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On dimensions of automorhic forms and zeta functions of prehomogenious vector space

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In this short note, we illustrate the rough idea to conncet the special value of zeta functions of prehomogeneous vector space to the parabolic contribution to the dimension of automorphic forms of bounded symmetric domain. The details will appear elsewhere.

1 Formally real Jordan algebras

The zeta functions of cones of formally real Jordan algebras has been studied by Satake and others in connection with geometric invariants of modular varieties. Their method is geometric. On the other hand, since the center of the unipotent radical of the maximal parabolic subgroups of the automorphism group of bounded symmetric domain has a structure of formally real Jordan algebra, it is natural that the zeta function associated with cones in formally real Jordan algebras appears in the dimension formula of automorphic forms given by trace formula. These Jordan algebras are a part of prehomogeneous vector spaces.

Formally real Jordan algebra is a (finite dimensional) non associative algebra J over the real number field R such that
(1) $xy=yx$,
(2) $x^2(xy)=x(x^2y)$ for all x, $y \in J$,
(3) If $x^2+y^2=0$, then $x=y=0$.
On the other hand, let V be a finite dimensional vector space over R. Let C be a non empty subset of V. We say that C is an open convex cone, if $ax+by\in C$ for any x, $y\in V$ and any positive real a, b. We say that C is non-degenerate, if C does not contain any straight line. Put $G(C)=\{g\in GL(V); gC=C\}$. If $G(C)$ acts transitively on C, we say that C is homogeneous. For a fixed positive definite inner product (,) on V, we define the dual of C by

$$C^*=\{x\in V; (x,y)>0 \text{ for all } y\in \bar{C}-\{0\}\}.$$

If $C^*=C$, then we say that C is self-dual. It is known that formally real Jordan algebras and non-degenerate self-dual homogeneous cones correspond one to one. The classification has been also known and is described as follows. There are five types.

(I) $V = \mathcal{H}_n(R)$, $C = \mathcal{P}_n(R)$.
(II) $V = \mathcal{H}_n(C)$, $C = \mathcal{P}_n(C)$.
(III) $V = \mathcal{H}_n(H)$, $C = \mathcal{P}_n(H)$.
(IV) $V = R^n$, $C = \{x \in V; x_1 > 0, x_1^2 - \sum_2^n x_i^2 > 0\}$
(V) exceptional type (3×3 matrices of the Cayley algebra),
where we denote by $\mathcal{H}_n(K)$ the symmetric matrices, hermitian matrices, or quaternion hermitian matrices, if $K = R$, or C, or H, respectively. (Here H is the real quaternion algebra.) We denote also by $\mathcal{P}_n(K)$ the positive definite symmetric matrices, positive definite hermitian matrices, or quaternion hermitian matrices, respectively for $K = R$, C, or H. The details such as the definition of multiplication, see Satake . We will treat only the classical case (I) to (IV) below. For type (IV), we sometimes replace C by a conected component of $\{x; Q[x] > 0\}$ when Q is a symmetric matrix of signature $(1, n-1)$.
Put $G = GL_n(K)$ for (I), (II), (III), where $K = R$, C, or H, and $G = G_m \times O(Q)$, where Q is a quadratic form of signature $(1, n-1)$. Then G acts on V by $gx = gx^tg$ for $g \in G$, $x \in V$ for type (I), (II), (III) and $x \to agx$ for $(a, g) \in G$ for type (IV). Then, (G, V) is a prehomogeneous vector space. The singular set is given by $S = \{x \in V; P(x) = 0\}$, where we denote by $P(x)$ the usual $\det(x)$ for (I) and (II), the root of the reduced norm of x (which is a polynomial function) for (III), and $Q[x] = {}^txQx$ for (IV). These are called Haupt norm in Braun and Koecher.
To define zeta functions, we must fix a Q form. We denote by $\mathcal{O}$ the ring of integers, maximal order of imaginary quadratic field, or maximal order of a definite quaternion algebra over Q, for (I), (II), or (III), respectively. We put $L = M_n(\mathcal{O}) \cap \mathcal{H}_n(K)$ and $\Gamma = GL_n(\mathcal{O})$ for (I), (II), (III). We put $\Gamma = O(Q) \cap SL_n(Z)$ and $L = Z^n$ for (IV). We put $L^+ = L \cap C$. We fix a Haar measure of G. For each $x \in V - S$, denote by G_x the stabilizer of x. Then G/G_x is regarded as an open subset of V and the Lebegue measure on V induces a measure on G/G_x. Then, we can decompose the measure of G as a product of the measure on G/G_x and G_x. For each $x \in L^+$, we put $\mu(x) = vol(G_x/\Gamma_x)$, where the measure on G_x is fixed as above. Actually, for $x \in L^+$, G_x is compact and Γ_x is a finite group. The definition of zeta function is given as follows.

$$\zeta(s, L) = \sum_{x \in \Gamma \backslash L^+} \frac{\mu(x)}{|P(x)|^s}.$$

2 New functional equations of zeta integral in the theory of prehomogeneous vector space

Related to the dimension formula and the trace formula, we introduce a new functional equations of zeta integral. The functional equation of this type was first obtained by Shintani for the case (I) and we shall treat the other cases. We would like to emphasize that the functional equation of this type is not contained in any standard theory of prehomogeneous vector space. In the usual theory, the test function in the zeta integral is rapidly decreasing and with compact support, but in our new equation, the test function is neither rapidly decreasing, nor with compact support, and actually even non-differentialbe. So, the argument on aonvergence is complicated and essential.

Let (G, V) be a prehomogeneous vector space. For a lattice $L \in V$ and any reasonable functions $f(x)$ of $x \in V$, we define a zeta integral by

$$Z(f(x), L, s) = \int_{G/\Gamma} \chi(g)^s \sum_{x \in L'} f(\rho(g)x)dg,$$

where $L' = \{x \in L; x \notin S\}$ and $\chi(g) = P(\rho(g)x)/P(x)$. We shall show that there exists a functional equation between two integrals of this type for two different functions $f_n(x, \lambda)$ and $f_n^*(x, \lambda)$, one of which is close to the zeta function and the other is to the kernel function of the trace formula.

We write $d = n$ for (I),(II),(III) and $d = 2$ for (IV). For each complex number $\lambda \in C$,each formally Jordan algebra V of type (I), (II), (III), or (IV), and an element $y_0 \in C$, we define two functions $f_n(x, \lambda, y_0)$ and $f_n^*(x, \lambda, y_0)$ as follows.

$$\begin{aligned} f_n(x, \lambda, y_0) &= \begin{cases} P(x)^{\lambda - \dim V/d} e^{-2\pi(x, y_0)} & \text{... if } x \in C, \\ 0 & \text{...}otherwise. \end{cases} \\ f_n^*(x, \lambda, y_0) &= P^*(y_0 - ix)^{-\lambda}. \end{aligned}$$

Here, the polynomials $P(x)$ and $P^*(x)$ are given by $P(x) = P^*(x) = Hm(x)$ for V of type I, II, III, (Hm(x) is the Haupt norm), and $P(x) = Q[x]$, $P^*(x) = Q^{-1}[x]$ for V of type IV. For abuse of notation, we sometimes omit y_0 and write $f_n(x, \lambda) = f_n(x, \lambda, y_0)$ and so on, since the argument on convergence or functinal equation does not depend on y_0 so much.

The generalized Gamma function $\Gamma_C(s)$ for each non-degenerate self-dual homogeneous cone C is defined as follows. (cf Satake)

$$\begin{aligned} \Gamma_\Omega(s) &= 2^{-1}\pi^{-2s+n/2-1}\Gamma(s)\Gamma(s - n/2 + 1)(\det Q)^{1/2} \text{ for type IV} \\ &= v(L)^{-1}\pi^{r_0 n(n-1)/4}(2\pi)^{-sn}\prod_{l=0}^{n-1}\Gamma(s - r_0 l/2) \text{ for type I, II, III,} \end{aligned}$$

where we write $r_0 = 1$, 2, or 4, when $K = R$, C, or H, respectively.

Theorem 1 *We take Jordan algebras of type (II), (III), or (IV). We assume $n \geq 4$ for type (IV). Notation and assumption being as above, we get*
(1) Two functions $Z(f_n(x, \lambda, y_0), L, s)$ and $Z(f_n^*(x, \lambda, y_0), L^*, s)$ of s and λ are continued meromorphically to the whole C^2 plane and satisfy the following functional equation.

$$Z(f_n^*(x, \lambda, y_0), L^*, (\dim V)/d - s) = \Gamma_C(\lambda)^{-1} Z(f_n(x, \lambda, y_0), L, s).$$

(2) The original integral which defines $Z(f_n^*(x,\lambda,y_0),L^*,s)$ is absolutely convergent, if $Re(s) > (\dim V)/d-1$ and $Re(\lambda) > 2(\dim V)/d-1$.
(3)For any s and λ, we have

$$Z(f_n(x,\lambda,y_0),L,s)=\Gamma_C(\lambda+s-(\dim V)/d).$$

Remark 1. As for the Jordan algebra of type (I), the similar theorem as above has been obtained by Shintani.
Remark 2. The above result (2) is important and essential for our purpose, but the proof is very subtle.
Remark 3. We omit the proof here, but the rough idea of the proof is as follows. First we can give rough estimate of the convergence of the zeta integral. Then as in the usual theory of prehomogeneous vector space, we divide the integral into det ≥ 1 part $Z_+$and det ≤ 1 part Z_-. Then we apply the Poisson summation formula to Z_-. We do this both for f and f^*, and get an analytic continuation to some wider region and then compare them. Then we get the functional equation. Here we use the following fact. $I(a):=\int_0^1 x^{a-1}dx=1/a$ for $a>0$, and $I'(a)=\int_1^\infty x^{a-1}dx=-1/a$ for $a<0$. The both integral converges only those a described above. But after analytic continuation, the equality $I(a)=I'(-a)$ has a meaning.

3 Relation to the dimension formula

Let Γ be a discrete subgroup of automorphism group of a bounded symmetric domain D. Then the dimension of automorphic forms of a fixed (sufficiently large) weight is given by an integral of some kernel function which is given as a sum over elements of Γ. To calculate this integral, we usually decompose the kernel function into partial sum on elements of Γ. In this sense, we can define the contribution of elements of Γ. Now, we treat the contribution of "central" unipotent elements. That is, let G be an algebraic group defined over Q. We call unipotent elements $u\in G$ as central, when u is contained in the center of the unipotent radical of some maximal Q-parabolic subgroup of G. We consider the contribution of the central unipotent elements of Γ. This is fairly important part of the dimension formula. If Γ is torsion free, one may conjecture that non vanishing term of the integral come only from ± 1 and central unipotent elements, and all the other part is zero. (This seems a forklore conjecture.) In this section, we shall explain that this contribution is related to the special value of the zeta function described in section 1. We sketch the case (IV) and omit the details for the other three cases.

Let Q be a half integral symmetric matrix with signature $(1,n-1)$. Put

$$G=\{g\in M_{n+2}(Q);\ {}^tg\begin{pmatrix}0&0&1\\0&Q&0\\1&0&0\end{pmatrix}g=\begin{pmatrix}0&0&1\\0&Q&0\\1&0&0\end{pmatrix}\}.$$

and

$$D = \{z \in C^n; z = x + iy, y \in C\},$$

where C = a connected component of $\{x \in R^n; Q[x] > 0\}$. The domain D is the bounded symmetric domain of type (IV). The action of the group G on D described as follows. Put

$$\tilde{z} = \begin{pmatrix} -\frac{1}{2}Q[z] \\ z \\ 1 \end{pmatrix}.$$

Then, the action $z \to g < z >$ and the factor of automorphy $j(g, z)$ are defined by

$$g \cdot \tilde{z} = \begin{pmatrix} * \\ g < z > \\ 1 \end{pmatrix} j(g, z).$$

We put $\Gamma = SL_{n+2}(Z) \cap G(R)^0$. The space $\mathcal{H}^\infty(k, \Gamma)$ of holomorphic cusp forms of weight k is the set of all holomorphic functions f on D such that (1) $f(\gamma z) = f(z)j(\gamma, z)^{-k}$, and (2)$\sup_{\Gamma \backslash D} |f(z)Q[z]^{k/2}| < \infty\}$.

Theorem 2 *Assume that $k > 2n - 2$. Put*

$$K(z, w) = 2^{2k-2n}\pi^{-1}(2k - n)\prod_{i=1}^{n-1}(2k - 2i)|\det(Q)|(-Q[z - \bar{w}])^{-k}$$

and

$$k_\Gamma(z) = \sum_{\gamma \in \Gamma} K(\gamma z, z)j(\gamma, z)^{-k}.$$

Then we have the following dimension formula.

$$\dim \mathcal{H}^\infty(k, \Gamma) = \int_{\Gamma \backslash D} k_\Gamma(z)Q[y]^{k-n}\,dx\,dy.$$

Now, there are two conjugacy class of maximal parabolic subgroup of G. We denote each representative (the standard parabolic) by P_0 or P_1. Denote the center of the unipotent radical of P_0 and P_1 by U_0 and U_1. Centers of the unipotent radical of the standard maximal parabolic subgroups are linearly ordered by inclusion, so we may assume $U_1 \subset U_0$. Then P_0 or P_1 corresponds to 0 dimensional or 1 dimensional cusp. We say that a central unipotent element u corresponds to 0 dimensional cusp, or to 1 dimensional, if any $G(Q)$-conjugate of u is not contained in U_1, or some contained in U_1. Our claim in this section is as follows.

The contribution to the dimension of central unipotent elements which correspond to 1 dimensional cusp is given by $\zeta(2-n)\times$ elementary factor, and those which correspond to 0 dimensional cusp is given by $\zeta(Q^{-1}, 0)\times$ elementary factor, where we denote by $\zeta(Q^{-1}, s)$ the zeta function corresponging to $P^*(x) = Q^{-1}[x]$.

We explain the outline of our argument for 0 dimensional case. The corresponding parablic P_0 and the center U_0 is given by

$$P_0 = \{ \begin{pmatrix} 1 & -{}^t x Q & -Q[x]/2 \\ 0 & 1_n & x \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a^{-1} \end{pmatrix} ; {}^t AQA = Q \},$$

and

$$U_0 = \{ \begin{pmatrix} 1 & -{}^t x Q & -Q[x]/2 \\ 0 & 1_n & x \\ 0 & 0 & 1 \end{pmatrix} \}.$$

We take $R \in GL_n(Q)$ such that

$${}^t RQR = \begin{pmatrix} 1 & 0 \\ 0 & -1_{n-1} \end{pmatrix}.$$

Put

$$z_0 = R \begin{pmatrix} i \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then the dimension formula is rewritten as follows.

$$\int_{\Gamma \backslash G(R)} \sum_{\gamma \in \Gamma} K(g^{-1}\gamma g z_0, z_0) j(g^{-1}\gamma g, z_0)^{-k} dg.$$

Now, denote by Π the set of central unipotent elements of Γ which correspond to 0 dimensional cusp. Then for any reasonable function f, we get

$$\sum_{\gamma \in \Pi} f(\gamma) = \sum_{\gamma \in \Gamma / \Gamma \cap P_0} \sum_{x \in Z^n - S} f(\gamma x \gamma^{-1}),$$

where $S = \{x \in R^n; Q[x] = 0\}$. Denote by K the maximal compact subgroup of $G(R)$ and by dp the left invariant measure of P_0 given by $dp = a^{-(n+1)} da dA dx$, where a, A and x are as in the definition of P_0. Then we get

$$\begin{aligned} & \int_{\Gamma \backslash G(R)} \sum_{\gamma \in \Pi} K(g^{-1}\gamma g z_0, z_0) j(g^{-1}\gamma g, z_0)^{-k} dg. \\ = & \int_{\Gamma \cap P_0 \backslash G(R)} \sum_{\gamma \in Z^n - S} K(g^{-1}\gamma g z_0, z_0) j(g^{-1}\gamma g, z_0)^{-k} dg. \\ = & \int_{\Gamma \cap P_0 \backslash P_0} \sum_{\gamma \in \Pi} K(p^{-1}\gamma p z_0, z_0) j(p^{-1}\gamma p, z_0)^{-k} dp \times \operatorname{vol}(P \cap K)^{-1}. \end{aligned}$$

hence the above integral reduces to the following integral

$$\int_{\Gamma \cap P_0 \backslash P_0} \sum_{x \in Z^n - S} (-Q[iy_0 - a^{-1} A^{-1} x])^{-k} a^{-n-1} da dA dx,$$

and this is equal to

$$\begin{aligned}
&\int_{P_0/\Gamma\cap P_0} \sum_{x\in Z^n-S} (-Q[iy_0 - aAx])^{-k} a^n (a^{-1}dadAdx), \\
= \quad & \mathrm{vol}(U_0(R)/(U_0(R)\cap\Gamma) \int_{M/M\cap\Gamma} \sum_{x\in Z^n-S} f_n^*(x,k)a^n dm \\
= \quad & \text{volume} \ \times Z(f_n^*(x,k), n/2),
\end{aligned}$$

where M is the Levi part of P_0 (as in the definition) and dm is the Haar measure of M. Hence the final integral is given by easy constant times $\zeta(s, Q^{-1})$. So, we get the desired result.

Each tube domain of type (I), (II), or (III) corresponds to the Jordan algebra of type (II), (III), or (I) respectively. We can show that the contribution of central unipotent elements is again given by $\zeta(r_0(r-n), L^*)\times$ elementary factor, where r describes "rank" (i.e. matrix size of the corresponding center) of corresponding cusps and $r_0 = 1, 2, or 4$ for Jordan algebra type (I), (II), or (III), respectively. We omit the details here.

Remark. All the special values which appear above can be calculated explicitly. This is one of the main part of our new theory. Hence the above results mean that we can get large part (conjecturally "all" if torsion free) of the dimension formula explicitly.

Reference

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