数理解析研究所講究録
925巻 1995年 120-128

A Theory of Ordinary p-adic Curves

Shinichi Mochizuki (望月 新一)
RIMS, Kyoto University (京都大学数理解析研究所)

General Introduction

In this talk, we gave a brief exposition of the theory developed in [Mzk]. This theory is a theory of r-pointed stable curves of genus g over p-adic schemes (for p odd), which, on the one hand, generalizes the Serre-Tate theory of ordinary elliptic curves to the hyperbolic case (i.e., $2g-2+r \geq 1$), and, on the other hand, generalizes the complex uniformization theory of hyperbolic Riemann surfaces (reviewed briefly below) due to Ahlfors, Bers, et al. to the p-adic case. In order to discuss this theory, it is thus first necessary to reformulate the complex theory in such a way that the translation into the p-adic case becomes more natural. The most fundamental tool for doing this, which is, in fact, of an algebraic, not an arithmetic nature, is the systematic use of the notion of an *indigenous bundle*, due to [Gunning]. The use of indigenous bundles enables one to get rid of the upper half plane, and thus to bring uniformization theory into a somewhat more algebraic setting. In any sort of nontrivial *arithmetic* theory of this nature, however, algebraic manipulations alone can never be enough. Thus, the fundamental *arithmetic* observation is the following:

> *Kähler metrics in the complex case correspond to Frobenius actions in the p-adic case.*

Since one typically gets a natural Frobenius action for free modulo p, a Frobenius action typically means a canonical lifting (over the p-adic integers) of the natural Frobenius action modulo p. In fact, in some sense, the complex analytic theory of uniformizations both of individual hyperbolic curves and of the moduli space of such curves can essentially be distilled down to two objects, both of which happen to be Kähler metrics:

(1) the hyperbolic metric on a hyperbolic Riemann surface (which encodes the upper half plane uniformization); and

(2) the Weil-Petersson metric on the moduli space (which encodes the Bers uniformization).

Moreover, these two metrics are related to each other in the sense that the latter is essentially the push-forward of the former. In a similar way, the p-adic theory revolves around two fundamental Frobenius liftings:

(1) the canonical Frobenius lifting on a canonical hyperbolic curve; and

(2) the canonical Frobenius lifting on a certain stack which is étale over the moduli stack.

Gunning's Theory of Indigenous Bundles

Let X be a compact hyperbolic Riemann surface. Let $H \to X$ be its uniformization by the upper half plane. Then by considering the covering transformations of $H \to X$, we get a homomorphism (well-defined up to conjugation)

$$\rho : \pi_1(X) \to Aut(H) = PSL_2(\mathbf{R})$$

which we call the *canonical representation of* X. If we regard ρ as defining a morphism into $PSL_2(\mathbf{C})$, then we obtain (in the usual fashion), a local system of $\mathbf{P}^1$-bundles on X, which thus gives us a holomorphic $\mathbf{P}^1$-bundle with connection (P, ∇_P) on X. By Serre's GAGA, (P, ∇_P) is necessarily *algebraic.* It turns out that P is always isomorphic to a certain $\mathbf{P}^1$-bundle of jets (which is also entirely algebraic). Thus, the upper half plane uniformization may be thought of as just being a special choice of connection ∇_P. A pair "like" (P, ∇_P) (satisfying certain technical properties discussed in Chapter I, Section 2 of [Mzk]) is called an *indigenous bundle.* By working with the log structures of [Kato], one can also define indigenous bundles in a natural way for smooth X with punctures, as well as for nodal X.

As emphasized earlier, the point of dealing with indigenous bundles is that they allow one to translate the upper half plane uniformization into the purely algebraic information of a connection on P. Of course, how one chooses this particular special connection on P is very nontrivial *arithmetic* issue. We shall call the pair consisting (P, ∇_P) consisting of this particular connection the *canonical indigenous bundle* on X.

Complex Uniformization Theory in Terms of Kähler Metrics

In this section we discuss how a Kähler metric on a complex manifold can be used to define canonical affine, holomorphic coordinates on the manifold locally in a neighborhood of a given point, and describe the two examples of this phenomenon that are important here, i.e., the ones that encode the uniformization theory of hyperbolic curves and their moduli. Everything that is discussed here is well-known, but our point of view is somewhat different from that usually taken in the literature.

Let M be a smooth complex manifold of complex dimension m. The complex analytic structure on M defines, in particular, a real analytic structure on M. Let μ be a real analytic $(1,1)$-form on M that defines a *Kähler metric* on M. Thus, in particular, μ is a closed differential form. Let M^c be the conjugate complex manifold to M: that is to say, we take M^c to be that complex manifold which has the same underlying real analytic manifold structure as M, but whose holomorphic functions are the anti-holomorphic functions of M. Let us fix a point $e \in M$. Let N be the germ of a complex manifold obtained by localizing the complex manifold $M^c \times M$ at $(e,e) \in M^c \times M$ (where this last expression makes sense since M^c has the same underlying set as M). Let Ω^{hol} (respectively, Ω^{ant}) be the holomorphic vector bundle on N obtained by pulling back the bundle Ω_M (respectively, Ω_{M^c}) of holomorphic differentials on M (respectively, M^c) to $M^c \times M$ via the projection $M^c \times M \to M$ (respectively, $M^c \times M \to M^c$), and then restricting to N. Thus, in summary, we have a $2m$-dimensional germ of a complex manifold N, together with two m-dimensional holomorphic vector bundles (locally free sheaves) Ω^{hol} and Ω^{ant} on N.

Note that locally at $e \in M$, the fact that μ is real analytic means that we can write μ as a convergent power series in holomorphic and anti-holomorphic local coordinates at e. In other words, if we restrict μ to N, we may regard $\mu|_N$ as defining a *holomorphic* section of $\Omega^{ant} \otimes_{\mathcal{O}_N} \Omega^{hol}$ (where $\mathcal{O}_N$ is the sheaf of holomorphic functions on N). Let d^{hol} (respectively, d^{ant}) be the exterior derivative on N with respect to the variables coming from M (respectively, M^c). Note that since Ω^{hol} is constructed via pull-back from M, we can apply d^{hol} to sections of Ω^{ant}. We thus obtain a sort of de Rham complex with respect to d^{hol}:

$$0 \longrightarrow \Omega^{ant} \xrightarrow{d^{hol}} \Omega^{ant} \otimes_{\mathcal{O}_N} \Omega^{hol} \xrightarrow{d^{hol}} \Omega^{ant} \otimes_{\mathcal{O}_N} (\wedge^2 \Omega^{hol}) \xrightarrow{d^{hol}} \dots$$

Relative to this complex, the section $\mu|_N$ of $\Omega^{ant} \otimes \Omega^{hol}$ satisfies $d^{hol}\ \mu|_N = 0$ (since μ is a closed form). It thus follows from the Poincaré Lemma that there exists a (holomorphic) section α of Ω^{ant} that vanishes at $(e,e) \in N$ and satisfies $d^{hol}\ \alpha = \mu|_N$. Let M_e be the germ of a complex manifold obtained by localizing M at $e \in M$. Let

$$\iota : M_e \hookrightarrow N$$

be the inclusion induced by the map $M \to M^c \times M$ that takes $f \in M$ to $(e,f) \in M^c \times M$. Then $\iota^*(\alpha)$ defines a holomorphic morphism $\beta : M_e \to \Omega_{M^c,e} = \Omega^c_{M,e}$, where $\Omega_{M,e}$ is the affine complex analytic space defined by the cotangent space of M at e. Note, moreover, that although α (as chosen above) is not unique, β is nonetheless independent of the choice of α. Moreover, β is an immersion: Indeed, to see this is suffices to check that the map induced by β on tangent spaces is an isomorphism, but this follows from the fact that $d^{hol}\ \alpha = \mu|_N$, and the fact that the Hermitian form defined by μ is nondegenerate.

In summary, we see that from the Kähler metric μ, we obtain a canonical holomorphic local affine uniformization

$$\beta : M_e \hookrightarrow \Omega^c_{M,e}$$

Pulling back the standard affine coordinates on $\Omega^c_{M,e}$ gives us a canonical collection of holomorphic coordinates on M_e.

Definition : *We shall refer to these coordinates as the canonical holomorphic local coordinates of the Kähler manifold* (M,μ) *at* e. *We shall refer to* β *as as the canonical local affine uniformization of the Kähler manifold* (M,μ) *at* e.

From our point of view, the two important examples of canonical holomorphic local coordinates defined by a Kähler metric are the following:

Example 1: Let $M = \{z \in \mathbf{C} |\ |z| < 1\}$, with the standard hyperbolic metric $\frac{dz \wedge d\overline{z}}{\sqrt{1+(z\cdot\overline{z})}}$. Then z is a canonical coordinate at 0. Indeed, to see this it suffices to note that $d^{hol}(z \cdot d\overline{z}) = dz \wedge d\overline{z}$, which is equal to the metric modulo the ideal generated by $\overline{z}$ in $\mathcal{O}_N$. Note that by the Köbe uniformization theorem of classical complex analysis, if X is an arbitrary hyperbolic Riemann surface *of finite type* (i.e., a compact Riemann surface minus a finite number of points), then the universal covering space $\widetilde{X}$ of X is holomorphically isomorphic to the open unit disk M. Since the standard hyperbolic metric on $M \cong \widetilde{X}$ is preserved by the deck transformations, it thus descends to X, hence defines a canonical metric μ_X on X. Thus, the canonical local coordinate at a point x of X associated to the Kähler metric μ_X is precisely the local coordinate at x obtained by descending the coordinate "z" via $M \cong \widetilde{X}$ to X. In other words, one can think of (at least the local coordinate obtained from) the isomorphism $M \cong \widetilde{X}$ as being encoded in the canonical metric μ_X.

Example 2: Let M be the moduli stack of compact Riemann surfaces of genus g, where $g \geq 2$ (so M is a smooth complex analytic stack). Let X be a Riemann surfaces of genus g. Let $Q_X = H^0(X, \omega_X^{\otimes 2})$ be the space of holomorphic quadratic differentials on X. Thus, if we denote by $[X] \in M$ the point of M defined by X, the vector space Q_X is canonically isomorphic to the holomorphic cotangent space to M at $[X]$. By using the canonical metric μ_X on X of the preceding example, we obtain the *Weil-Petersson inner product*:

$$\langle \phi, \psi \rangle \stackrel{\text{def}}{=} \int_X \frac{\phi \cdot \overline{\psi}}{\mu_X}$$

for $\phi, \psi \in Q_X$. (Here the bar over the ψ denotes complex conjugation.) This inner product on the vector space Q_X clearly varies real analytically with respect to $[X]$, hence defines a real analytic Hermitian metric μ_M on M. It is a result of Weil and Ahlfors that this metric μ_M, which is called the *Weil-Petersson metric* on M, is Kähler. Moreover, it is a result of Royden ([Royd]) that the coordinates arising from the Bers embedding (see, e.g., [Gard]) are canonical local coordinates with respect to μ_M. In other words, even though the Bers embedding is quite difficult to construct, one can already construct it locally quite easily by applying the above construction to the Weil-Petersson metric which is very easy to define.

Ordinary Frobenius Liftings

Let p be an odd prime number. Let k be an algebraically closed field of characteristic p. Let $A = W(k)$ be the ring of Witt vectors with coefficients in k. Let S be a smooth formal scheme of relative dimension d over A. Let $\Phi_A : A \to A$ denote the Frobenius morphism on A. Let $\Phi : S \to S$ be a Φ_A-linear morphism. We shall call Φ a *Frobenius lifting* if its reduction $\Phi_{\mathbf{F}_p} : S_{\mathbf{F}_p} \to S_{\mathbf{F}_p}$ modulo p is equal to the absolute Frobenius morphism (i.e., given by raising sections of the structure sheaf to the p^{th} power). Suppose that Φ is a Frobenius lifting. Then the pull-back morphism that it induces on differentials

$$d\Phi : \Phi^*\Omega_{S/A} \to \Omega_{S/A}$$

is equal to zero modulo p. Thus, we can consider the morphism

$$\frac{1}{p} d\Phi : \Phi^*\Omega_{S/A} \to \Omega_{S/A}$$

obtained by dividing $d\Phi$ by p. We shall call Φ an *ordinary Frobenius lifting* if $\frac{1}{p}d\Phi$ is an isomorphism.

Let $z \in S(k)$ be a k-valued point of S. Let R_z be the completion of the local ring of S at z. Thus, R_z is noncanonically isomorphic to the power series ring $A[[t_1, \ldots, t_d]]$, where the t_i are indeterminates. Then it is shown in [Mzk], Chapter III, Section 1, that there exists a free $\mathbf{Z}_p$-module Ω^{et} of rank d, along with the following:

(1) a canonical isomorphism $\Omega^{et} \otimes_{\mathbf{Z}_p} R_z \cong \Omega_{S/A} \otimes_{\mathcal{O}_S} R_z$;

(2) a canonical continuous homomorphism $\mathcal{Q} : \Omega^{et} \to R_z^{\times}$ such that for any $\omega \in \Omega^{et}$, $q_\omega \stackrel{\text{def}}{=} \mathcal{Q}(\omega)$ satisfies the property: $\Phi^{-1}(q_\omega) = q_\omega^p$, i.e., the units q_ω *diagonalize* the Frobenius lifting Φ.

Thus, one can think of the q_ω (or more properly, their logarithms) as *canonical local coordinates at z associated to Φ.*

It is the fact that both Kähler metrics and ordinary Frobenius liftings define canonical local coordinates that is the essence of the claimed analogy between these two types of objects.

Statement of the Main Results of [Mzk]

We are now ready to discuss what is done in [Mzk] in a bit more detail. The first step is to note that one can define indigenous bundles in the p-adic context (as a $\mathbf{P}^1$-bundle with connection satisfying certain properties), and that in this context they enjoy many of the same properties as their complex analytic forebears. We then study the p-curvature of indigenous bundles in characteristic p, and show that a generic r-pointed stable curve of genus g has a finite, nonzero number of distinguished indigenous bundles (P, ∇_P), which are characterized by the following two properties:

(1) the p-curvature of (P, ∇_P) is nilpotent;

(2) the space of indigenous bundles with nilpotent p-curvature is étale over the moduli stack of curves at (P, ∇_P).

We call such (P, ∇_P) *nilpotent and ordinary*, and we call curves *ordinary* if they admit at least one such nilpotent, ordinary indigenous bundle. If a curve is ordinary, then choosing any one of the finite number of nilpotent, ordinary indigenous bundles on the curve completely determines the "uniformization theory of the curve" – to be described in the following paragraphs. Because of this, we refer to this choice as the choice of a *p-adic quasiconformal equivalence class* to which the curve belongs.

After studying various basic properties of ordinary curves and ordinary indigenous bundles in characteristic p, we then consider the p-adic theory. Let $\overline{\mathcal{M}}_{g,r}$ be the moduli stack of r-pointed stable curves of genus g over $\mathbf{Z}_p$. Then we show that there exists a canonical p-adic (nonempty) formal stack $\overline{\mathcal{N}}^{ord}_{g,r}$ together with an étale morphism

$$\overline{\mathcal{N}}^{ord}_{g,r} \to \overline{\mathcal{M}}_{g,r}$$

such that modulo p, $\overline{\mathcal{N}}^{ord}_{g,r}$ is the moduli stack of ordinary r-pointed curves of genus g, together with a choice of p-adic quasiconformal equivalence class. Moreover, the generic degree of $\overline{\mathcal{N}}^{ord}_{g,r}$ over $\overline{\mathcal{M}}_{g,r}$ is > 1 (as long as $2g - 2 + r \geq 1$, and p is sufficiently large). It is over $\overline{\mathcal{N}}^{ord}_{g,r}$ that most of our theory will take place. Our first main result is the following:

Theorem 0.1: *Let $\mathcal{C}^{log} \to (\overline{\mathcal{N}}^{ord}_{g,r})^{log}$ (where the "log" refers to canonical log scheme structures) be the tautological ordinary r-pointed stable curve of genus g. Then there exists a canonical Frobenius lifting $\Phi^{log}_{\mathcal{N}}$ on $(\overline{\mathcal{N}}^{ord}_{g,r})^{log}$, together with a canonical indigenous bundle (P, ∇_P) on $\mathcal{C}^{log}$. Moreover, $\Phi^{log}_{\mathcal{N}}$ and (P, ∇_P) are uniquely characterized by the fact (P, ∇_P) is "Frobenius invariant" (in some suitable sense) with respect to $\Phi^{log}_{\mathcal{N}}$.*

Moreover, there is an open p-adic formal substack $\mathcal{C}^{ord} \subseteq \mathcal{C}$ of "ordinary points" of the curve. The open formal substack $\mathcal{C}^{ord} \subseteq \mathcal{C}$ is dense in every fiber of $\mathcal{C}$ over $\overline{\mathcal{N}}^{ord}_{g,r}$. Also, there is a unique canonical Frobenius lifting

$$\Phi^{log}_{\mathcal{C}} : (\mathcal{C}^{log})^{ord} \to (\mathcal{C}^{log})^{ord}$$

which is $\Phi_{\mathcal{N}}^{log}$-linear and compatible with the Hodge section of the canonical indigenous bundle (P, ∇_P). Finally, $\Phi_{\mathcal{C}}^{log}$ and $\Phi_{\mathcal{N}}^{log}$ have various functoriality properties, such as functoriality with respect to "log admissible coverings of $\mathcal{C}^{log}$" and with respect to restriction to the boundary of $\overline{\mathcal{M}}_{g,r}$.

This Theorem is an amalgamation of Theorem 2.8 of Chapter III and Theorem 2.6 of Chapter V of [Mzk]. In some sense all other results in this paper are formal consequences of the above Theorem. For instance,

Corollary 0.2: *The Frobenius lifting $\Phi_{\mathcal{N}}^{log}$ allows one to define canonical affine local coordinates on $\mathcal{M}_{g,r}$ at an ordinary point α valued in k, a perfect field of characteristic p. These coordinates are well-defined as soon as one chooses a quasiconformal equivalence class to which α belongs. Also, at a point $\alpha \in \overline{\mathcal{M}}_{g,r}(k)$ corresponding to a totally degenerate curve, $\Phi_{\mathcal{N}}^{log}$ defines canonical multiplicative local coordinates.*

This Corollary follows from Chapter III, Theorem 3.8 and Definition 3.13 of [Mzk].

Let $\alpha \in \overline{\mathcal{N}}_{g,r}^{ord}(A)$, where $A = W(k)$, the ring of Witt vectors with coefficients in a perfect field of characteristic p. If α corresponds to a morphism $Spec(A) \to \overline{\mathcal{N}}_{g,r}^{ord}$ which is *Frobenius equivariant* (with respect to the natural Frobenius on A and the Frobenius lifting $\Phi_{\mathcal{N}}^{log}$ on $\overline{\mathcal{N}}_{g,r}^{ord}$), then we call the curve corresponding to α *canonical.* Let K be the quotient field of A. Let $GL_2^{\pm}(\mathbf{Z}_p)$ be the quotient of $GL_2(\mathbf{Z}_p)$ by $\{\pm 1\}$.

Theorem 0.3: *Once one fixes a k-valued point α_0 of $\overline{\mathcal{N}}_{g,r}^{ord}$, there is a unique canonical $\alpha \in \overline{\mathcal{N}}_{g,r}^{ord}(A)$ that lifts α_0. Moreover, if a curve $X^{log} \to Spec(A)$ is canonical, it admits*

(1) *A canonical dual crystalline (in the sense of [Falt], §2) Galois representation $\rho : \pi_1(X_K) \to GL_2^{\pm}(\mathbf{Z}_p)$;*

(2) *A canonical log p-divisible group G^{log} (up to $\{\pm 1\}$) on X^{log} whose Tate module defines the representation ρ;*

(3) *A canonical Frobenius lifting $\Phi_X^{log} : (X^{log})^{ord} \to (X^{log})^{ord}$ over the ordinary locus (which satisfies certain properties).*

Moreover, if a lifting $X^{log} \to Spec(A)$ of α_0 has any one of these objects (1) through (3), then it is canonical.

This Theorem results from Chapter III, Theorem 3.2, Corollary 3.4; Chapter IV, Theorem 1.1, Theorem 1.6, Definition 2.2, Proposition 2.3, Theorem 4.17 of [Mzk].

The case of curves with ordinary reduction modulo p which are not canonical is more complicated. Let us consider the universal case. Thus, let $S^{log} = (\overline{\mathcal{N}}_{g,r}^{ord})^{log}$; let f^{log} :

$X^{log} \to S^{log}$ be the universal r-pointed stable curve of genus g. Let $T^{log} \to S^{log}$ be the finite covering (log étale in characteristic zero) which is the Frobenius lifting $\Phi_{\mathcal{N}}^{log}$ of Theorem 0.1. Let $P^{log} \to S^{log}$ be the inverse limit of the coverings of S^{log} which are iterates of the Frobenius lifting $\Phi_{\mathcal{N}}^{log}$. Let $X_T^{log} = X^{log} \times_{S^{log}} T^{log}$; $X_P^{log} = X^{log} \times_{S^{log}} P^{log}$. We would like to consider the arithmetic fundamental groups

$$\Pi_1 \stackrel{\text{def}}{=} \pi_1((X_T^{log})_{\mathbf{Q}_p}); \quad \Pi_\infty \stackrel{\text{def}}{=} \pi_1((X_P^{log})_{\mathbf{Q}_p})$$

Unlike the case of canonical curves, we do not get a canonical Galois representation of Π_1 into $GL_2^{\pm}(\mathbf{Z}_p)$. Instead, we have the following

Theorem 0.4: *There is a canonical Galois representation*

$$\rho_\infty : \Pi_\infty \to GL_2^{\pm}(\mathbf{Z}_p)$$

Now suppose that $p \geq 5$. Then the obstruction to extending ρ_∞ to Π_1 is nontrivial and is measured precisely by the extent to which the canonical affine coordinates (of Corollary 0.2) are nonzero. Also, there is a ring $\mathcal{D}_S^{Gal}$ with a continuous action of $\pi_1(T_{\mathbf{Q}_p}^{log})$ such that we have a canonical dual crystalline representation

$$\rho_1 : \Pi_1 \to GL_2^{\pm}(\mathcal{D}_S^{Gal})$$

(i.e., this is a twisted homomorphism, with respect to the action of Π_1 (acting through $\pi_1(T_{\mathbf{Q}_p}^{log})$) on $\mathcal{D}_S^{Gal}$). Finally, the ring $\mathcal{D}_S^{Gal}$ has an augmentation $\mathcal{D}_S^{Gal} \to \mathbf{Z}_p$ which is Π_∞-equivariant (for the trivial action on $\mathbf{Z}_p$) and which is such that after restricting to Π_∞, and base changing by means of this augmentation, ρ_1 reduces to ρ_∞.

This follows from Chapter V, Theorems 1.4 and 1.7 of [Mzk].

All along, we note that when one specializes the theory to the case of elliptic curves, one recovers the familiar classical theory of Serre-Tate. For instance, the definitions of "ordinary curves" and "canonical liftings" specialize to the objects with the same names in Serre-Tate theory. The p-adic canonical coordinates on the moduli stack $\mathcal{M}_{g,r}$ (Corollary 0.2) specialize to the Serre-Tate parameter. The Galois obstruction to extending ρ_∞ to a representation of Π_1 specializes to the obstruction to splitting the well-known exact sequence of Galois modules that the p-adic Tate module of an ordinary elliptic curve fits into.

Bibliography

[Falt] G. Faltings, *Crystalline Cohomology and p-adic Galois Representations*, *JAMI Conference*, Johns Hopkins Univ. Press, pp. 25-79 (1990).

[Gard] F. Gardiner, *Teichmüller Theory and Quadratic Differentials*, John Wiley and Sons (1987).

[Gunning] R. C. Gunning, *Special Coordinate Coverings of Riemann Surfaces*, *Math. Annalen* **170**, pp. 67-86 (1967).

[Kato] K. Kato, *Logarithmic Structures of Fontaine-Illusie*, *Proceedings of the First JAMI Conference*, Johns-Hopkins University Press (1990), pp. 191-224.

[Mzk] S. Mochizuki, *A Theory of Ordinary p-adic Curves*, manuscript.

[Royd] H. L. Royden, *Intrinsic Metrics on Teichmüller Space*, *Proc. International Congress of Mathematicians* **2**, Vancouver, pp. 217-221 (1974).