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Report on the convergence of zeta functions associated with prehomogeneous vector spaces

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The aim of the present note is to give a survey of results and techniques of proving the convergence of zeta functions.

§1 Direct approach

1.1 Explicit calculation

The convergence of zeta functions is obvious if one can calculate the zeta functions under consideration explicitly and obtain an expression in terms of known zeta functions such as the Riemann zeta function. This is the case, for example, for the zeta functions associated with the space $(GL(n), M(n))$. In this case the zeta function is given by

$$\zeta(s)\zeta(s-1)\cdots\zeta(s-n+1).$$

1.2 Use of reduction theory

Zeta functions associated with prehomogeneous space are of the form of Dirichlet series

$$\sum_{m=1}^{\infty}\frac{N_i(L;m)}{m^s},\quad N_i(L;m)=\sum_{\substack{v\in\Gamma\backslash L\cap V_i\\ P(v)=\epsilon_i m}}\mu(v),$$

where the notation is the same as in [S3], §1. Hence, if we can get an estimate like

$$N_i(L;m)<c\cdot m^{\alpha},\quad m=1,2,\ldots,$$

then we know that the zeta function is absolutely convergent for $\mathrm{Re}(s) > \alpha + 1$.

In case the density $\mu(v)$ is simple, then, it is often possible to obtain a necessary estimate of $N_i(L; m)$, by using the reduction theory of algebraic groups (= the description of the fundamental domain G^+/Γ with the Siegel domain, for this see [BH]). An example of this kind of argument is found in [Sh], Proposition 2.1, where Shintani obtained an estimate necessary to the proof of the zeta functions associated with the space of binary cubic forms.

However, if the density $\mu(v)$ is not so simple, then it is rather difficult to carry out the proof of convergence through this method. For example, the convergence of the Siegel zeta functions of indefinite quadratic forms is not at all obvious, though Siegel wrote in his paper [Si] just "Die Konvergents der Reihe entnimmt man der Reduktiontheorie". A detailed proof of the convergence of the Siegel zeta functions can be found in Tamagawa [Ta]. In §3 (resp. §4), we explain another method that can be applied to the Siegel zeta functions if the number of variables of a quadratic form is > 4 (resp. > 3).

§2 Mean value theorem

The following integral formula is due to Siegel:

$$\int_{SL_n(\mathbb{R})/SL_n(\mathbb{Z})} \sum_{x \in \mathbb{Z}^n - \{0\}} f(gx)\, dg = c \int_{\mathbb{R}^n} f(x)\, dx \quad (f \in L^1(\mathbb{R}^n)). \tag{2.1}$$

Using this formula, we give a proof of the convergence of the Epstein zeta function. Let Y be a positive definite symmetric matrix of size n. Then the Epstein zeta function is defined by

$$\zeta(Y, s) = \sum_{x \in \mathbb{Z}^n - \{0\}} Y[x]^{-s},$$

where we employ the usual notaion $Y[x] = {}^t x Y x$. The zeta function is nothing but the zeta function associated with the space $(GO(Y), V(n))$. We define a modification $\zeta_1(Y, s)$ of the Epstein zeta function by setting

$$\zeta_1(Y, s) = \sum_{\substack{x \in \mathbb{Z}^n - \{0\} \\ Y[x] \geq 1}} |Y[x]|^{-s}.$$

The series $\zeta_1(Y, s)$ differs from the Epstein zeta function only by a finite number of terms. We consider the integral

$$I(s) = \int_{SL_n(\mathbb{R})/SL_n(\mathbb{Z})} \zeta_1(Y[g], s)\, dg.$$

Put

$$f_s(x) = \begin{cases} Y[x]^{-s} & \text{if } Y[x] \geq 1, \\ 0 & \text{if } Y[x] < 1. \end{cases}$$

Then, by (2.1), we have

$$\begin{aligned} I(s) &= \int_{SL_n(\mathbb{R})/SL_n(\mathbb{Z})} \sum_{x\in\mathbb{Z}-\{0\}} f_s(gx)\,dg \\ &= c\int_{\mathbb{R}^n} f_s(x)\,dx \end{aligned}$$

Since

$$\begin{aligned} \int_{\mathbb{R}^n} f_s(x)\,dx &= \int_{Y[x]\geq 1} Y[x]^{-s}\,dx \\ &= (\det Y)^{-1/2}\int_{{}^txx\geq 1} ({}^txx)^{-s}\,dx \\ &= c'(\det Y)^{-1/2}\int_1^\infty r^{-(2s-n+1)}\,dr. \end{aligned}$$

The last integral is absolutely convergent if $\mathrm{Re}(s) > n/2$. This implies that $I(s)$ is absolutely convergent for $\mathrm{Re}(s) > n/2$. By the Fubini theorem, $\zeta_1(Y[g];s)$ is absolutely convergent for almost everywhere on $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ under the same condition. The convergence of $\zeta(Y;s)$ for every Y and s with $\mathrm{Re}(s) > n/2$ follows immediately from this.

The above argument can be generalized further to, e.g., $(SO_m \times GL_n, M_{m,n})$ if the real form of SO_m is compact (see [Te]). Much more generally, the Godement criterion on the convergence of the Eisenstein series of reductive groups is based on a similar idea (see [B], Chapter 11 §2). However, if Y is indefinite, then the argument above does not apply to the zeta function (the Siegel zeta function) attached to $(GO(Y), V(n))$. The reason is that the Siegel zeta functions are defined only when Y is a rational symmetric matrix.

§3 Admissible representations in the sense of Igusa

Let k be an algebraic number field. Let G^1 be a linear algebraic group defined over k and $\rho : G^1 \to GL(V)$ a k-rational representation of G^1 on a finite dimensional vector space V with k-structure. Following Igusa [I], we call $\rho : G^1 \to GL(V)$ an *admissible* representation if the integral

$$I(\Phi) = \int_{G^1(\mathbb{A})/G^1(k)} \sum_{x\in V(k)} \Phi(\rho(g)x)\,dg$$

is absolutely convergent for any Schwartz-Bruhat function Φ on $V(\mathbb{A})$.

Remark. Do not confuse the notion of "admissible representation" in this sense with that in the theory of infinite-dimensional representations.

Let (G,ρ,V) be an irreducible regular p.v. defined over $\mathbb{Q}$. Let $P(v)$ be a irreducible relative invariant and χ the corresponding character. Denote by G^1 the kernel of χ. One of the basic assumptions in [SS] is that

(3.1) *the restriction of ρ to G^1 is admissible* (cf. [SS], (2.6)).

Remarks. (1) In [SS], the authors employed the classical language, not the adelic language. However the adelic formulation above is equivalent to [SS], (2.6).

(2) In [SS], (2.6), it is assumed further that

$$\mathcal{S}(V(\mathbb{A})) \ni \Phi \longmapsto I(\Phi) \in \mathbb{C}$$

is a tempered distribution. This is a consequence of the admissibility as is noted at the beginning of the proof of [W], Lemma 5.

The assumption (3.1) implies the convegence of the zeta functions ([SS], Corollary to Lemma 2.3 and Lemma 2.5).

Since no proof of Corollary to Lemma 2.3 is given in [SS], we give here a proof, in which any undefined symbols have the same meanings as in [SS].

Proof of Corollary to Lemma 2.3 in [SS]. Since $f \mapsto I'(f, L)$ is a tempered distribution, for a given positive constant C, there exist positive integers k, l and a positive number ϵ such that

$$|I'(f, L)| < C \quad \text{if} \quad \|f\|_{k,l} < \epsilon,$$

where

$$\|f\|_{k,l} = \sup_{v \in V(\mathbb{R})} \left(1 + \|v\|^2\right)^k \sum_{|\alpha| \leq l} |D^\alpha(f)(v)|.$$

For any positive number t, we put $f^t(v) = f(tv)$. Then, if $t \geq 1$, we have $\|f^t\|_{k,l} \leq t^l \|f\|_{k,l}$. Take an everywhere non-negative f_0 in $C_0^\infty(\{v \in V_i \mid |P(v)| = 1\})$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a C^∞-function such that $\psi(x) = 1$ if $\frac{1}{2} \leq x \leq 1$ and the support is contained in $\mathbb{R}_+^\times$. Define a function $f(v)$ by setting

$$f(v) = f_0(|P(v)|^{1/d} v)\psi(|P(v)|).$$

We choose f_0 so that $\|f\|_{k,l} < \epsilon$. Then we have

$$\begin{aligned}
C &> \left|I'(t^{-l} f^t, L)\right| \\
&= t^{-l} \sum_{x \in L_i/\sim} \mu(x) \int_{\rho(G_+^1)\cdot x} f^t(y)\tilde{\omega}(y) \\
&= t^{-l} \int_{\{v \in V_i \mid |P(v)|=1\}} f_0(y)\tilde{\omega}(y) \sum_{x \in L_i/\sim} \mu(x)\psi(t^d P(x)) \\
&\geq t^{-l} \int_{\{v \in V_i \mid |P(v)|=1\}} f_0(y)\tilde{\omega}(y) \sum_{\substack{x \in L_i/\sim \\ \frac{1}{2} \leq |P(x)| \leq t}} \mu(x).
\end{aligned}$$

This implies the estimate

$$\sum_{\substack{x \in L_i/\sim \\ |P(x)| \leq t}} \mu(x) = O(t^l). \qquad \blacksquare$$

A natural problem arising here is how to check the condition. The answer was given by Weil [W], Lemma 5 and Igusa [I], §2 and was used by Shintani in [SS] to prove the convergence of the zeta functions attached to the space of hermitian matrices.

We may assume that G^1 is reductive. Let T be a maximal $\mathbb{Q}$-split torus of G^1 and P a minimal $\mathbb{Q}$-parabolic subgroup containing T. Let Θ be the identity component of $T(\mathbb{R})$. We consider Θ as a subgroup of $T(\mathbb{A})$. We denote by θ a general element of Θ. Let Θ_1 be the subset of Θ defined by $|\alpha(\theta)| \leq 1$ for every positive root α (with respect to P). We put

$$D_\rho(\theta) = \prod_\lambda \sup(1, |\lambda(\theta)|^{-1}),$$

where λ runs over the set of weights of ρ each repeated with its multiplicity. Then the representation ρ of G^1 is admissible if and only if the integral

$$\int_{\Theta_1} D_\rho(\theta) \cdot D_{Ad}(\theta)^{-1}\, d\theta$$

is convergent, where Ad denotes the adjoint representation of G^1. The "if" part was proved by Weil and the "only if " part was proved by Igusa.

Remark. The admissibility is a quite strong condition. A much weaker condition is sufficient to ensure the convergence of zeta functions. In fact, as the above proof of Corollary to Lemma 2.3 in [SS] shows, the convergence of

$$I'(\Phi) = \int_{G^1(\mathbb{A})/G^1(k)} \sum_{x \in (V-S)(k)} \Phi(\rho(g)x)\, dg \quad (\Phi \in \mathcal{S}(V(\mathbb{A}))$$

implies the convergence of zeta functions (see [SS], p.169, Additional remark 2). Perhaps it is better to say that the sum defining zeta functions should be taken over the maximal subset V' of $V(k)$ such that the integral

$$I(V', \Phi) = \int_{G^1(\mathbb{A})/G^1(k)} \sum_{x \in V'} \Phi(\rho(g)x)\, dg$$

is absolutely convergent for any $\Phi \in \mathcal{S}(V(\mathbb{A}))$.

§4 The case of "connected" isotropy subgroups

Let (G, ρ, V) be a prehomogeneous vector space defined over an algebraic number field k. Let Ω be a right invariant gauge form on G and define a character Δ by $\Delta(g) = \Omega(gx)/\Omega(x)$. Let $P_1, \cdots, P_n$ be the basic relative invariants over k and $\chi_1, \cdots, \chi_n$ the corresponding characters. Denote by G^1 the identity component of $\bigcap_{i=1}^n \mathrm{Ker}(\chi_i)$. Let $\delta_1, \cdots, \delta_n$ be the rational numbers such that $\left(\chi_1(g)^{\delta_1} \cdots \chi_n(g)^{\delta_n}\right)^e = (\det \rho(g)/\Delta(g))^e$ for some non-zero integer e.

Theorem 4.1 ([S2]) *Assume that*

(4.1) $P_1, \cdots, P_n$ *are absolutely irreducible,*

(4.2) *for an* $x \in V - S$, $G_x^1 := \{g \in G^1 \mid \rho(g)x = x\}$ *is connected semisimple or trivial.*

Then the zeta functions associated with (G, ρ, V) *are absolutely convergent if* $\mathrm{Re}(s_i) > \delta_i$ $(i = 1, \cdots, n)$.

Remarks. (1) Note that the condition (4.2) is much weaker than the condition that $G_x = \{g \in G \mid \rho(g)x = x\}$ is connected. For example the prehomogeneous vector space $(GL_n, 2\Lambda_1, Sym(n))$ satisfies the condition (4.2), while $G_x = O(n)$ is not connected. Therefore the title of this section might be mileading.

(2) The abscissa $(\delta_1, \ldots, \delta_n)$ of absolute convergence is best possible in the known cases. We can prove the convergence of zeta functions without the assumption (4.1); however the result on the abscissa of absolute convergence is less precise.

(3) If a prehomogeneous vector space (G, ρ, V) satisfies the assumptions in the theorem above, then so does its castling transform.

The theorem was proved in [S2] in the case $k = \mathbb{Q}$. The generalization to the case of arbitrary algebraic number fields requires only obvious modifications of the proof.

Recently K.Ying ([Yi1]) rediscovered the theorem for irreducible regular prehomogeneous vector spaces and gave a proof different from the one in [S2]. He formulated the theorem for irreducible regular prehomogeneous vector spaces over an arbitrary algebraic number field. Moreover he classified the irreducible regular prehomogeneous vector spaces satisfying the assumptions in the theorem. (The first condition in Theorem 4.1 is obvious for the irreducible case.) In [Yi2], he generalized the theorem to irreducible regular prehomogeneous vector spaces satisfying the following weaker assumption

(4.3) for an $x \in V - S$, $G_x^1 := \{g \in G^1 \mid \rho(g)x = x\}$ is connected reductive.

In this case the sum defining zeta functions should be taken over the set

$$\left\{x \in (V-S)(k) \mid X_k(G_x^1) = \{1\}\right\},$$

not on the whole $(V - S)(k)$.

§5 Estimate of Theta series à la Weil

Following Weil's method of estimating theta series in [W], Yukie [Yu1] developed another method of proving the convergence of zeta functions.

For simplicity, let us assume that the prehomogeneous vector space under consideration is irreducible, defined over $\mathbb{Q}$ and of the form $(T_0 \times G_1, V)$, where G_1 is semisimple, T_0 is isomorphic to GL_1 and acts on V as scalar multiplication. Let T_1 be a maximal $\mathbb{Q}$-split torus of G_1 and P_1 a minimal parabolic subgroup of G_1 containing T_1.

We choose a basis $\{v_1, \ldots, v_N\}$ of V consisting of weight vectors of T_1. Namely we assume that there exists a character λ_i of T_1 satisfying $t_1 \cdot v_i = \lambda_i(t_1)v_i$ for every $t_1 \in T_1$. We denote by $X(T_1)$ the group of rational characters of T_1 and put $\mathfrak{t}_1^* = X(T_1) \otimes_{\mathbb{Z}} \mathbb{R}$. We may identify $\mathfrak{t}_1^*$ the dual space of the Lie algebra of T_1. We consider λ_i as an element of $\mathfrak{t}_1^*$.

For an $v = \sum_{i=1}^N x_i v_i \in V(\mathbb{Q})$, we put $I_v = \{i \mid 1 \leq i \leq N, x_i \neq 0\}$. Let C_v be the convex hull of $\{\lambda_i \mid i \in I_v\}$ in $\mathfrak{t}_1^*$. A $\mathbb{Q}$-rational point $v \in V(\mathbb{Q})$ is called *$\mathbb{Q}$-stable* if $C_{g_1 \cdot v}$ contains a neighbourhood of the origin for any $g_1 \in G_1(\mathbb{Q})$. Denote by $V(\mathbb{Q})^{st}$ the set of $\mathbb{Q}$-stable points.

Theorem 5.1 ([Yu1], Proposition 3.1.4) *The adelic zeta function*

$$\int_{T_0(\mathbb{A})/T_0(\mathbb{Q})} |t|^s \left|d^\times t\right| \int_{G_1(\mathbb{A})/G_1(\mathbb{Q})} \sum_{v \in V(\mathbb{Q})^{st}} \Phi(tg_1 \cdot v)\ |dg_1|$$

is absolutely convergent if the real part of s is sufficiently large.

Corollary 5.2 ([Yu1], p.67) *If* $\dim G = \dim V$, *then* $V(\mathbb{Q})^{st} = (V-S)(\mathbb{Q})$ *and the adelic zeta function*

$$\int_{T_0(\mathbb{A})/T_0(\mathbb{Q})} |t|^s \left|d^\times t\right| \int_{G_1(\mathbb{A})/G_1(\mathbb{Q})} \sum_{v \in (V-S)(\mathbb{Q})} \Phi(tg_1 \cdot v)\ |dg_1|$$

is absolutely convergent if the real part of s is sufficiently large.

The adelic integral in the theorem is transformed into the sum of several zeta integrals of the form discussed in [S3] (by using the reduction theory) and the convergence of the adelic zeta integral is equivalent to the convergence of zeta functions discussed in [S3].

§6 Irreducible regular reduced prehomogeneous vector spaces

In this section, we explain how the convergence criterions in §2–4 are applied to the irreducible regular reduced prehomogeneous vector spaces classified by [SK].

6.1 Weil-Igusa criterion

First we note that, if the split k-form of a representation (G^1, ρ, V) is admissible, then any k-form of (G^1, ρ, V) is admissible. However the converse is not true in general. For example, as is proved in [SS], the prehomogeneous vector space of hermitian matrices $(GL_n(K), Herm_n(K))$, K = a quadratic number field, is admissible over $\mathbb{Q}$; nevertheless its split $\mathbb{Q}$-form $(GL_n \times GL_n, M(n))$ is not admissible. Igusa [I] classified the admissible representations of split type. It seems that the classification of admissible representations of non-split type is still open. We also note that the castling transform does not preserve the admissibility.

The following is the list of irreducible regular reduced prehomogeneous vector spaces for which the split $\mathbb{Q}$-form of (G^1, ρ, V) is admissible. Whole theory of [SS] can be applied to any $\mathbb{Q}$-form of these prehomogeneous vector spaces.

Admissible PV

(3) $(GL_{2m}, \Lambda_2, Alt_{2m})$

(14) $(GL_1 \times Sp_3, \Lambda_1 \otimes \Lambda_3, V(1) \otimes V(14))$

(15) $(SO_m \times GL_1, \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(1))$ $(m \geq 5)$

(16) $(GL_1 \times Spin_7, \Lambda_1 \otimes \mathrm{spin}, V(1) \otimes V(8))$

(19) $(GL_1 \times Spin_9, \Lambda_1 \otimes \mathrm{spin}, V(1) \otimes V(16))$

(22) $(GL_1 \times Spin_{11}, \Lambda_1 \otimes \mathrm{spin}, V(1) \otimes V(32))$

(23) $(GL_1 \times Spin_{12}, \Lambda_1 \otimes \mathrm{spin}, V(1) \otimes V(32))$

(25) $(GL_1 \times G_2, \Lambda_1 \otimes \Lambda_2, V(1) \otimes V(7))$

(27) $(GL_1 \times E_6, \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(27))$

(29) $(GL_1 \times E_7, \Lambda_1 \otimes \Lambda_1, V(1) \otimes V(56))$

6.2 Sato-Ying criterion

In Theorem 4.1, the first condition is always satisfied by irreducible prehomogeneous vector spaces. The second condition can be checked over the algebraic closure. All the admissible prehomogeneous vector spaces listed above satisfy the assumptions in the theorem.

Now we give the list of non-admissible irreducible regular prehomogeneous vector spaces satisfying the assumptions of Theorem 4.1, which is due to K.Ying [Yi1].

Non-admissible PV with isotropy subgroup satisfying (4.2)

(1) $(G \times GL_m, \rho \otimes \Lambda_1, V(m) \otimes V(m))$

(2) $(GL_n, 2\Lambda_1, Sym_n)$ $(n \geq 3)$

(5) $(GL_6, \Lambda_3, V(20))$

(6) $(GL_7, \Lambda_3, V(35))$

(7) $(GL_8, \Lambda_3, V(56))$

(10) $(SL_5 \times GL_3, \Lambda_1 \otimes \Lambda_1, V(10) \otimes V(3))$

(13) $(Sp_n \times GL_{2m}, \Lambda_1 \otimes \Lambda_1, V(2n) \otimes V(2m))$

(15) $(SO_m \times GL_n, \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(n))$ ($m > n$, and $n, m - n \neq 2$)

(18) $(GL_3 \times Spin_7, \Lambda_1 \otimes \text{spin}, V(3) \otimes V(8))$

(20) $(GL_2 \times Spin_{10}, \Lambda_1 \otimes \text{half-spin}, V(2) \otimes V(16))$

(21) $(GL_3 \times Spin_{10}, \Lambda_1 \otimes \text{half-spin}, V(3) \otimes V(16))$

(24) $(GL_1 \times Spin_{14}, \Lambda_1 \otimes \text{spin}, V(1) \otimes V(64))$

The generalization of the theorem by Ying [Yi2] applies to the prehomogeneous vector spaces in the following list.

Non-admissible PV with isotropy subgroup satisfying (4.3)

(2) $(GL_2, 2\Lambda_1, Sym_2)$

(15) $(SO_m \times GL_n, \Lambda_1 \otimes \Lambda_1, V(m) \otimes V(n))$ $(m > n, n = 2 or m - n = 2)$

(17) $(GL_2 \times Spin_7, \Lambda_1 \otimes \text{spin}, V(2) \otimes V(8))$

(26) $(GL_2 \times G_2, \Lambda_1 \otimes \Lambda_2, V(2) \otimes V(7))$

In [Yi1], Ying observed the following interesting fact:

Fact *for an irreducible regular prehomogeneous vector space (G, ρ, V), the group G_x^1 ($x \in V - S$) is connected if and only if the largest root of the b-function is equal to -1.*

Recall that the b-function controls the location of poles of zeta functions, and the latter condition is equivalent to that the first (possible) pole of the zeta functions predicted by the b-function is $s = \dim V / \deg P$. This is consistent with the result on the abscissa of the convergence obtained in Theorem 4.1.

6.3 Yukie's criterion

By Corollary 5.2, we can check the convergence of the zeta functions associated to the following prehomogeneous vector spaces.

PV with finite isotropy

(4) $(GL_2, 3\Lambda_1, V(4))$

(8) $(SL_3 \times GL_2, 2\Lambda_1 \otimes \Lambda_1, V(6) \otimes V(2))$

(11) $(SL_5 \times GL_4, \Lambda_2 \otimes \Lambda_1, V(10) \otimes V(4))$

In [Yu2], Yukie proved that $V_k^{st} = (V - S)_k$ for the prehomogeneous vector space

(12) $(SL_3 \times SL_3 \times GL_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$

Hence Theorem 5.1 implies the convergence of the zeta functions associated with this space. By an argument similar to that leading to Theorem 5.1, he further proved the convergence of the zeta functions associated with the prehomogeneous vector space

(9) $(SL_6 \times GL_2, \Lambda_2 \otimes \Lambda_1)$.

Yukie dealt only with the split forms of (9) and (12); however a careful analysis of his method might lead to the convergence of zeta functions for arbitrary k-forms.

6.4 Conclusion

The results above can be summarized in the following table. The case (28) is the only space for which the convergence of the zeta functions has not yet been settled.

	PV		[W], [I]	[S2], [Yi1]	[Yi2]	[Yu1]	
(1)	$(G \times GL_m, \rho \otimes \Lambda_1)$, $G =$ semisimple			○			
(2)	$(GL_n, 2\Lambda_1)$ $(n \geq 3)$	$n \geq 3$		○			
		$n = 2$			○		
(3)	(GL_{2m}, Λ_2)		○	○			
(4)	$(GL_2, 3\Lambda_1)$					○	
(5)	(GL_6, Λ_3)			○			
(6)	(GL_7, Λ_3)			○			
(7)	(GL_8, Λ_3)			○			
(8)	$(SL_3 \times GL_2, 2\Lambda_1 \otimes \Lambda_1)$					○	
(9)	$(SL_6 \times GL_2, \Lambda_2 \otimes \Lambda_1)$						[Yu2]
(10)	$(SL_5 \times GL_3, \Lambda_2 \otimes \Lambda_1)$			○			
(11)	$(SL_5 \times GL_4, \Lambda_2 \otimes \Lambda_1)$					○	
(12)	$(SL_3 \times SL_3 \times GL_2, \Lambda_1 \otimes \Lambda_1 \otimes \Lambda_1)$						[Yu2]
(13)	$(Sp_n \times GL_{2m}, \Lambda_1 \otimes \Lambda_1)$			○			
(14)	$(GL_1 \times Sp_3, \Lambda_1 \otimes \Lambda_3)$		○	○			
(15)	$(SO_m \times GL_n, \Lambda_1 \otimes \Lambda_1)$	$m \geq 5, n = 1$	○	○			
		$n, m - n \neq 2$		○			
		n or $m - n = 2$			○		
(16)	$(GL_1 \times Spin_7, \Lambda_1 \otimes \text{spin})$		○	○			
(17)	$(GL_2 \times Spin_7, \Lambda_1 \otimes \text{spin})$				○		
(18)	$(GL_3 \times Spin_7, \Lambda_1 \otimes \text{spin})$			○			
(19)	$(GL_1 \times Spin_9, \Lambda_1 \otimes \text{spin})$		○	○			
(20)	$(GL_2 \times Spin_{10}, \Lambda_1 \otimes \text{half-spin})$			○			
(21)	$(GL_3 \times Spin_{10}, \Lambda_1 \otimes \text{half-spin})$			○			
(22)	$(GL_1 \times Spin_{11}, \Lambda_1 \otimes \text{spin})$		○	○			
(23)	$(GL_1 \times Spin_{12}, \Lambda_1 \otimes \text{half-spin})$		○	○			
(24)	$(GL_1 \times Spin_{14}, \Lambda_1 \otimes \text{half-spin})$			○			
(25)	$(GL_1 \times G_2, \Lambda_1 \otimes \text{7-dim})$		○	○			
(26)	$(GL_2 \times G_2, \Lambda_1 \otimes \text{7-dim})$				○		
(27)	$(GL_1 \times E_6, \Lambda_1 \otimes \text{27-dim})$		○	○			
(28)	$(GL_2 \times E_6, \Lambda_1 \otimes \text{27-dim})$						
(29)	$(GL_1 \times E_7, \Lambda_1 \otimes \text{56-dim})$		○	○			

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