q^{-s_1},\cdots,q^{-s_l}$ where
$\omega = (\omega^{(1)},\cdots,\omega^{(l)})$ *with* $\omega^{(i)} = \omega_{s_i}$ ($1 \leqq i \leqq l$).
Then we have the functional equation

$$Z_i(\omega, \hat{\Phi}^*) = \sum_j \Gamma_{ij}(\omega) Z_j^*(\omega^* \omega_{\lambda^*}, \Phi^*)$$

for all $\Phi^* \in \mathfrak{S}(V_K^*)$ *for* $i, j = 1, \cdots, \nu$ *where* $\nu = \sharp\ \rho^*(G)_K \backslash Y_K^*$.

[Proof]

The condition *(C2)* corresponds to Lemma 2.2 in F.Sato [15] p474 for the case of $ch(K) = 0$. Then the proof is just similar as the case of $ch(K) = 0$ (using Proposition 4.3) (See Igusa [9] and F.Sato [15] p.477). □

Now let (G, ρ, V) be a reductive **Q**-regular P.V. Then for almost all p, we have a reduction modulo p and we obtain K_p-regular P.V. (G_p, ρ_p, V_p) where K_p is a local field with the constant field $\mathbf{F}_p$.

(Assumption A)

Assume that $\sharp\ \rho_p(G)_{K_p} \backslash S_{K_p} < +\infty$ *and for* $x \in S_{K_p}$, *there exists* $\chi \in X_1(G_p)_{K_p}$ *satisfying* $\chi(G_{p,x,K_p}) \not\subseteq R_p^\times$ *for almost all* p.

Let (G, ρ, V) be a reductive **Q**-regular P.V. with (**Assumption A**). Let $f_1, \cdots, f_l$ be basic **Q**-relative invariants with **Z**-coefficients. Denote $|f_1 \ mod\ p|_{K_p}^{s_1} \cdots |f_l \ mod\ p|_{K_p}^{s_l}$ by $|f^{(p)}(x)|_{K_p}^s$ and

$Z_i^p(s, \Phi_p) = \int_{(Y_{K_p})_i} |f^{(p)}(x)|_{K_p}^s \Phi_p(x) dy_{Y_p}(x)$

for $\Phi_p \in \mathfrak{S}(V_{K_p})$.

Theorem 4.5

Let (G, ρ, V) *be a reductive* **Q**-*regular P.V. with* (**Assumption A**). *Then for almost all rational prime* p, *the integral* $Z_i^p(s, \Phi_p)$ ($i = 1, \cdots, \nu_p$, $Y_{K_p} = Y_1 \cup \cdots \cup Y_{\nu_p}$) *is a rational function and satisfies the functional equation:*

$$Z_i^p(s, \hat{\Phi}_p) = \sum_{j=1}^{\nu_p} \Gamma_{ij}(s) Z_j^p(s^*, \Phi_p)$$

($i = 1, ..., \nu_p$ *).*

When $l = 1$, *we have* $s^* = \frac{n}{d} - s$ *with* $n = dim\ V$ *and* $d = deg\ f$. *In general, for* $\omega = \omega_s = \omega_{s_1} \cdots \omega_{s_l}$, *we have* $\omega_{s^*} = \omega^* \omega_{\lambda^*}$.

[Proof]

By Theorem 4.4 and using the results of §1 and §3, we obtain our result. □

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