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$\mathbb{Z}$-forms of representations of reductive groups and prehomogeneous vector spaces

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In [**1**], the representations of reductive group schemes are discussed, and especially the concept of the 'split form' is defined. In the present article, first we review [**1**] in a most elementary way, restricting ourselves to the case where the base scheme is $\operatorname{Spec}\mathbb{Z}$. Then we discuss how such a general theory can be applied to the theory of prehomogeneous vector spaces.

1. Reductive group scheme.

1.1. A reductive group scheme over $\mathbb{Z}$ is by definition a group scheme which is affine and smooth over $\mathbb{Z}$ whose geometric fibres are *connected* reductive [**5**, exposé 19, 2.7]. (More generally, for any commutative ring A or for any scheme S, we can similarly define the concept of reductive group scheme over A or S.)

1.2. Remark. If the connectedness is not assumed, I do not know how to define the concept of 'reductive group scheme'. If the fibre dimension is 0, then it would be natural to assume that it is finite étale.

1.3.. Remark. In order to consider the bad reduction, it is interesting to remove the smoothness assumption from the definition of the reductive group scheme.

1.4. Example.

$$GL_{n,\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[\{x_{ij}\}_{1\le i,j\le n}, \det(x_{ij})^{-1}]) \quad \text{and} \tag{1}$$

$$SL_{n,\mathbb{Z}} = \operatorname{Spec}(\mathbb{Z}[\{x_{ij}\}_{1\le i,j\le n}]/(\det(x_{ij})-1)_{\text{ideal}})$$

are reductive group schemes over $\mathbb{Z}$. We shall denote their coordinate rings (i.e., the inside of Spec( )) by $\mathbb{Z}[GL_{n,\mathbb{Z}}]$ etc.

(2) Let $f := u_1^2 + \cdots + u_n^2$ and $G(\mathbb{C}) := SO_n(\mathbb{C})$ be the special orthogonal group with respect to f (i.e., the group of the usual orthogonal matrices with determinant 1). Put

$$
\begin{aligned}
&I := \{\varphi \in \mathbb{Z}[SL_{n,\mathbb{Z}}] \mid \varphi \equiv 0 \text{ on } SO_n(\mathbb{C})\}, \\
&G_{\mathbb{Z}} := \operatorname{Spec}(\mathbb{Z}[SL_{n,\mathbb{Z}}]/I), \\
&G_A := G_{\mathbb{Z}} \otimes_{\mathbb{Z}} A \; (= G_{\mathbb{Z}} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec} A)
\end{aligned}
$$

for any commutative ring A. Then $G_{\mathbb{Z}[1/2]}$ is a reductive group scheme over $\mathbb{Z}[1/2]$, and $G_{\mathbb{Z}}(\mathbb{C}) = SO_n(\mathbb{C})$ is a reductive algebraic group, but $G_{\mathbb{Z}}$ is *not* a reductive group scheme over $\mathbb{Z}$. In fact, $G_{\mathbb{Z}}(\overline{\mathbb{F}}_2) \subset GL_n(\overline{\mathbb{F}}_2)$ is conjugate with

$$
\left\{ \begin{pmatrix} 1 & 0 & \cdots & 0 \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix} \in SL_n(\overline{\mathbb{F}}_2) \right\}
$$

in $GL_n(\overline{\mathbb{F}}_2)$, where $\overline{\mathbb{F}}_2$ is an algebraic closure of $\mathbb{F}_2$, especially the geometric fibre of $G_{\mathbb{Z}}$ at $\operatorname{Spec}\overline{\mathbb{F}}_2$ is not reductive. This phenomenon occurs because f becomes a *degenerate* quadratic form after reduction modulo 2.

It would be worth noting here that $G_{\mathbb{Q}}$ has no model over $\mathbb{Z}$ which is reductive over $\mathbb{Z}$.

(3) Note that, in (2), we can construct the group scheme $G_{\mathbb{Z}}$ from any quadratic form f. If we start from

$$
f = \begin{cases} \sum_{i=1}^{m} x_i x_{m+i} & (n = 2m) \\ \sum_{i=1}^{m} x_i x_{m+i} + x_{2m+1}^2 & (n = 2m+1), \end{cases}
$$

then the resulting group scheme is reductive over $\mathbb{Z}$, whose geometric fibres are always special orthogonal groups.

2. Split form of a representation.

2.1. Definition of a split $\mathbb{Z}$-form.

Notation.

$G = G_{\mathbb{C}} =$ a connected complex reductive group.

$T = T_{\mathbb{C}} =$ a maximal torus.

$M = \mathrm{Hom}(T, \mathbb{C}^\times)$.

$V = V_{\mathbb{C}} =$ a finite dimensional *multiplicity free* rational G-module.

$V = \bigoplus_i V_i =$ irreducible decomposition.

$V = \bigoplus_{\mu \in M} V_\mu =$ the weight space decomposition with respect to T.

$\mathcal{U}_{\mathbb{Z}} =$ the $\mathbb{Z}$-subalgebra of the enveloping algebra $U(\mathfrak{g})$ generated by $X_\alpha^m/m!$, where $\{X_\alpha \mid \alpha = \text{root}\}$ is a Chevalley system. See [**3**, §1] for Chevalley system. Consider a triple $(T, \mathcal{U}_{\mathbb{Z}}, V(\mathbb{Z}))$, where $V(\mathbb{Z})$ is a free $\mathbb{Z}$-submodule of V such that

(1) $\mathrm{rank}_{\mathbb{Z}} V(\mathbb{Z}) = \dim_{\mathbb{C}} V$,

(2) $\mathcal{U}_{\mathbb{Z}} \cdot V(\mathbb{Z}) \subset V(\mathbb{Z})$,

(3) $V(\mathbb{Z}) = \bigoplus_{\mu \in M} V(\mathbb{Z}) \cap V_\mu$.

(If G is semisimple, then the condition (3) is redundant [**8**, p.17, Corollary 1], but it is necessary in general.) Consider the equivalence relation $(T, \mathcal{U}_{\mathbb{Z}}, V(\mathbb{Z})) \sim (gTg^{-1}, g\mathcal{U}_{\mathbb{Z}}g^{-1}, \sigma g V(\mathbb{Z}))$ for $g \in G(\mathbb{C})$ and $\sigma \in \mathrm{Aut}_G V$. We call each of the equivalence classes (or $(T, \mathcal{U}_{\mathbb{Z}}, V(\mathbb{Z}))$ itself) a *split $\mathbb{Z}$-form* of (G, V). (If we can understand T and $\mathcal{U}_{\mathbb{Z}}$ from the context, we sometimes call abusively $V(\mathbb{Z})$ a split $\mathbb{Z}$-form.)

2.2. Dual. Let $(G, V^\vee) = (G, \bigoplus_i V_i^\vee)$ be the dual of (G, V). Put $V^\vee(\mathbb{Z}) := \{v^\vee \in V^\vee \mid \langle v^\vee, V(\mathbb{Z})\rangle \subset \mathbb{Z}\}$ (= the dual lattice). Then $V^\vee(\mathbb{Z})$ is a split $\mathbb{Z}$-form of $(G, V^\vee)$.

2.3. Minimal split $\mathbb{Z}$-form. Let μ_i be a weight of (G, V_i) which is highest with respect to some fixed Borel subgroup B containing T. Take $0 \neq v_i \in V_i \cap V_{\mu_i}$ and put $V_{\min}(\mathbb{Z}) := \bigoplus_i \mathcal{U}_{\mathbb{Z}} \cdot v_i$. Then $V_{\min}(\mathbb{Z})$ is a split $\mathbb{Z}$-form such that $V_{\min}(\mathbb{Z}) \cap \bigoplus_i V_{\mu_i} = \bigoplus_i \mathbb{Z} v_i$.

2.4. Maximal split $\mathbb{Z}$-form. Take $v_i^\vee \in V_i^\vee \cap V_{-\mu_i}^\vee$ so that $\langle v_i^\vee, v_i \rangle = 1$. Put $V_{\min}^\vee(\mathbb{Z}) := \bigoplus_i \mathcal{U}_{\mathbb{Z}} \cdot v_i^\vee$ and $V_{\max}(\mathbb{Z}) := \{v \in V \mid \langle v, V_{\min}^\vee(\mathbb{Z}) \rangle \subset \mathbb{Z}\}$. Then $V_{\max}(\mathbb{Z})$ is a split $\mathbb{Z}$-form such that $V_{\max}(\mathbb{Z}) \cap \bigoplus_i V_{\mu_i} = \bigoplus_i \mathbb{Z} v_i$.

2.5. (1) Every split $\mathbb{Z}$-form $V(\mathbb{Z})$ normalized so that $\bigoplus_i V(\mathbb{Z}) \cap V_{\mu_i} = \bigoplus_i \mathbb{Z} v_i$ satisfies

$$V_{\min}(\mathbb{Z}) \subset V(\mathbb{Z}) \subset V_{\max}(\mathbb{Z}).$$

Conversely, any $\mathbb{Z}$-submodule $V(\mathbb{Z})$ of $V(\mathbb{C})$ such that

$$V_{\min}(\mathbb{Z}) \subset V(\mathbb{Z}) \subset V_{\max}(\mathbb{Z}), \text{ and}$$
$$\mathcal{U}_{\mathbb{Z}} V(\mathbb{Z}) \subset V(\mathbb{Z})$$
$$V(\mathbb{Z}) = \bigoplus_{\mu \in M} V(\mathbb{Z}) \cap V_\mu(\mathbb{C})$$

gives a split $\mathbb{Z}$-form.

Problem A. *When two such $V(\mathbb{Z})$'s are equivalent?*

If the answer of the following problem is affirmative, then different $V(\mathbb{Z})$'s are never equivalent.

Problem B. $V(\mathbb{Z}) = \bigoplus_i (V_i(\mathbb{C}) \cap V(\mathbb{Z}))$?

(2) If (G, V) is a (not necessarily reduced) saturated [**3**, Introduction], irreducible, regular, prehomogeneous vector space, then $V_{\max}(\mathbb{Z})/V_{\min}(\mathbb{Z}) = 0$ or $= \mathbb{Z}/2\mathbb{Z}$. (The proof uses the classification of M.Sato and T.Kimura. Note that $V_{\min}(\mathbb{Z})$ and

$V_{\max}(\mathbb{Z})$ depends only on $\mathcal{U}_{\mathbb{Z}}$, and that they behave well under the castling transformation.) Hence there are at most 2 split $\mathbb{Z}$-forms. More precisely, there is only one split $\mathbb{Z}$-form for the prehomogeneous vector space of type (1), (3), (5), (6), (7), (9), (10), (11), (12), (13), (15D), (20), (21), (23), (24), (27), (28), (29). There are 2 split $\mathbb{Z}$-forms for the type (2), (4), (8), (14), (15B). Here the number refers to that of [**6**, §7]. The type (15), i.e, $(SO_n \times GL_m, \mathbb{C}^n \otimes \mathbb{C}^m)$ is referred to as (15B) (resp. (15D)) if n is odd (resp. even). See [**3**] for the detail.

2.6. Example. If $G = GL_n$ and V is the totality of $n \times n$ symmetric matrices. Then

$$V_{\min}(\mathbb{Z}) = \{(x_{ij}) \in V \mid x_{ij} \in \mathbb{Z}\},$$
$$V_{\max}(\mathbb{Z}) = \{(x_{ij}) \in V \mid x_{ii} \in \mathbb{Z},\ 2x_{ij} \in \mathbb{Z}\ (i \neq j)\}.$$

2.7. Geometric meaning of split $\mathbb{Z}$-form. Let $(T, \mathcal{U}_{\mathbb{Z}}, V(\mathbb{Z}))$ be a split $\mathbb{Z}$-form. Then, we get

(1) a Chevalley-Demazure group scheme $G_{\mathbb{Z}}$ (= a split reductive group scheme) such that $G_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C} = G_{\mathbb{C}}$ and which contains $T_{\mathbb{Z}} = \operatorname{Spec} \mathbb{Z}M$ as a maximal torus, where $\mathbb{Z}M$ is the group ring of M (cf. (2.1) for M),

(2) a vector bundle $V_{\mathbb{Z}} = \operatorname{Spec} S(V^{\vee}(\mathbb{Z}))$, where $S(V^{\vee}(\mathbb{Z}))$ is the symmetric algebra of $V^{\vee}(\mathbb{Z})$, and

(3) the action $G_{\mathbb{Z}} \times V_{\mathbb{Z}} \to V_{\mathbb{Z}}$ which becomes the original action $G \times V \to V$ after $\otimes_{\mathbb{Z}} \mathbb{C}$.

Let us add some explanation about (3). By (2.1, (3)), from a split $\mathbb{Z}$-form we can get a $T_{\mathbb{Z}}$-action on $V_{\mathbb{Z}}$. Roughly speaking, $G_{\mathbb{Z}}$ consists of $T_{\mathbb{Z}}$ (=maximal torus) and $\mathcal{U}_{\mathbb{Z}}$ ($\fallingdotseq$ semisimple part), and hence we get a $G_{\mathbb{Z}}$-action on $V_{\mathbb{Z}}$ combining the above $T_{\mathbb{Z}}$-action and the $\mathcal{U}_{\mathbb{Z}}$-action on $V(\mathbb{Z})$.

2.8. General $\mathbb{Z}$-forms. Now, let $(G'_{\mathbb{Z}}, V'_{\mathbb{Z}})$ be an arbitrary pair of a reductive $\mathbb{Z}$-group scheme and a vector bundle over $\operatorname{Spec}\mathbb{Z}$ (i.e., a $\mathbb{Z}$-lattice, since the class number of $\mathbb{Z}$ is 1) such that $(G'_{\mathbb{Z}}, V'_{\mathbb{Z}}) \otimes \mathbb{C} = (G, V)$. Such a pair $(G'_{\mathbb{Z}}, V'_{\mathbb{Z}})$ is called *a $\mathbb{Z}$-form* of (G, V), and can be obtained from a split $(G_{\mathbb{Z}}, V_{\mathbb{Z}})$ by twisting it using non-abelian étale cohomology.

2.9. Remark. (1) In [**1**], we obtained (2.8) assuming the irreducibility of (G, V). In order to obtain (2.8) assuming only that (G, V) is multiplicity free, it is enough to replace the "highest weight vector v_0" appearing in the definition of the "épinglage of a representation of a reductive group scheme" [**1**, (3.6)] by the "maximal weight vectors $\{v_i\}_i$" (see (2.3) for $\{v_i\}_i$).

Problem C. *Prove (2.8) without assuming the multiplicity freeness.*

The essential difficulty is how to define "épinglage of a representation of a reductive group scheme". (Even without assuming the multiplicity freeness, we can prove that, étale locally with respect to the base scheme, a representation of a reductive group scheme over $\mathbb{Z}$ can be obtained similarly as in (2.7). An "épinglage" is a device which is used to patch together these local data to obtain a globally split object.)

(3) Although Problem C is unsettled, "the multiplicity freeness" does not seem to be very harmful for our application in the theory of prehomogeneous vector spaces. In fact, if (G, V) is a prehomogeneous vector space, and $V = \bigoplus_{i=1}^{N} V_i$ is an irreducible decomposition of V, consider $(\tilde{G}, V) := (G \times GL_1^N, \bigoplus_{i=1}^{N} V_i)$, where the i-th factor of $GL_1^N = \{(x_1, \cdots, x_N)\}$ acts on V_i as a scalar multiplication and trivially on the remaining V_j's. Then $(\tilde{G}, V)$ is a multiplicity free, and as is easily seen, the relative invariant polynomials on the prehomogeneous vector spaces (G, V) and $(\tilde{G}, V)$ are the same. Since the relative invariants are of our main interest, replacing G with the larger group $\tilde{G}$, we can escape the difficulty.

2.10. Polynomial with $\mathbb{Z}$-coefficients. Assume that a $\mathbb{Z}$-form of (G, V) is given (cf. (2.8)). Then we can consider the lattice $V(\mathbb{Z})$ and its dual lattice $V^\vee(\mathbb{Z})$. An element of $V^\vee(\mathbb{Z})$ ($\subset V^\vee$) gives a linear function on V, and hence the symmetric algebra $\mathbb{Z}[V_\mathbb{Z}] := S_\mathbb{Z}(V^\vee(\mathbb{Z}))$ generated by $V^\vee(\mathbb{Z})$ (over $\mathbb{Z}$) can be regarded as a ring of polynomial functions on V. We shall consider an element of $\mathbb{Z}[V_\mathbb{Z}]$ as *a polynomial functions on V with $\mathbb{Z}$-coefficients.* In the same way, we can consider a polynomial function on $V^\vee$ with $\mathbb{Z}$-coefficients.

3. Application to the theory of prehomogeneous vector spaces —— Leading coefficients of $b(s)$.

From now on, we assume that (G, V) (cf. §2) is a prehomogeneous vector space. Concerning the prehomogeneous vector spaces, we use the notations of [**2**, (1.4)] freely.

3.1. Take $\phi \in \mathrm{Hom}(G, \mathbb{C}^\times)$. Let $f \in \mathbb{C}[V]$ (resp. $f^\vee \in \mathbb{C}[V^\vee]$) be a relative invariant whose character is ϕ (resp. ϕ^{-1}). (See [**2**, (1.4, (10), and (11)].) If we do not consider a $\mathbb{Z}$-form of (G, V), f and $f^\vee$ are determined only up to $\mathbb{C}^\times$. Hence the leading coefficient of $b(s)$ does not have a much significance. Now consider a $\mathbb{Z}$-form $(G_\mathbb{Z}, V_\mathbb{Z})$ of (G, V), and assume that

(1) some constant multiples of f and $f^\vee$ are polynomial functions with $\mathbb{Z}$-coefficients.

(This condition is automatically satisfied if (G, V) is irreducible.) Then first assume that f and $f^\vee$ are of $\mathbb{Z}$-coefficients, and next single out the common factor of the coefficients. In this way, we can normalize f and $f^\vee$ up to ± 1. Then the leading coefficient b_0 of $b(s)$ has a meaning up to ± 1. Now multiplying suitable ± 1 to f and $f^\vee$, we may assume $b_0 > 0$, and then the leading coefficient b_0 of the b-function is uniquely determined without any ambiguity. In (3.2)–(3.4) below, we assume that f, $f^\vee$, $b(s)$ and b_0 are normalized in this way.

3.2. A strange formula. If (G, V) is a (not necessarily reduced nor regular) irreducible prehomogeneous vector space and if the $\mathbb{Z}$-form $(G_{\mathbb{Z}}, V_{\mathbb{Z}})$ is split, then (3.1, (1)) is satisfied and

$$b_0 = \prod_{j\geq 1} (j^j)^{e(j)},$$

where

$$b^{\exp}(t) = \prod_{j\geq 1} (t^j - 1)^{e(j)}.$$

See [2, (1.4, (24))] for $b^{\exp}$ and $e(j)$.

3.3. Example. Let (G, V) be a reduced irreducible regular prehomogeneous vector space of type (11) [6, §7, Table I]. Then a conjecture of I.Ozeki says

$$\begin{aligned} b(s) =& (s+1)^8\{(s+\tfrac{2}{3})(s+\tfrac{4}{3})\}^4\{(s+\tfrac{3}{4})(s+\tfrac{5}{4})\}^4\{(s+\tfrac{5}{6})(s+\tfrac{7}{6})\}^4 \\ & \times \{(s+\tfrac{7}{10})(s+\tfrac{9}{10})(s+\tfrac{11}{10})(s+\tfrac{13}{10})\}^2. \end{aligned}$$

Hence

$$b^{\exp}(t) = \phi_1^8\phi_3^4\phi_4^4\phi_6^4\phi_{10}^2, \text{ and } \prod_{j\geq 1}(j^j)^{e(j)} = 2^{56}3^{24}5^{10},$$

where ϕ_j is the j-th cyclotomic polynomial (e.g., $\phi_3 = t^2 + t + 1$). On the other hand b_0 is calculated by J.Murakami (1984.8.20) using a computer based on the method (3.7) below: $b_0 = 2^{56}3^{24}5^{10}$.

3.4. Remark. I expect that (3.2) holds without assuming the irreducibility. See [4, Remarks 7–9].

3.5. Even if we admit a degeneration of the geometric fibres of $G_{\mathbb{Z}}$, the leading coefficient b_0 of $b(s)$ seems to be divisible by $\prod_{j\geq 1}(j^j)^{e(j)}$, where $b^{\exp}(t) = \prod_{j\geq 1}(t^j - 1)^{e(j)}$, and moreover the quotient seems to be a product of (powers of) primes at which $G_{\mathbb{Z}}$ degenerates. In other words, $b_0/\prod_{j\geq 1}(j^j)^{e(j)}$ seems to control the bad reduction of a prehomogeneous vector space (G, V) together with f.

3.6. Example. Let $f = x_1^2 + \cdots + x_n^2$ and $f^\vee = y_1^2 + \cdots + y_n^2$. Then $b(s) = 4(s+1)(s+\frac{n}{2})$, $b_0 = 4$,

$$b^{\exp}(t) = \begin{cases} (t-1)^2 & \text{if } n \text{ is even,} \\ (t^2-1) & \text{if } n \text{ is odd,} \end{cases}$$

$$\prod_{j\geq 1}(j^j)^{e(j)} = \begin{cases} 1 & \text{if } n \text{ is even,} \\ 4 & \text{if } n \text{ is odd.} \end{cases}$$

3.7. The leading coefficient b_0 of $b(s)$ can be calculated by the method used in the proof of Proposition 2.7 of [**7**]. Let us explain it. In our notation, $b_0 f(v)^{-1} = f^\vee((\operatorname{grad}\log f)(v)) = f^\vee(f(v)^{-1}\cdot(\operatorname{grad} f)(v)) = f(v)^{-d}\cdot f^\vee((\operatorname{grad} f)(v))$, i.e.,

$$b_0 = f(v)^{-d+1} f^\vee((\operatorname{grad} f)(v)). \tag{1}$$

Take some v, which is suitable for the calculation, and then evaluate the right hand side of (1).

4. Second application —— Hessian of $\log f$.

Take a $\mathbb{Z}$-form of a prehomogeneous vector space (G, V). Then we can consider $V(\mathbb{Z})$ and its dual lattice $V^\vee(\mathbb{Z})$. Let $\{v_1^\vee, \cdots, v_n^\vee\}$ be a free $\mathbb{Z}$-basis of $V^\vee(\mathbb{Z})$, put $x_i := v_i^\vee$ and regard $\{x_1, \cdots, x_n\}$ as a linear coordinate system of V. Then we can consider

$$\operatorname{Hess}(\log f) = \det\left(\frac{\partial \log f}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq n}. \tag{1}$$

If we took an arbitrary linear coordinate system $\{x_1, \cdots, x_n\}$ defined over $\mathbb{C}$, then (1) depends on the choice of the coordinate and hence $\operatorname{Hess}(\log f)$ has a meaning only up to a constant multiple. However, under the normalization as above, for two coordinate systems $\{x_1, \cdots, x_n\}$ and $\{x'_1, \cdots, x'_n\}$, the Jacobian $\det(\partial x_i/\partial x'_j)$ is ± 1, and especially $\mathrm{Hess}(\log f)$ is independent of the choice of the coordinate. Therefore it is interesting to know its explicit form. This calculation is complicated, but can be somewhat simplified by using

$$\mathrm{Hess}(\log f) = (1-d)^{-1} f(x)^{-n}\, \mathrm{Hess}(f), \tag{1}$$

(cf. the proof of Proposition 10 of [**6**, pp.62–64]). Indeed, the right hand side is easier to calculate, although it is still difficult. Note that this quantity and some other related quantity appear in [**2**, Theorem C].

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