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COUNTING SINGULARITIES IN STABLE PERTURBATIONS OF MAP-GERMS

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Introduction. Let $f:(\mathbf{C}^n,0)\to(\mathbf{C},0)$ be a holomorphic function-germ which defines an isolated singularity at 0. An important invariant of f is its Milnor number, $\mu(f)$, which we can define to be $\mu(f)=\dim_{\mathbf{C}}\mathcal{O}_n/J(f)$ where $\mathcal{O}_n$ is the ring of holomorphic function-germs of $(\mathbf{C}^n,0)$ and $J(f)$ denote the jacobian ideal of f, generated by the germs of partial derivatives $\frac{\partial f}{\partial x_i}$ $(1\leq i\leq n)$. It is well-known that $\mu(f)$ is the number of critical points (Σ^n-points, in Thom-Boardman notation) of a Morse function near f.

Let $f:(\mathbf{C}^2,0)\to(\mathbf{C}^2,0)$ be a $\mathcal{K}$-finite holomorphic map-germ. We define the number $c(f)$ by $c(f)=\dim_{\mathbf{C}}\mathcal{O}_2/(J,\frac{\partial(p,J)}{\partial(x_1,x_2)},\frac{\partial(q,J)}{\partial(x_1,x_2)})$ where $f=(p,q)$, $J=\frac{\partial(p,q)}{\partial(x_1,x_2)}$. Then, $c(f)$ is the number of cusps ($\Sigma^{1,1}$-points) in a stable perturbation of f. This fact was proved in [Fukuda-Ishikawa] and [Gaffney-Mond].

Thus, we can expect that the number of Σ^I-points of a stable perturbation of a map-germ $f:(\mathbf{C}^n,0)\to(\mathbf{C}^p,0)$ could be expressed by some languages of algebras obtained by f, under some mild conditions. We consider this problem using the ideal Δ^I introduced by B.Morin in [Morin], which we review in §1. In §2, we review some facts in commutative algebra which we use later. In §3, we discuss Cohen-Macaulay property of the zero locus of Δ^I. Our main results are described in §4. This asserts that the number $s_I(f)$ of Σ^I-points in a generic approximation f_u of f is equal to the number $c_I(f)$, which is the length of Artinian ring determined by some procedure by f, under some conditions. We should remark that J.Nuño Ballesteros and M.Saia have first considered this problem in [BS] using the ideals Δ^I due to B.Morin and described several results. In the next section, we present an example which satisfies that $c_I(f)>s_I(f_u)$. In the last section, we give some open problems in this direction.

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§1. Thom-Boardman singularities. In this section, we review the Thom-Boardman singularity set Σ^I following the work of Morin [Morin]. Let $\mathbf{K}$ denote the field of real numbers $\mathbf{R}$ or that of complex numbers $\mathbf{C}$. Let U be an open subset of $\mathbf{K}^n$ and V that of $\mathbf{K}^p$, $J=J^r(U,V)$ the jet space of order r and $\pi_n:J\to U$, $\pi_p:J\to V$ the natural projections. Let $\mathcal{F}$ denote the foliation in J whose leaves are fibers of π_p, and $\mathcal{G}$ the foliation whose leaves are images of jet sections $j^rf:U\to J$ where f are polynomial maps of degree less than or equal to r. Let I be a Boardman symbol of length $k(=|I|)$, i.e. $I=(i_1,i_2,\dots,i_k)$ be a k-tuple of integers with $n\geq i_1\geq i_2\geq\dots\geq i_k\geq 0$. We set $tI=(i_1,\dots,i_{k-1})$ if $|I|=k\geq 2$ and $tI=\emptyset$ if $|I|=1$. Then we define inductively the *Thom-Boardman submanifold* Σ^I in J by

$$\Sigma^I=\{z\in\Sigma^{tI}:\dim(T_z\mathcal{F}\cap T_z\mathcal{G}\cap T_z\Sigma^{tI})=i_k,k=|I|\},$$

where $T_z\mathcal{F},T_z\mathcal{G}$ denote the tangent spaces of the leaves of $\mathcal{F},\mathcal{G}$ including $z\in J$. Here we understand that $\Sigma^{\emptyset}=J$. This is well-defined because Σ^I is nonsingular. See

[Boardman,(6.1)], [Morin, p.15, p.97]. By custom, we denote $\nu_I(n,p)$ the codimension of Σ^I in J. By [AGV .p46], [Boardman, (6.5)], [Morin, p.15],

$$\nu_I(n,p) = (p-n+i_1)\mu(I)-(i_1-i_2)\mu(i_2,...,i_k)-(i_2-i_3)\mu(i_3,...,i_k)-\cdots-(i_{k-1}-i_k)\mu(i_k)$$

where $\mu(i_1,...,i_k)$ denote the number of k-tuples $(j_1,...,j_k)$ of integers so that $i_r \geq j_r \geq 0$ $(0 \leq r \leq k)$, $j_1 \geq j_2 \geq ... \geq j_k \geq 0$, and $j_1 > 0$.

Let B be the ring of differentiable functions which are constant on each leaf of $\mathcal{F}$, and $\mathcal{D}$ the submodule of the vector fields of J, whose elements are tangent to all leaves of $\mathcal{G}$. For a Boardman symbol $I = (i_1,...,i_k)$, we denote by Δ^I the ideal generated by Δ^{tI} and the determinants $\det(d_i\varphi_j)_{1\leq i,j\leq n-i_k+1}$ where $d_i \in \mathcal{D}, \varphi_j \in B + \Delta^{tI}$. Here we understand that $\Delta^{\emptyset}$ is the zero ideal.

It is sometimes convenient to write down an explicit coordinate system of J. Let $x = (x_1,...,x_n)$ denote a coordinate system of $U \subset \mathbf{K}^n$, and $y = (y_1,...,y_p)$ that of $V \subset \mathbf{K}^p$. Then we can write down the canonical coordinate functions on J, namely the function X_i, Y_j, Z^j_σ for $1 \leq i \leq n$, $1 \leq j \leq p$, and $1 \leq |\sigma| \leq r$ where $\sigma = (\sigma_1,...,\sigma_n)$ is a n-tuple of non-negative integers and $|\sigma| = \sum_i \sigma_i$; these are defined by

$$X_i = x_i \circ \pi_n,\ Y_j = y_j \circ \pi_p,\ Z^j_\sigma(j^r f(p)) = \frac{\partial^{|\sigma|}(y_j \circ f)}{\partial x_\sigma}(p),$$

where f is the germ at p of any map from U to V. We define vector fields D_i $(1 \leq i \leq n)$ by

$$D_i = \frac{\partial}{\partial X_i} + \sum_{\sigma:|\sigma|<r} \left(\sum_{j=1}^{p} Z^j_{\sigma(i)} \frac{\partial}{\partial Z^j_\sigma} \right),$$

where $\sigma(i) = (\sigma_1,...,\sigma_{i-1},\sigma_i+1,\sigma_{i+1},...,\sigma_n)$. These D_i $(1 \leq i \leq n)$ generate $\mathcal{D}$. For a Boardman symbol $I = (i_1,...,i_k)$, Δ^I is the ideal generated by Δ^{tI} and subdeterminants of order $n - i_k + 1$ of the matrix ${}^t(D_iY_j, D_ig_s)$ where $g = (g_1,...,g_t)$ is a system of generators of Δ^{tI}. For the proof of this fact, see Lemma 2 in I.(b) and Lemma 3 in I.(c) in [Morin]. Let I' denote the smallest (in the lexicographic order) Boardman symbol J which is larger (in the lexicographic order) than I with $|J| \leq |I|$. Remark that I' is not defined if $I = (n,...,n)$.

THEOREM (1.1). *Let Z_I be the zero locus of Δ^I in the jet space J. Then Z_I is nonsingular along Σ^I, and, as underlying topological spaces,*

$$Z_I = \Sigma^I \cup Z'_I \text{ (disjoint union)} \quad \left(\text{where } Z'_I = \begin{cases} Z_{I'} & \text{if } I' \text{ is defined} \\ \emptyset & \text{if } I' \text{ is not defined} \end{cases} \right).$$

REFERENCE FOR PROOF: See [Morin], Théorème on p.15, Corollaire on p.97. ∎

Let $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ be a $\mathbf{K}$-analytic map-germ and $\mathcal{O} = \mathcal{O}_n$ the ring of $\mathbf{K}$-analytic function-germs of $(\mathbf{K}^n, 0)$. For a Boardman symbol $I = (i_1,...,i_k)$, we define a set-germs $\Sigma^I(f)$ in $(\mathbf{K}^n, 0)$ and an ideal $\Delta^I(f)$ by $\Sigma^I(f) = (j^k f)^{-1}(\Sigma^I)$, $\Delta^I(f) = (j^k f)^*(\Delta^I)$.

A map-germ $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ is said to be a *singularity of class* Σ^I if $0 \in \Sigma^I(f)$. Then, $0 \in \Sigma^i(f)$ iff f has kernel rank i at 0. A map-germ is said to be *generic* if its jet extension is transversal to each Boardman submanifolds. If f is generic, then $\Sigma^I(f)$'s are nonsingular and $\Sigma^{i_1,\dots,i_k}(f) = \Sigma^{i_k}(f|\Sigma^{i_1,\dots,i_{k-1}})$. Any map-germ may be approximated as accurately as one wishes by a generic map. See [AGV, p.45], [Boardman], [Mather4], and [Morin], for its proof. In this paper, we consider the following question: how many Σ^I-points appear in a generic approximation of a map-germ $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$. Thus we are interested in the Boardman symbol I with $\nu_I(n,p) = n$. The first few such I's are listed in the following table.

$n \setminus p$	1	2	3	4	5	6	7	8
1	(1)							
2	(2)	(1_2)	(1)					
3	(3)	(21)	(1_3)		(1)			
4	(4)	(31)	(21_2)	$(1_4)(2)$	(1_2)		(1)	
5	(5)	(41)	(31_2)	$(21_3)(2_2)$	(1_5)			
6	(6)	(51)	(41_2)	$(31_3)(32)$	$(21_4)(3)$	(1_6)	$(1_3)(2)$	(1_2)
7	(7)	(61)	(51_2)	$(41_3)(42)$	(31_4)	$(21_5)(2_21)$	$(1_7)(21)$	
8	(8)	(71)	(61_2)	$(51_3)(52)$	(41_4)	$(31_5)(321)(4)$	(21_6)	(1_8)

Here (1_k) denote $\overbrace{(1,\dots,1)}^{k \text{ times}}$, and so on.

If (n,p) belongs to the region of stability (that is, the nice range of dimensions due to J.Mather [Mather2]), any map-germ $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ can be approximated by a stable (precisely speaking, locally infinitesimal stable) map in the sense of J.Mather [Mather2]. Remark that a stable map is generic.

REMARK (1.2). *Let I be a Boardman symbol and $H(I)$ the set of all Boardman symbols J which is not smaller (in the lexicographical order) than I with $|J| \le |I|$. Then the height of the ideal Δ^I at $z := j^r f(0)$ is the minimal number of $\nu_J(n,p)$ so that $z \in \overline{\Sigma^J}$ and $J \in H(I)$. Here $\overline{\Sigma^I}$ denotes the Zariski closure of Σ^I. If the height $ht({\Delta^I}_z) = n$ and f is an $\mathcal{A}$-finite (i.e. finitely determined) map-germ, then $c_I(f) := \dim_{\mathbf{K}} \mathcal{O}_n/\Delta^I(f) < \infty$. This was shown in [Ballesteros-Saia, Lemma(4.1)]*

§2. **Commutative algebra.** Let A be a commutative Noetherian local ring with identity. We say that $a_1, \dots, a_r$ is an *A-sequence* if the following conditions satisfied:

(i) $(a_1, \dots, a_{i-1}) : a_i = (a_1, \dots, a_{i-1})$, or equivalently, a_i does not represent a zero-divisor of $A/(a_1, \dots, a_{i-1})$, for $i = 1, 2, \dots, r$.

(ii) $(a_1, \dots, a_r) \ne A$.

A local ring A called to be *Cohen-Macaulay* if there is an A-sequence in the maximal ideal of A whose length is equal to the Krull dimension $\dim A$ of A.

THEOREM (2.1).

(i) *A regular local ring is Cohen-Macaulay.*

(ii) *Let $a_1, \dots, a_r$ be an A-sequence in the maximal ideal. Then A is Cohen-Macaulay iff $A/(a_1, \dots, a_r)$ is.*

(iii) *Let A be a Cohen-Macaulay ring, and I a proper ideal of A. Then we have*

$$ht(I) = \dim A - \dim A/I = \text{ the maximal length of } A\text{-sequences in } I.$$

(iv) *Let A be a Cohen-Macaulay ring, and $a_1, \dots, a_r$ be in the maximal ideal of A. Then $a_1, \dots, a_r$ is an A-sequence iff $ht(a_1, \dots, a_r) = r$.*

Reference for Proof: See [Mat] 17.8 for (i), 17.3 ii) for (ii), 17.4 i) for (iii), and 17.4 iii) for (iv). ∎

Theorem (2.2). *Let A be a Cohen-Macaulay local ring, and $X = (c_{ij})$ an r by s matrix with entries in A. If I denotes the ideal generated by subdeterminants of order k of the matrix X, then $ht(I) \leq (r-k+1)(s-k+1)$. If equality holds, then A/I is Cohen-Macaulay.*

Reference for Proof: See [Hochster-Eagon], Corollary 4. (p.1024). ∎

Theorem (2.3). *Let A be a Cohen-Macaulay local ring, and $X = (c_{ij})$ an s by s symmetric matrix with entries in A. If I denotes the ideal generated by subdeterminants of order k of the matrix X, then $ht(I) \leq \frac{1}{2}(s-k+1)(s-k+2)$. If equality holds, then A/I is Cohen-Macaulay.*

Reference for Proof: See [Kutz]. ∎

§**3. Cohen-Macaulay property of Z_I.** We continue the notation in §1. Let $S(I)$ be the set of all Boardman symbols J which is not smaller (in the lexicographic order) than I so that $\nu_J(n,p) = \mathrm{ht}(\Delta^I)$ and $|J| \leq |I|$. Let $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ be a map-germ, and $z = j^k f(0)$, $k = |I|$. Set $S(I;z) = \{J \in S(I) : z \in \overline{\Sigma^J}\}$. We also denote $S(I;z)$ by $S(I;f)$. We say that a variety is said to be *Cohen-Macaulay* at a point if its local ring at that point is Cohen-Macaulay.

Proposition (3.1). *Let $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ be a $\mathbf{K}$-analytic map-germ and I a Boardman symbol with $ht({\Delta^I}_z) = n$. Here $z = j^r f(0)$ and r is the order of jet space we consider. Remark that $r \geq |I|$. Let $s_I(f_u)$ denote the number of Σ^I-points in a generic approximation f_u (or stable perturbation f_u if (n,p) is nice) of $f = f_0$. and $s_{I;J}(f_u)$ the number of intersections of Z_I and the jet section of f_u in Σ^J counting multiplicities. Obviously we have $s_{I;J}(f_u) \geq s_J(f_u)$ for $J \in S(I)$, and $s_{I;I}(f_u) = s_I(f_u)$ by (1.1). If $c_I(f) < \infty$, then $\sum_{J \in S(I;f)} s_{I;J}(f_u) \leq c_I(f)$. In the case $\mathbf{K} = \mathbf{C}$, equality holds iff Z_I is Cohen-Macaulay at z.*

Proof: Let $\mathcal{S}$ be the structure sheaf of $j^r f(U)$ and $\mathcal{I}$ the sheaf of ideals in $\mathcal{S}$ defining the subspace $j^r f(U) \cap Z_I$. The natural sheaf homomorphism $\mathcal{S} \to \mathcal{O}_n$ induces an isomorphism $\mathcal{S}_{j^r f(0)}/\mathcal{I}_{j^r f(0)} \simeq \mathcal{O}_n/\Delta^I(f)$. Thus, in the case $\mathbf{K} = \mathbf{C}$, the theorem is a consequence of (1.1) and Proposition 7.1, and Example 7.1.3 in [Fulton]. In the case $\mathbf{K} = \mathbf{R}$, we may assume that f_u is represented by a real polynomial map. Thus, the assertion is immediate from the case $\mathbf{K} = \mathbf{C}$. ∎

Proposition (3.2). *Let $f : (\mathbf{K}^n, 0) \to (\mathbf{K}^p, 0)$ be a $\mathbf{K}$-analytic map-germ, $F : (\mathbf{K}^{n+t}, 0) \to (\mathbf{K}^{p+t}, 0)$ an unfolding of f so that f_u is generic (or stable if (n,p) is nice) for general u, where we write $F(x,u) = (f_u(x), u)$, and I a Boardman symbol with $ht(\Delta^I(F)_0) = n$. If $c_I(f) < \infty$, then $\sum_{J \in S(I;f)} s_{I;J}(f_u) \leq c_I(f)$. In the case $\mathbf{K} = \mathbf{C}$, equality holds iff $\Delta^I(F)$ define a Cohen-Macaulay variety in $(\mathbf{C}^{n+t}, 0)$.*

Proof: We first assume that $\mathbf{K} = \mathbf{C}$. Let $u = (u_1, ..., u_t)$ be the unfolding parameter of F. Since f_u is generic for general u, we have $\sum_{J \in S(I;f)} s_{I;J}(f_u)$ is the intersection multiplicity of $\{u_1 = \cdots = u_t = 0\}$ and the zero locus of $\Delta^I(F)$, and $\mathcal{O}_{n+t}/\Delta^I(F) + (u_1, ..., u_t)$ has length $c_I(f)$. Thus, the assertion is a consequence of Proposition 7.1 and

Example 7.1.3 in [Fulton]. In the case $\mathbf{K} = \mathbf{R}$, we may assume that f_u is represented by a real polynomial map. Thus, the assertion is immediate from the case $\mathbf{K} = \mathbf{C}$. ∎

COROLLARY (3.3). *Under the assumptions of (3.2), we have the following:*

(i) *If* $(n,p) = (4,4)$, *then* $5s_2(f_u) + s_{1_4}(f_u) \leq c_{1_4}(f)$.
(ii) *If* $n \geq 5$ *and* $p = 4$, *then* $4s_{n-3,2}(f_u) + s_{n-3,1_3}(f_u) \leq c_{n-3,1_3}(f)$.

PROOF: (i): It is enough to see that $c_{1_4}(f) = 5$, if $f : (\mathbf{C}^4, 0) \to (\mathbf{C}^4, 0)$ is a stable map-germ with $0 \in \Sigma^2(f)$. Such germ is well-known as $I_{2,2}$-singularity, and is given by the following normal form: $f(x_1, ..., x_4) = (x_1x_2 + x_1x_3 + x_2x_4, x_1^2 + x_2^2, x_3, x_4)$. By elementary computation, we have $\Delta^1(f) = (x_2(x_2 + x_3) - x_1(x_1 + x_4))$, $\Delta^{1_2}(f) = \Delta^1(f) + (x_1x_3 - x_2x_4, x_4(x_1+x_4) - x_3(x_2+x_3))$, $\Delta^{1_3}(f)$ is generated by subdeterminants of order 2 of

$$\begin{pmatrix} x_2 & x_1 & 0 & x_4 & x_3 \\ x_1 & x_2 & x_3 & 0 & x_4 \end{pmatrix},$$

and $\Delta^{1_4}(f) = (x_1, ..., x_4)^2$. Thus, $c_{1_4}(f) = 5$.

(ii): It is enough to see that $c_{n-3,1_3}(f) = 4$, if $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^4, 0)$ is a stable map-germ with $0 \in \Sigma^{n-3,2}(f)$. This is D_4-singularity, and is given by $f(x_1, ..., x_n) = (\frac{1}{2}x_1^2x_2 + \frac{1}{6}x_2^3 + \frac{1}{2}x_2^2x_3 + x_2x_4 + x_1x_5 + x_6^2 + \cdots + x_n^2, x_3, x_4, x_5)$. By elementary computation, we have $\Delta^{n-3}(f) = (x_1x_2 + x_5, \frac{1}{2}(x_1^2 + x_2^2) + x_2x_3 + x_4, x_6, ..., x_n)$, $\Delta^{n-3,1}(f) = \Delta^{n-3}(f) + (x_2^2 + x_2x_3 - x_1^2)$, $\Delta^{n-3,1_2}(f) = \Delta^{n-3,1}(f) + (x_1(4x_2 + 3x_3), x_2(4x_2 + 3x_3))$, and $\Delta^{n-3,1_3}(f) = (x_1, x_2, x_3)^2 + (x_4, ..., x_n)$. Thus, $c_{n-3,1_3}(f) = 4$. ∎

REMARK (3.4). *Since* Δ^I *is generated by polynomials in variables* Z_σ^j *with* $1 \leq j \leq p, |\sigma| \leq k = |I|$, Δ^I *defines a Cohen-Macaulay variety at* $j^r f(0)$ *in the jet space of order* r *for each* $r \geq k$, *iff it does for some* $r \geq k$. *We say that* Δ^I *define a Cohen-Macaulay variety at* $j^r f(0)$ *if these equivalent conditions hold.*

LEMMA (3.5). *Let* $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ *be a map-germ, and* $F : (\mathbf{C}^{n+t}, 0) \to (\mathbf{C}^{p+t}, 0)$ *an unfolding of* f *with* $t \geq 1$. *If* $ht(\Delta^I{}_z) = ht(\Delta^I(F)_0) = n$ $(z := j^r f(0))$ *then the following conditions are equivalent.*

(i) Δ^I *defines a Cohen-Macaulay variety at* z.
(ii) $\Delta^I(F)$ *defines a Cohen-Macaulay variety at* 0.

PROOF: By (2.1.iv) and (2.1.ii), (i) implies (ii). Assume that (ii) holds. By projection formula, the intersection number of $j^r f(\mathbf{C}^n)$ and Z_I in $J^r(\mathbf{C}^n, \mathbf{C}^p)$ equals that of $j^r F(\mathbf{C}^n \times 0)$ and Z_I in $J^r(\mathbf{C}^{n+t}, \mathbf{C}^{p+t})$. This equals $(\mathbf{C}^n \times 0).(j^r F)^{-1}(Z_I)$ by projection formula again. By Cohen-Macaulay property, it is $\dim \mathcal{O}_{n+t}/\Delta^I(F) + (u_1, ..., u_t) = c_I(f)$. This shows that the equality holds in (3.1). Thus, Z_I is Cohen-Macaulay by (3.1). ∎

§4. **When the equality $s_I(f_u) = c_I(f)$ holds?** Let $I = (i_1, ..., i_k)$ be a Boardman symbol with length k. Let us state the following condition (A):

(A) $c_I(f)$ is finite and f has an unfolding F so that $\mathrm{ht}(\Delta^{i_1,...,i_s}(F)) = \mathrm{ht}(\Delta^{i_1,...,i_s}{}_z)$ $(z = j^r F(0))$ for $s = 1, 2, ..., k$.

LEMMA (4.1). *Condition (A) holds, if one of the following conditions is satisfied.*

(i) *f is K-finite and $c_I(f)$ is finite.*

(ii) *$ht(\Delta^{i_1,\ldots,i_s}(f)) = ht(\Delta^{i_1,\ldots,i_s}{}_{j^r f(0)})$ for $s = 1, \ldots, k$.*

PROOF: (i): If f is a K-finite map-germ, then f has an unfolding F so that F is generic as a map-germ. In fact, since f is K-finite, f has a versal unfolding F. Then F is stable map-germ, by the last paragraph in page 501 in [W]. Comparing the definition of (infinitesimal) stability with the last paragraph of [Mather5], we obtain that F is generic. Thus, $\mathrm{ht}(\Delta^I(F)) = \mathrm{ht}(\Delta^I{}_{j^r F(0)})$ for each Boardman symbol I.

(ii): Since $\mathrm{ht}(\Delta^I(f)) = \mathrm{ht}(\Delta^I{}_{j^r f(0)})$, $c_I(f)$ is finite. By Theorem 15.1 in [Mat], we complete the proof. ∎

We are ready to state the main result.

THEOREM (4.2). *Suppose that $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ is a holomorphic map-germ satisfying Condition (A). Let I be a Boardman symbol, and $s_I(f_u)$ denote the number of Σ^I-points in a generic approximation f_u (or stable perturbation f_u if (n,p) is nice) of f. Then we have $s_I(f_u) = c_I(f)$, if one of the following conditions holds.*

(0) $0 \in \Sigma^I(f)$, $\nu_I(n,p) = n$.

(1) $I = (i)$, $i(p - n + i) = n$.

(2.0) $I = (1,1)$, $n = p = 2$.

(2.0') $I = (2,1)$, $0 \in \Sigma^2(f)$, $p = \frac{5n-7}{4}$.

(2.1) $I = (n-p+1, j)$, $0 \in \Sigma^{n-p+1}(f)$, $p = \frac{1}{2}j(j+1) + 1$.

(2.j) $I = (n-p+j, 1)$, $0 \in \Sigma^{n-p+j}(f)$, $p = \frac{2n(j-1)+1}{2j-1} + j$.

(3.1) $I = (n-2, 1, 1)$, $0 \in \Sigma^{n-2}(f)$, $p = 3$.

(3.j) $I = (n-p+1, j, 1)$, $0 \in \Sigma^{n-p+1,j}(f)$, $p = j^2 + 2$.

(k.0) $I = (1_k)$, $0 \in \Sigma^1(f)$, $k(p-n+1) = n$.

(k.1) $I = (n-p+1, 1_{k-1})$, $0 \in \Sigma^{n-p+1,1}(f)$.

We first remark that the Boardman symbols I above satisfy $\nu_I(n,p) = n$. When $p = 1, I = (n-p+1)$ in (1), $c_I(f)$ is the Milnor number, and the result is classical. (2.0) is originally due to [Fukuda-Ishikawa], see [Gaffney-Mond (1.6)] also. In the case $(n,p) = (2,3)$, this result can be found in [Mond (2.4)]. We should remark that (0), (1), and (k.0) were obtained by [Ballesteros-Saia].

PROOF: It is easy to see that $S(I; f) = \{I\}$ in each cases. By (3.1-2), the only thing we have to do is to show that Δ^I (or $\Delta^I(F)$) defines a Cohen-Macaulay variety at $z := j^k f(0)$ (or 0) of codimension $\nu_I(n,p)$.

The following shows that (0), (1), (2.0) of (4.2).

LEMMA (4.3).

(i) *Z_i is Cohen-Macaulay and of codimension $i(p-n+i)$.*

(ii) *If $z \in \Sigma^I$, then Z_I is Cohen-Macaulay at z and of codimension $\nu_I(n,p)$.*

(iii) *If $n = p$, $Z_{1,1}$ is Cohen-Macaulay and of codimension 2.*

PROOF: Since $\nu_i(n,p) = i(n-p+i)$, we obtain (i) using (2.2). By (1.1) and (2.1.i), (ii) holds. We show (iii). Assume that $n = p$. In the notation in §1, the ideal Δ^1 is generated by $\delta = \det(Z_i^j)_{1 \le i,j \le n}$ and its height is 1. Thus, $\Delta^{1,1}$ is generated by subdeterminants of order n of the matrix ${}^t(Z_i^j, D_i\delta)$. Since $\mathrm{ht}(\Delta^{1,1}) = \nu_I(n,p) = 2$, we complete the proof of (iii) by (2.2). ∎

LEMMA (4.4). *Let $F : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a map-germ with $0 \in \Sigma^k(f)$, i.e. rank $n-k$. Then there are systems of local coordinates $x = (x_1, ..., x_n)$ of $\mathbf{C}^n$ at 0, and $y = (y_1, ..., y_p)$ of $\mathbf{C}^p$ at 0 such that $y \circ F(x) = (f^1(x), ..., f^{p-n+k}(x), x_{k+1}, ..., x_n)$ where $f^j(x)$ $(1 \leq j \leq p-n+k)$ are some function-germs.*

PROOF: See p.161 in [AGV], and so on. ∎

LEMMA (4.5). *Let $F : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a map-germ with $0 \in \Sigma^{n-p+1,k}(f)$. Then there are systems of local coordinates $x = (x_1, ..., x_n)$ of $\mathbf{C}^n$ at 0, and $y = (y_1, ..., y_p)$ of $\mathbf{C}^p$ at 0 such that $y \circ F(x) = (g(x_1, ..., x_{p+k-1}) + \sum_{i=p+k}^n x_i^2, x_{k+1}, ..., x_{p+k-1})$ where g is a function-germ with variables $x_1, ..., x_{p+k-1}$.*

PROOF: Consequence of (4.4) and parametrized Morse lemma. ∎

The following implies (2.1),(3.1), (3.j) and (k.1) of (4.2), because of (3.5).

LEMMA (4.6). *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a map-germ with $n \geq p$, and $F : (\mathbf{C}^{n+t}, 0) \to (\mathbf{C}^{p+t}, 0)$ an unfolding of f in Condition (A). If f has rank $p-1$, i.e. $0 \in \Sigma^{n-p+1}(f)$, then the following hold.*

(i) *$\Delta^{n-p+1}(F)$ defines a Cohen-Macaulay variety at 0 of codimension $n-p+1$.*
(ii) *$\Delta^{n-p+1,j}(F)$ defines a Cohen-Macaulay variety at 0 of codimension $n-p+1+\frac{1}{2}j(j+1)$, for j with $1 \leq j \leq n-p+1$,*
(iii) *$\Delta^{n-p+1,1,1}(F)$ defines a Cohen-Macaulay variety at 0 of codimension $n-p+3$.*
(iv) *If $0 \in \Sigma^{n-p+1,k}(f)$, then $\Delta^{n-p+1,k,1}(F)$ defines a Cohen-Macaulay variety at 0 of codimension $n-p+k^2+2$.*
(v) *If $0 \in \Sigma^{n-p+1,1}(f)$, then $\Delta^{n-p+1,1_{k-1}}(F)$ defines a Cohen-Macaulay variety at 0 of codimension $n-p+k$.*

PROOF: Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a map-germ with rank $p-1$, i.e. kernel rank $n-p+1$. By (4.4), we may assume that $f(x) = (g(x), x_{n-p+2}, ..., x_n)$ and $F(x,u) = (G(x,u), x_{n-p+2}, ..., x_n)$. Then $I_1 := \Delta^{n-p+1}(F)$ is generated by $G_i := \frac{\partial G}{\partial x_i}$ $(1 \leq i \leq n-p+1)$. Since $\mathrm{ht}(I_1) = \nu_{n-p+1}(n,p) = n-p+1$, $A_1 := \mathcal{O}_{n+t}/I_1$ is Cohen-Macaulay with $\dim A_1 = t+p-1$ by (2.2), which shows (i). The ideal $\Delta^{n-p+1,j}$ is generated by I_1 and I_2 where I_2 is the ideal generated by the subdeterminants of order $n-p-j+2$ of the symmetric matrix $(G_{i_1,i_2})_{1 \leq i_1, i_2 \leq n-p+1}$ where $G_{i_1,i_2} = \frac{\partial^2 G}{\partial x_{i_1} \partial x_{i_2}}$. We denote by $\overline{I}_2$ the ideal generated by the image of I_2 in A_1. Since $\mathrm{ht}(\Delta^{n-p+1,j}(F)) = \nu_{n-p+1,1}(n,p) = n-p+1+\frac{1}{2}j(j+1)$, $A_2 := \mathcal{O}_{n+t}/\Delta^{n-p+1,j}(F) \simeq A_1/\overline{I}_2$ has dimension $t+p-1-\frac{1}{2}j(j+1)$ and $\mathrm{ht}(\overline{I}_2) = \dim A_1 - \dim A_1/\overline{I}_2 = \frac{1}{2}j(j+1)$, A_2 is Cohen-Macaulay by (2.3), which shows (ii). Setting $H = \det(G_{i_1,i_2})_{1 \leq i_1, i_2 \leq n-p+1}$, the ideal $\Delta^{n-p+1,1,1}(F)$ is generated by I_1 and I_3 where I_3 is the ideal generated by the subdeterminants of order $n-p+1$ of the matrix ${}^t(G_{i_1,i_2}, \frac{\partial H}{\partial x_{i_1}})_{1 \leq i_1, i_2 \leq n-p+1}$. We denote by $\overline{I}_3$ the ideal generated by the image of I_3 in A_1. Because $\mathrm{ht}(\Delta^{n-p+1,1,1}) = \nu_{n-p+1,1,1}(n,p) = n-p+3$, $A_3 := \mathcal{O}_{n+t}/\Delta^{n-p+1,1,1}(F) \simeq A_1/\overline{I}_3$ has dimension $t+p-3$ and $\mathrm{ht}(\overline{I}_3) = \dim A_1 - \dim A_1/\overline{I}_3 = 2$. Thus, A_3 is Cohen-Macaulay by (2.2), and this implies (iii). Assume that $0 \in \Sigma^{n-p+1,k}(F)$. Then, by (4.5), there are systems of local coordinates $x = (x_1, ..., x_n)$ of $\mathbf{C}^n$ at 0, and $y = (y_1, ..., y_p)$ of $\mathbf{C}^p$ at 0 such that $y \circ F(x) = (g(x_1, ..., x_{p+k-1}) + \sum_{i=p+k}^{n} x_i^2, x_{k+1}, ..., x_{p+k-1})$. Then, the ideal $\Delta^{n-p+1,k}(F)$ is generated by $G_1, ..., G_k, x_{k+p}, ..., x_n, G_{1,1}, ..., G_{1,k}, ..., G_{k,1}, ..., G_{k,k}$. We denote its quotient ring by A_4. Note that this is a Cohen-Macaulay ring and the dimension is $t + p - \frac{1}{2}k(k+1) - 1$. The ideal $\Delta^{n-p+1,k,1}(F)$ is generated by I_4 and I_5 where I_5 is the ideal generated by the subdeterminants of order k of the matrix $(G_{i_1,i_2;j})_{1\leq i_1\leq i_2\leq k; 1\leq j\leq k}$. Here $G_{i_1,i_2;j} = \frac{\partial}{\partial x_j} G_{i_1,i_2}$. We denote by $\overline{I}_5$ the ideal generated by the image of I_5 in A_4. Because $\mathrm{ht}(\Delta^{n-p+1,k,1}) = \nu_{n-p+1,k,1}(n,p) = n-p+1+\frac{1}{2}k(k+1)+\frac{1}{2}k(k+1)-k+1$, $\mathrm{ht}(\overline{I}_5) = \dim A_4 - \dim A_4/\overline{I}_5 = \frac{1}{2}k(k+1)-k+1$. This completes the proof of (iv) because of (2.2). In the above proof of (iv), we set $k = 1$. Continuing the discussion similar to the above, we obtain (v). ∎

Using similar computation, it is not hard to see the following (4.7) which is, essentially, due to [Ballesteros-Saia]. This implies (k.0) of (4.2).

LEMMA (4.7). *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a map-germ, and $F : (\mathbf{C}^{n+t}, 0) \to (\mathbf{C}^{p+t}, 0)$ an unfolding of f in Condition (A). If f is rank $n-1$, i.e. $0 \in \Sigma^1(f)$, then $\Delta^{1_k}(F)$ define a Cohen-Macaulay variety of codimension $k(p-n+1)$.*

The following implies (2.0') of (4.2).

LEMMA (4.8). *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a map-germ, and $F : (\mathbf{C}^{n+t}, 0) \to (\mathbf{C}^{p+t}, 0)$ an unfolding of f in Condotion (A). If f is rank $n-2$, i.e. $0 \in \Sigma^2(F)$, then $\Delta^{2,1}(F)$ define a Cohen-Macaulay variety of codimension $4(p-n)+7$.*

PROOF: Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a map-germ with rank $n-2$, i.e. kernel rank 2. By (4.4), we may assume that $f(x) = (f^1(x), ..., f^{p-n+2}(x), x_3, ..., x_n)$ and $F(u, x) = (F^1(x, u), ..., F^{p-n+2}(x, u), x_3, ..., x_n, u)$. Then $I_1 := \Delta^2(F)$ is generated by $F_i^j := \frac{\partial F^j}{\partial x_i}$, $(1 \leq i \leq 2, 1 \leq j \leq p-n+2)$. Since $\mathrm{ht}(I_1) = \nu_2(n,p) = 2(p-n+2)$, $A_1 := \mathcal{O}_{n+t}/I_1$ is Cohen-Macaulay with $\dim A_1 = t + 3n - 2p - 2$. Let I_2 denote the ideal generated by the subdeterminants of order 2 of the 2 by $2(p-n+2)$ matrix

$$\begin{pmatrix} F_{1,1}^j & F_{1,2}^j \\ F_{2,1}^j & F_{2,2}^j \end{pmatrix}_{1\leq j\leq p-n+2} \quad \text{where } F_{i_1,i_2}^j = \frac{\partial^2 F^j}{\partial x_{i_1} \partial x_{i_2}}.$$

Then the ideal $\Delta^{2,1}(F)$ is generated by I_1 and I_2. Let $\overline{I}_2$ denote the ideal generated by the image of I_2 in A_1. Since $\mathrm{ht}(\Delta^{2,1}(F)) = \nu_{2,1}(n,p) = 4(p-n)+7$, $A_2 := \mathcal{O}_{n+t}/\Delta^{2,1}(F) \simeq A_1/\overline{I}_2$ has dimension $t + 5n - 4p - 7$. Since $\mathrm{ht}(\overline{I}_2) = \dim A_1 - \dim A_1/\overline{I}_2 = 2(p-n)+5$, A_2 is Cohen-Macaulay by (2.2). ∎

The following implies (2.j) of (4.2).

LEMMA (4.9). *Let $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$ be a map-germ, and $F : (\mathbf{C}^{n+t}, 0) \to (\mathbf{C}^{p+t}, 0)$ an unfolding of f in Condition (A). If f is rank $p-j$, i.e. $0 \in \Sigma^{n-p+j}(f)$, then $\Delta^{n-p+j,1}(F)$ define a Cohen-Macaulay variety of codimension $(2j-1)(n-p+j)+1$.*

The proof of (4.9) is similar to the discussion above, and we omit the details.

§5. **Examples.** In this section, we see that the consequence $s_I(f_u) = c_I(f)$ of (4.2) does not always hold, even if $(n,p) = (3,2)$. We denote by $C(f)$ the critical locus of a map f.

Example (5.1). *Let* $f_u : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ *be a map-germ defined by* $f_u(x,y,z) = \frac{1}{2}(x^2+y^2, y^2+z^2) + u(z, x+y)$.

(i) f_0 *is* $\mathcal{K}$*-finite, i.e.* $f^{-1}(0) \cap C(f_0) = \{0\}$ *near* 0.
(ii) $s_{2,1}(f_u)$*(the number of cusps)* $= 6$, *for general* u.
(iii) $\dim_{\mathbf{C}} \mathcal{O}_3/\Delta^{2,1}(f_0) = 7$.

Proof: Since the jacobi matrix of f_u is

$$\begin{pmatrix} x & y & u \\ u & u+y & z \end{pmatrix},$$

the ideal $\Delta^2(f_u)$ is generated by $xy + u(x-y)$, $yz - u(u+y)$, $xz - u^2$. This shows that $f_0^{-1}(0) \cap C(f_0) = \{0\}$. Moreover, the image of the map $\lambda : \mathbf{C} - \{0, -u\} \to \mathbf{C}^3$ defined by $\lambda(y) = (uy(u+y)^{-1}, y, uy^{-1}(u+y))$ is the critical set $C(f_u)$. Thus, the tangent space of $C(f_u)$ is generated by the vector $v = {}^t(u^2(u+y)^{-2}, 1, -u^2y^{-2})$. Since the restriction X of the Jacobi matrix of f_t to $C(f_u)$ is written by

$$X = \begin{pmatrix} uy(u+y)^{-1} & y & u \\ u & y+u & u(u+y)y^{-1} \end{pmatrix},$$

we obtain

$$Xv = \frac{u^3y^3 + (y^3 - u^3)(u+y)^3}{y^3(u+y)^3} \begin{pmatrix} y \\ u+y \end{pmatrix},$$

which is zero iff $u^3y^3 + (y^3 - u^3)(u+y)^3 = 0$. Therefore we obtain $s_{2,1}(f_u) = 6$ for general u. Since $\Delta^{2,1}(f_0)$ is generated by $xy, zx, yz, x^3, y^3, z^3$, $\dim_{\mathbf{C}} \mathcal{O}_3/\Delta^{2,1}(f_0) = 7$. ∎

This example shows that $Z_{2,1}$ is not Cohen-Macaulay at $j^r f_0(0)$ for $r \geq 2$. The following example shows the consequence of (4.2) does not always hold, even if f is finitely determined (i.e. $\mathcal{A}$-finite).

Example (5.2). *With the notation in the preceding example, we set* $g_{u,a} : (\mathbf{C}^3, 0) \to (\mathbf{C}^2, 0)$ *the map-germ defined by* $g_{u,a}(x,y,z) = f_u(x,y,z) + a(x^n + y^n, z^n)$ *where* n *is an integer greater than* 4.

(i) $g_{0,a}$ *is* $\mathcal{A}$*-finite, if* $a \neq 0$.
(ii) $s_{2,1}(g_{u,a})$*(the number of cusps)* $= 6$.
(iii) $\dim_{\mathbf{C}} \mathcal{O}_3/\Delta^{2,1}(g_{0,a}) = 7$.

Proof: The proof is similar to that of (4.2), but rather complicated. Since the jacobi matrix of f_u is

$$\begin{pmatrix} x + anx^{n-1} & y + any^{n-1} & u \\ u & u+y & z + anz^{n-1} \end{pmatrix},$$

the ideal $\Delta^2(f_u)$ is generated by its 2 by 2 minors. This show that $g_{0,a}|C(g_{0,a})$ is generically one to one map for general a, which implies that $g_{0,a}$ represents a $\mathcal{A}$-finite map-germ at 0 by Theorem 2.1 in [Wall]. We define a map $\varphi : \mathbf{C} \to \mathbf{C}$, by $\varphi(y) = y + any^{n-1}$. Let $\varphi_1 : D \to \mathbf{C}$ denote the inverse map of φ with $\varphi_1(0) = 0$ where D is some small disc in $\mathbf{C}$ centered at 0. Then, the image of the map $\lambda : D_1 \to \mathbf{C}^3$

defined by $\lambda(y) = (\varphi_1 \circ \alpha(y), y, \varphi_1 \circ \beta(y))$ where $D_1 = \{y \in \mathbf{C} : \varphi(y) \neq 0, y+u \neq 0, \alpha(y) \in D, \beta(y) \in D\}$ is the critical set $C(g_{u,a})$ near 0, $\alpha(y) = u(u+y)^{-1}\varphi(y)$, and $\beta(y) = u(u+y)\varphi(y)^{-1}$. Then the tangent vector of $C(g_{u,a})$ is generated by the vector $v = {}^t(u(u+y)^{-2}P_1P,\ 1,\ -u\varphi(y)^{-2}P_2P)$, where $P_1 = (1+an(n-1)(\varphi_1 \circ \alpha(y))^{n-2})^{-1}$, $P_2 = (1+an(n-1)(\varphi_1 \circ \beta(y))^{n-2})^{-1}$, and $P = (u+y)(1+an(n-1)y^{n-2}) - (y+any^{n-1}) = u + an(n-1)uy^{n-2} + an(n-2)y^{n-1}$. Since the restriction X of the Jacobi matrix of $g_{u,a}$ to $C(g_{u,a})$ is written by

$$X = \begin{pmatrix} \alpha(y) & \varphi(y) & u \\ u & u+y & \beta(y) \end{pmatrix},$$

we obtain

$$Xv = \begin{pmatrix} u(u+y)^{-2}\alpha(y)P_1P + \varphi(y) - u^2\varphi(y)^{-2}P_2P \\ u^2(u+y)^{-2}P_1P + (u+y) - u\varphi(y)^{-2}\beta(y)P_2P \end{pmatrix} = \frac{Q(y)}{(u+y)^3\varphi(y)^3}\begin{pmatrix} \varphi(y) \\ u+y \end{pmatrix}$$

where $Q(y) = (u+y)^3(y+nay^{n-1})^3 + u^2P((y+nay^{n-1})^3P_1 - (u+y)^3P_2)$. Six zeros of $Q(y)$ tend to 0, if u tends 0. Therefore, we obtain $s_{2,1}(g_{u,a}) = 6$ for general t near 0. Since $\Delta^{2,1}(f_0)$ is generated by $xy, zx, yz, x^3, y^3, z^3$, and $\Delta^{2,1}(g_{0,a}) \equiv \Delta^{2,1}(f_0)$ mod $(x,y,z)^3$, $\dim_{\mathbf{C}} \mathcal{O}_3/\Delta^{2,1}(g_{0,a}) = 7$. ∎

§6. **Problems.** To end the paper, we state some problems in our direction. The examples above show that the number $c_I(f)$ is not enough to describe the number $s_I(f_u)$ of Σ^I-points of a generic approximation f_u of map-germ $f : (\mathbf{C}^n, 0) \to (\mathbf{C}^p, 0)$, even if $(n,p) = (3,2)$. Thus, we need the Serre's definition of intersection multiplicity to describe $s_I(f_u)$ in algebraic languages. See Example 7.1.2 in [Fulton] for its definition. Let $F : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}^{p+1}, 0)$ be an unfolding of f defined by $F(x,u) = (f_u(x), u)$, and I a Boardman symbol with $\mathrm{ht}(\Delta^I(F)_0) = n$. Suppose that f_u is generic for general u. We see $c_I(f) = \dim_{\mathbf{K}} \mathcal{O}_{n+1}/\Delta^I(F) + (u)$. If $c_I(f) < \infty$, then

$$\sum_{J \in S(I;f)} s_{I,J}(f_u) = c_I(f) - t_I(F)$$

where $t_I(F)$ denotes the length of $\mathcal{O}_{n+1}$-module $\{g \in \mathcal{O}_{n+1}/\Delta^I(F) : ug = 0\}$.

PROBLEMS.

(i) *Describe the Cohen-Macaulay locus of Z_I. Is it possible to describe it as a union of some Thom-Boardman submanifolds Σ^J's?*

(ii) *Describe the multiplicities of Z_I along irreducible components of the Zariski closure of Σ^J with $J \in S(I)$.*

(iii) *Describe the number $t_I(F)$ above in terms of f, not F.*

By (1.1), the ideal Δ^I does not describe the Zariski closure of the Thom-Boardman submanifold Σ^I. Since Thom-Boardman submanifold is a key word to describe singularities, we have the following question.

(iv) Find a system of generators of the ideal defining the Zariski closure of Σ^I in the jet space. Consider when this ideal defines a Cohen-Macaulay space, like (i). Remark that the Morin's ideal Δ^I is an answer under some restriction e.g. $p \leq 3$.

By J.Mather's theorem in [Mather1], A-class of a stable map-germ is determined by its K-class. Thom-Boardman submanifolds are invariant by the action of K, and a Thom-Boardman submanifold contains the union of some K-orbits. Thus, we are interested in studying the location of K-orbits and the jet section of map-germ.

(v) Find systems of generators of the ideals defining the Zariski closures of K-orbits in the jet space. Consider when this ideal defines a Cohen-Macaulay space.

In the following table, we describe simple K-orbits in the Thom-Boardman strata Σ^I with $\nu_I(n,p) = n$ and $0 \leq n-p \leq 5$, $0 \leq p \leq 9$. Here, we use the notation of simple K-orbits in the last page. The Thom-Boardman submanifolds Σ^I contain no stable jets, if the Boardman symbols I are in brackets.

n-p	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	$p=6$	$p=7$	$p=8$	$p=9$
0	$(1)\ A_1$	$(1_2)\ A_2$	$(1_3)\ A_3$	$(1_4)\ A_4$ $(2)\ I_{22}$	$(1_5)\ A_5$ I_{32}	$(1_6)\ A_6$ $I_{42}\ I_{33}$	$(1_7)\ A_7$ $I_{52}\ I_{43}$ $(21)\ I_7$	$(1_8)\ A_8$ $I_{62}\ I_{53}\ I_{44}$ I_8	$(1_9)\ A_9$ $I_{72}\ I_{63}\ I_{54}$ $I_9\ H_9$ [3]
1	$(2)\ A_1$	$(21)\ A_2$	$(21_2)\ A_3$	$(21_3)\ A_4$ $(22)\ D_4$	$(21_4)\ A_5$ D_5 $(3)\ S_5$	$(21_5)\ A_6$ D_6 $(221)\ E_6$ S_6	$(21_6)\ A_7$ D_7 E_7 S_7 $T_7\ U_7$	$(21_7)\ A_8$ D_8 $(2211)\ E_8$ [222] S_8 $T_8\ U_8\ W_8$	$(21_8)\ A_9$ D_9 S_9 $T_9\ U_9\ W_9$ $(31)\ Z_9$
2	$(3)\ A_1$	$(31)\ A_2$	$(31_2)\ A_3$	$(31_3)\ A_4$ $(32)\ D_4$	$(31_4)\ A_5$ D_5	$(31_5)\ A_6$ D_6 $(321)\ E_6$ [4]	$(31_6)\ A_7$ D_7 E_7 [33]	$(31_7)\ A_8$ D_8 $(3211)\ E_8$ [322]	$(31_8)\ A_9$ D_9
3	$(4)\ A_1$	$(41)\ A_2$	$(41_2)\ A_3$	$(41_3)\ A_4$ $(42)\ D_4$	$(41_4)\ A_5$ D_5	$(41_5)\ A_6$ D_6 $(421)\ E_6$	$(41_6)\ A_7$ D_7 E_7 [43][5]	$(41_7)\ A_8$ D_8 $(4211)\ E_8$ [422]	$(41_8)\ A_9$ D_9
4	$(5)\ A_1$	$(51)\ A_2$	$(51_2)\ A_3$	$(51_3)\ A_4$ $(52)\ D_4$	$(51_4)\ A_5$ D_5	$(51_5)\ A_6$ D_6 $(521)\ E_6$	$(51_6)\ A_7$ D_7 E_7 [53]	$(51_7)\ A_8$ D_8 $(5211)\ E_8$ [522][6]	$(51_8)\ A_9$ D_9
5	$(6)\ A_1$	$(61)\ A_2$	$(61_2)\ A_3$	$(61_3)\ A_4$ $(62)\ D_4$	$(61_4)\ A_5$ D_5	$(61_5)\ A_6$ D_6 $(621)\ E_6$	$(61_6)\ A_7$ D_7 E_7 [63]	$(61_7)\ A_8$ D_8 $(6211)\ E_8$ [622]	$(61_8)\ A_9$ D_9 [7]

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$n-p$	Notation	Normal form	Restrictions
$s-1$	A_k	$x_1^{k+1}+x_2^2+\cdots+x_s^2$	$k\geq 1, s\geq 1$
$s-1$	D_k	$x_1^2x_2+x_2^{k-1}+x_3^2+\cdots+x_s^2$	$k\geq 4, s\geq 2$
$s-1$	E_6	$x_1^3+x_2^4+x_3^2+\cdots+x_s^2$	$s\geq 2$
$s-1$	E_7	$x_1^3+x_1x_2^3+x_3^2+\cdots+x_s^2$	$s\geq 2$
$s-1$	E_8	$x_1^3+x_2^5+x_3^2+\cdots+x_s^2$	$s\geq 2$
0	I_{ab}	$x_1x_2, x_1^a+x_2^b$	$a\geq b\geq 2$
0	I_{2a+1}	$x_1^2+x_2^3, x_2^a$	$a\geq 3$
0	I_{2a}	$x_1^2+x_2^3, x_1x_2^{a-2}$	$a\geq 4$
0	I_{10}^*	x_1^2, x_2^4	
0	H_a	$x_1^2+x_2^{a-5}, x_1x_2^2$	$a\geq 9$
1	S_k	$x_1^2+x_2^2+x_3^{k-3}, x_2x_3$	$k\geq 5$
1	T_k	$x_1^2+x_2^3+x_3^{k-4}, x_2x_3$	$7\leq k\leq 9$
1	U_7	$x_1^2+x_2x_3, x_1x_2+x_3^3$	
1	U_8	$x_1^2+x_2x_3, x_1x_2+x_1x_3^2$	
1	U_9	$x_1^2+x_2x_3, x_1x_2+x_3^4$	
1	W_8	$x_1^2+x_2^3, x_2^2+x_1x_3$	
1	W_9	$x_1^2+x_2x_3^2, x_2^2+x_1x_3$	
1	Z_9	$x_1^2+x_3^3, x_2^2+x_3^3$	
1	Z_{10}	$x_1^2+x_2x_3^2, x_2^2+x_3^3$	

Notations of simple contact orbits

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付録：可換環論より

以下は可換環論, 特にCM環の周辺, について勉強したことを私の個人的な好みでまとめたものである. 私自身, 可換環論について全くの素人なので思わぬ間違いを書いていることを恐れる. しかし私と似たような興味から可換環論を勉強する人にとって何かの参考になればと思い, ここに付録として掲載することにした. 私自身の怠惰と能力不足から証明はいっさい省いてある. 証明については後掲の文献を参照して下さい.

A を単位元 1 をもつ可換環, M を A-加群とする. $\mathrm{Spec}(A)$ で A の素イデアル全体の集合に Zariski 位相をいれた位相空間, 即ち A のイデアル I に対し部分集合

$$V(I) = \{P \in \mathrm{Spec}(A) : P \supset I\}$$

を閉集合とするような (最弱の) 位相空間を表す.

$$\mathrm{Ass}(M) := \{P \in \mathrm{Spec}(A) : P = \mathrm{ann}(x), \exists x \in M\},$$
$$\mathrm{Supp}(M) := \{P \in \mathrm{Spec}(A) : M_P \neq 0\}$$

とおく. M がネータ環 A 上の有限生成 A 加群ならば $\mathrm{Ass}(M)$ は有限集合で,

$$\mathrm{Ass}(M) \subset \mathrm{Supp}(M) = V(\mathrm{ann}(M))$$

かつ $\mathrm{Supp}(M)$ の極小元全体は $\mathrm{Ass}(M)$ の極小元全体と一致する.

§**1. 射影加群** ([松村, 付録 B]). A-加群 P が次の条件を満たすとき**射影的** (または**射影加群**) であるという: 任意の全射 $f : M \to N$ と任意の射 $g : P \to N$ に対し, $g = f \circ h$ を満たす $h : P \to M$ が存在する.

定理.

(i) 自由加群は射影的で, 射影加群は自由加群の直和因子である.
(ii) 任意の加群は射影加群 *(例えば自由加群)* の剰余加群として表せる.
(iii) A を局所環とする. このとき射影 A-加群と自由 A-加群は同じ概念 *([松村, 2.5])*. さらに有限生成加群については, 平坦 A-加群も自由 A-加群と同じ概念 *([松村, 7.10])*.

A-加群 M に対し射影加群 P_0 から M への全射 $P_0 \to M$ をとりその核を K_0 とすれば $0 \to K_0 \to P_0 \to M \to 0$ なる完全列を得る. K_0 について同様に射影加群 P_1 からの全射 $P_1 \to K_0$ をとれば $0 \to K_1 \to P_1 \to K_0 \to 0$ なる完全列を得る. 以下同様にして $i = 1, 2, \ldots$ に対して射影加群 P_i と完全列 $0 \to K_i \to P_i \to K_{i-1} \to 0$ が作られる. これらをつないで得られる複体

$$\text{(P.)} \qquad \cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to 0$$

を M の**射影分解** (projective resolution) という. 右端の $P_0 \to 0$ を $P_0 \to M \to 0$ にかえたものは完全列になる.

補題. A が ネータ環で M が有限生成 A-加群ならば P_0 を有限生成自由加群とすることができ そのときは K_0 も有限生成になる. 以下同様であるから M の射影分解として各 P_i が有限生成自由加群であるものが存在する.

A-加群 M の射影分解 (または自由分解) $P.$ が $P_d \neq 0$ かつ $P_k = 0$ $(k > d)$ をみたすとき $P.$ の長さは d であるという. A-加群 M の長さ最小の射影分解の長さを M の射影次元といい $\mathrm{pd}(M)$ で表す. M が射影加群であることと $\mathrm{pd}(M) = 0$ は同値である.

補題. A が ネータ局所環で 有限生成 A-加群の完全列 $0 \to M' \to M \to M'' \to 0$ があれば, $\mathrm{pd}(M) \leq \max\{\mathrm{pd}(M'), \mathrm{pd}(M'')\}$ で等号が成立しないときは $\mathrm{pd}(M'') = \mathrm{pd}(M') + 1$.

§**2. Tor.** M, N を A-加群とする. M の射影分解 $(P.)$ の各項に $\otimes_A N$ を施して得られる複体を $P. \otimes N$ とかく.

$$(P. \otimes N) \qquad \cdots \to P_n \otimes_A N \to P_{n-1} \otimes_A N \to \cdots \to P_1 \otimes_A N \to P_0 \otimes_A N \to 0$$

このとき A-加群 $\mathrm{Tor}_n^A(M, N)$ を次で定義する.

$$\mathrm{Tor}_n^A(M, N) := \frac{\mathrm{Ker}\{P_n \otimes_A N \to P_{n-1} \otimes_A N\}}{\mathrm{Im}\{P_{n+1} \otimes_A N \to P_n \otimes_A N\}}.$$

この加群 $\mathrm{Tor}_n^A(M, N)$ は射影分解 $(P.)$ のとり方に依存せず M と N だけで決まる.

Tor の主な性質.

(i) $\mathrm{Tor}_0^A(M, N) = M \otimes_A N$, $\mathrm{Tor}_n^A(M, N) = \mathrm{Tor}_n^A(N, M)$.

(ii) M が射影的または平坦ならば任意の A-加群 N に対し $\mathrm{Tor}_n^A(M, N) = 0$ $(n > 0)$.

(iii) 短完全列 $0 \to M' \to M \to M'' \to 0$ に対して長完全列 $\cdots \to \mathrm{Tor}_n^A(M', N) \to \mathrm{Tor}_n^A(M, N) \to \mathrm{Tor}_n^A(M'', N) \to \mathrm{Tor}_{n-1}^A(M', N) \to \cdots \to \mathrm{Tor}_1^A(M'', N) \to M' \otimes_A N \to M \otimes_A N \to M'' \otimes_A N \to 0$ が得られる.

§**3. Ext.** M, N を A-加群とする. M の射影分解 $(P.)$ の各項に $\mathrm{Hom}_A(-, N)$ を施して得られる複体を $\mathrm{Hom}_A(P., N)$ とかく.

$(\mathrm{Hom}_A(P., N))$

$$0 \to \mathrm{Hom}_A(P_0, N) \to \mathrm{Hom}_A(P_1, N) \to \cdots \to \mathrm{Hom}_A(P_{n-1}, N) \to \mathrm{Hom}_A(P_n, N) \to \cdots$$

このとき A-加群 $\mathrm{Ext}_A^n(M, N)$ を次で定義する.

$$\mathrm{Ext}_A^n(M, N) := \frac{\mathrm{Ker}\{\mathrm{Hom}_A(P_n, N) \to \mathrm{Hom}_A(P_{n+1}, N)\}}{\mathrm{Im}\{\mathrm{Hom}_A(P_{n-1}, N) \to \mathrm{Hom}_A(P_n, N)\}}.$$

この加群 $\mathrm{Ext}_A^n(M, N)$ は射影分解 $(P.)$ のとり方に依存せず M と N だけで決まる.

Ext の主な性質.

(i) $\mathrm{Ext}_A^0(M, N) = \mathrm{Hom}_A(M, N)$.

(ii) M が射影的ならば任意の A-加群 N に対し $\mathrm{Ext}_A^n(M, N) = 0$ $(n > 0)$.

(iii) 短完全列 $0 \to M' \to M \to M'' \to 0$ に対して長完全列 $0 \to \mathrm{Hom}_A(M'', N) \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M', N) \to \mathrm{Ext}_A^1(M'', N) \to \mathrm{Ext}_A^1(M, N) \to \mathrm{Ext}_A^1(M'', N) \to \mathrm{Ext}_A^2(M'', N) \to \mathrm{Ext}_A^2(M, N) \to \cdots$ が得られる.

(iv) 短完全列 $0 \to N' \to N \to N'' \to 0$ に対して長完全列 $0 \to \mathrm{Hom}_A(M, N') \to \mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N'') \to \mathrm{Ext}_A^1(M, N') \to \mathrm{Ext}_A^1(M, N) \to \mathrm{Ext}_A^1(M, N'') \to \mathrm{Ext}_A^2(M, N') \to \mathrm{Ext}_A^2(M, N) \to \cdots$ が得られる.

§4. 局所環のパラメータ. $(A, \mathbf{m})$ を n 次元ネータ局所環とすると n 個の元で生成された $\mathbf{m}$-準素イデアルが存在し n 個より少ない数の元で生成された $\mathbf{m}$-準素イデアルは存在しない. $x_1, ..., x_n$ が $\mathbf{m}$-準素イデアルを生成するとき $x_1, ..., x_n$ を A のパラメータ系 (s.o.p.) といい $x_1, ..., x_n$ の生成するイデアルをパラメータイデアルという. $\mathbf{m}$ を生成するのに必要な元の最小数は $\dim_{A/\mathbf{m}} \mathbf{m}/\mathbf{m}^2$ である. $\mathbf{m}$ が n 個の元で生成されるとき A は正則局所環であるといい $\mathbf{m}$ を生成するパラメータ系を正則パラメータ系という.

定理 ([松村, (14.1)]). $(A, \mathbf{m})$ を n 次元ネータ局所環 $x_1, .., x_n$ をパラメータ系とすると

(i) $\dim A/(x_1, ..., x_i) = n - i$ $(1 \le i \le n)$.
(ii) $\mathrm{ht}(x_1, ..., x_i) = i$ は任意の i について成り立つとはいえない.

定理 ([松村, (14.2)]). n 次元ネータ局所環 $(A, \mathbf{m})$ の $\mathbf{m}$ の元 $x_1, .., x_n$ に対し, 次は同値.

(i) $x_1, ..., x_i$ は正則パラメータ系の一部になる.
(ii) $x_1, ..., x_i$ の $\mathbf{m}/\mathbf{m}^2$ における像が一次独立.
(iii) $A/(x_1, ..., x_i)$ は $n - i$ 次元の正則局所環.

A-加群 M に対しては $\dim M/(x_1, ..., x_r)M = 0$ なる $\mathbf{m}$ の元 $x_1, ..., x_r$ を M のパラメータ系といい それの生成するイデアルを M のパラメータイデアルという.

§5. Samuel の重複度. $(A, \mathbf{m})$ を n 次元ネータ局所環 M を有限生成 A-加群, (x) を A の $\mathbf{m}$-準素イデアルとする. このとき 十分大きな k に対し $\ell(M/(x)^{k+1}M)$ は k の有理係数多項式と一致する. この多項式を **Samuel 多項式**という. Samuel 多項式の次数は A の次元 n に一致する. Samuel 多項式の最高次の係数の $n!$ 倍は整数でありこれを $e((x), M)$ とかく. $e((x), M)$ は **Samuel の重複度**と呼ばれる.

$$e((x), M) = \lim_{k \to \infty} \frac{n!}{k^n} \ell(M/(x)^k M).$$

特に $n = 0$ ならば $e((x), M) = \ell(M)$. また $e(A) := e(\mathbf{m}, A)$ を A の重複度という.

定理 ([松村, p.130]). $(A, \mathbf{m})$ を n 次元ネータ局所環 M を有限生成 A-加群, (x) (x') を A の $\mathbf{m}$-準素イデアルとする.

(i) $\dim M = n$ ならば $e((x), M) > 0$ で, $\dim M < n$ ならば $e((x), M) = 0$.
(ii) $(x') \subset (x)$ ならば $e((x), M) \le e((x'), M)$, 特に, $e((x)^r, M) = e((x), M) r^n$.
(iii) 有限生成 A-加群の完全列 $0 \to M' \to M \to M'' \to 0$ があれば $e((x), M) = e((x), M') + e((x), M'')$.

定理 ([松村,(14.9)], [S, V]). $(A, \mathbf{m})$ を n 次元ネータ局所環 M を有限生成 A-加群, $(x) = (x_1, ..., x_r)$ を M の パラメータイデアルとする. このとき,

$$\ell(M/(x)M) \ge e((x), M) = \textstyle\sum_i \ell(\mathrm{Tor}_i^A(A/(x), M)).$$

§6. Serre の交点数 ([S, V]). $(A, \mathbf{m})$ を ネータ局所環 I, J を A の イデアルとする. $i(I, J; A) := \sum_i (-1)^i \ell(\mathrm{Tor}_i^A(A/I, A/J))$ を I と J の A での交点数という. 幾何学的に重要なのは A が 多項式環 $\mathbf{C}[X_1, ..., X_N]$ をある素イデアル P で局所化したもののときである. このとき $\mathbf{C}[X_1, ..., X_N]$ のイデアル I, J で $I \subset P$, $J \subset P$ なものに対し $V(I)$ と $V(J)$ の $V(P)$ に沿っての交点数は $i(\overline{I}, \overline{J}; A)$ で与えられる. ここで $\overline{I}$ は I が生成する A のイデアルを表す.

定理. $(A,\mathbf{m})$ を ネータ局所環 M を有限生成 A-加群, $(x)=(x_1,...,x_n)$ を A/I の パラメータイデアルとする. このとき $e((x),A/I)=i((x),I;A)$.

§7. 極小自由分解と有限自由分解 ([松村, §19]). $(A,\mathbf{m})$ を局所環とする. 有限生成 A-加群 M に対し射影分解 $(P.)$ が**極小自由分解** (minimal free resolution) であるとは次の条件を満たすときをいう.

(i) 各 P_k は有限自由加群 A^{b_k} に同型.
(ii) 写像 $A^{b_k}=P_k\to P_{k-1}=A^{b_{k-1}}$ を行列表示したらその行列の各成分は A の極大イデアル $\mathbf{m}$ の元.

補題. A が ネータ局所環で M が有限生成 A-加群ならば M は極小自由分解をもつ.

補題. A が ネータ局所環で M が極小自由分解をもつ A-加群ならば
(i) $b_k=\dim_{A/\mathbf{m}}\mathrm{Tor}_k^A(M,A/\mathbf{m})$.
(ii) $\mathrm{pd}(M)=\sup\{i:\mathrm{Tor}_i^A(M,A/\mathbf{m})\neq 0\}\leq \mathrm{pd}(A/\mathbf{m})$.

$\mathrm{gl.dim}(A):=\sup\{\mathrm{pd}(M):M$は A-加群 $\}$ とおく.

定理. ネータ局所環 A に対し, A:正則 $\Longleftrightarrow \mathrm{gl.dim}(A)=\dim(A)\Longleftrightarrow \mathrm{gl.dim}(A)<\infty$.

長さ有限の自由分解を**有限自由分解** (FFR) という. A-加群 M が有限自由分解

$$0\to A^{b_n}\to A^{b_{n-1}}\to\cdots\to A^{b_1}\to A^{b_0}\to M\to 0$$

をもつとき $\chi(M)=\sum_i(-1)^ib_i$ を M の Euler 数という.

定理. A が可換環で A-加群 M が *FFR* をもてば $\chi(M)\geq 0$.

定理. A がネータ環で A-加群 M が *FFR* をもてば $\chi(M)=0\Longleftrightarrow \mathrm{ann}(M)\neq 0$.

§8. 正則列. A を単位元 1 をもつ可換環, M を A-加群とする. $a\in A$ が M-正則とは 任意の $m\in M$ に対し $am=0$ ならば $m=0$ となるときをいう. $0\to M\xrightarrow{\times a}M$ が完全列となるときと言い替えてもよい. $a_1,...,a_r$ が次の 2 条件をみたすとき M-列という.
(i) $i=1,...,r$ について a_i が $M/(a_1,...,a_{i-1})M$-正則.
(ii) $(a_1,...,a_r)M\neq M$.

A を ネータ環, M を有限生成 A-加群, I を A のイデアルとする. I の中の極大 M-列の長さを $\mathrm{dp}_A(I,M)$ とかく.

補題. $\mathrm{dp}_A(I,M)=\inf\{i:\mathrm{Ext}_A^i(A/I,M)\neq 0\}$.

$(A,\mathbf{m})$ を ネータ局所環, M を有限生成 A-加群のとき, $\mathrm{dp}_A(M)=\mathrm{dp}_A(\mathbf{m},M)$ と略記する. 局所コホモロジー群

$$\mathrm{H}^i_{\mathbf{m}}(M):=\varinjlim_n \mathrm{Ext}_A^i(A/\mathbf{m}^n,M)$$

を用いると $\mathrm{dp}_A(M)=\inf\{i:\mathrm{H}^i_{\mathbf{m}}(M)\neq 0\}$ である.

補題 ([AK, III, (3.16)]). ネータ局所環 $(A, \mathbf{m})$, $(B, \mathbf{n})$, の射 $f : (A, \mathbf{m}) \to (B, \mathbf{n})$ が $f(\mathbf{m}) \subset \mathbf{n}$ をみたせば任意の有限生成 B-加群について $\mathrm{dp}_A(M) = \mathrm{dp}_B(M)$ がなりたつ.

定理. ネータ局所環 $(A, \mathbf{m})$ と有限生成 A-加群 M について $\mathrm{dp}_A(M) \leq \dim(A/P), \forall P \in \mathrm{Ass}(M)$ で, $\mathrm{pd}_A(M) < \infty$ ならば $\mathrm{pd}_A(M) + \mathrm{dp}_A(M) = \mathrm{dp}_A(A)$.

$\mathrm{grade}(M) = \inf\{i : \mathrm{Ext}^i_A(M, A) \neq 0\}$ を A-加群 M の **grade** という.

定理. $\mathrm{grade}(M) \leq \mathrm{pd}(M)$, $\mathrm{grade}(A/I) = \mathrm{dp}_A(I, A) \leq \mathrm{ht}(I)$.

$\mathrm{grade}(I) := \mathrm{grade}(A/I)$ とかく. $\mathrm{grade}(M) = \mathrm{pd}(M)$ なる A-加群 M を完全加群 (perfect module) という. A/I が完全加群のとき イデアル I は完全であるという.

§9. Cohen-Macaulay 環 ([松村, §17]). A を ネータ局所環 M を有限生成 A-加群とする. $\mathrm{dp}(M) = \dim(M)$ のとき M を **Cohen-Macaulay(CM) 加群** という. また A が A-加群として CM のとき即ち $\mathrm{dp}(A) = \dim(A)$ のとき A を **Cohen-Macaulay(CM) 局所環**という.

定理. $(A, \mathbf{m})$ を ネータ局所環 M を有限生成 A-加群とする.
 (i) *CM* 加群 M と $P \in \mathrm{Ass}(M)$ に対し, $\dim(A/P) = \dim M = \mathrm{dp}(M)$.
 (ii) $a_1, ..., a_r \in \mathbf{m}$ が M-列ならば, M : *CM* $\Longleftrightarrow M/(a_1, ..., a_r)M$: *CM*.

定理. $(A, \mathbf{m})$ を *CM* ネータ局所環とする. I を A の真のイデアルとすると $\mathrm{ht}(I) = \mathrm{dp}_A(I, A) = \mathrm{grade}(I)$, $\mathrm{ht}(I) + \dim(A/I) = \dim(A)$.

定理. $(A, \mathbf{m})$ を *CM* ネータ局所環とする. $a_1, ..., a_r \in \mathbf{m}$ に対し次は同値.
$a_1, ..., a_r$ は A-列 $\Longleftrightarrow \mathrm{ht}(a_1, ..., a_r) = r \Longleftrightarrow a_1, ..., a_r$ はパラメータ系の一部.

定理. $(A, \mathbf{m})$ を ネータ局所環 M を有限生成 A-加群とする. 次は同値.
 (i) M は *CM* 加群.
 (ii) 任意の M のパラメータイデアル $(x) = (x_1, ..., x_n)$ に対し $e((x), M) = \ell(M/(x)M)$.
 (iii) ある M のパラメータイデアル $(x) = (x_1, ..., x_n)$ に対し $e((x), M) = \ell(M/(x)M)$.

ネータ環 A が すべての極大イデアル $\mathbf{m}$ に対して $\mathbf{m}$ での局所化 $A_\mathbf{m}$ が CM 環のとき **Cohen-Macaulay(CM) 環**であるという. ネータ環 A 上の 加群 M が すべての極大イデアル $\mathbf{m}$ に対して $\mathbf{m}$ での局所化 $M_\mathbf{m}$ が CM $A_\mathbf{m}$-加群のとき **Cohen-Macaulay(CM) 加群**であるという.

定理 ([BV, 16.C]). A を *CM* 環, M を有限生成 A-加群とする.
 (i) $A[X_1, ..., X_n]$ は *CM* 環.
 (ii) 完全 A-加群 M は *CM* A-加群.
 (iii) $\mathrm{pd}(M) < \infty$ なる *CM* A-加群 M で $\mathrm{Supp}(M)$ が連結ならば M は完全 A-加群. 特に A が正則局所環ならば *CM* A-加群は完全 A-加群.

§10. Cohen-Macaulay 環の type ([HK, 1]). $(A, \mathbf{m})$ を n 次元ネータ局所環, M を有限生成 CM 加群とする. $r_A(M) := \dim_{A/\mathbf{m}} \mathrm{Ext}^n_A(A/\mathbf{m}, M)$ を CM A-加群 M の **type** という.

定理. $d = \mathrm{dp}_A(M)$ とおくと, $r_A(M) = \dim_{A/\mathbf{m}} \mathrm{Tor}^A_{n-d}(M, A/\mathbf{m}) = \dim_{A/\mathbf{m}} H^d_{\mathbf{m}}(M)$.

定理.

(i) $A \to A'$ が全射なら $r_A(M) = r_{A'}(M)$.
(ii) $a \in A$ が M-正則なら $r_A(M) = r_A(M/aM)$.
(iii) $r_A(A) \leq e(A)$ で, 等号成立 $\Longleftrightarrow$ A:正則.
(iv) 任意の M のパラメータイデアル (x) に対し, $r_A(M) = \dim_{A/\mathbf{m}} \gamma(M/(x)M)$.

ここで $\gamma(M/(x)M)$ は $M/(x)M$ の **socle** と呼ばれるもので$\gamma(M/(x)M) = \{y \in M/(x)M : \mathbf{m}y = 0\}$ で定義される.

§11. Buchsbaum 加群 ([SV]). $(A, \mathbf{m})$ を ネータ局所環, M を有限生成 A-加群とする. 次の条件をみたすとき M は **Buchsbaum 加群**であるという: 任意の M のパラメータイデアル $(x) = (x_1, ..., x_r)$ に対し $\ell(M/(x)M) - e((x), M)$ が (x) に依存しない定数.

M が n 次元 Buchsbaum 加群ならば次が成立 ([SV, I.2.6]).

$$\ell(M/(x)M) - e((x), M) = \sum_{i=0}^{n-1} \binom{n-1}{i} \ell(\mathrm{H}^i_{\mathbf{m}}(M)).$$

定理 ([SV, IV.3.2]). $\mathrm{dp}(M) \neq 0$ のとき次は同値.

(i) M は *Buchsbaum* 加群.
(ii) M の任意のパラメータイデアル (x) に対し, 次数環 $R = R((x); M) := \sum_i (x)^i M$ を任意の $P \in \mathrm{Proj}(\bigoplus_n (x)^n)$ で局所化したものは *CM* A_P-加群.

§12. Koszul 複体 ([EN], [HE]). A を環, $x_1, ..., x_n \in A$ とするとき, 複体 $K.$ を次のように定義する. $K_0 = A$ とし $1 \leq p \leq n$ に対しては K_p は $\{e_{i_1 \cdots i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}$ を基底とする階数 $\binom{n}{p}$ の自由加群 $K_p = \bigoplus A e_{i_1 \cdots i_p}$ とし $0 \leq p \leq n$ 以外の p に対しては $K_p = 0$ とする. 微分作用素 $d : K_p \to K_{p-1}$ は $d(e_{i_1 \cdots i_p}) = \sum_r (-1)^{r-1} x_{i_r} e_{i_1 \cdots \widehat{i_r} \cdots i_p}$ ($p = 1$ なら $d(e_i) = x_i$) によって定める.

定理. $I = (x_1, ..., x_n)$ とおく.

(i) $\mathrm{grade}(I) \leq n$ で, 等号成立ならば $K.$ は A/I の自由分解で A/I は完全加群.
(ii) A が局所環で上で等号成立, かつ各 $x_i \in \mathbf{m}$ ならばこれは極小自由分解.

§13. 行列式イデアル. ネータ環 A の元を成分とする m 行 n 列の行列 $X = (x_{ji})$ を考える. 行列 X の t 次小行列式の生成するイデアルを $I_t(X)$ で表す.

定理 ([HE]). X を m 行 n 列の行列とする. このとき, $\mathrm{grade}(I_t(X)) \leq (m-t+1)(n-t+1)$ で, 等号成立ならば $A/I_t(X)$ は完全加群.

この定理で等号成立のとき $I_t(X)$ は**行列式イデアル** (determinantal ideal) という.

定理 ([K]). X を n 次対称行列とする. このとき, $\mathrm{grade}(I_t(X)) \leq \frac{1}{2}(n-t+1)(n-t+2)$ で, 等号成立ならば $A/I_t(X)$ は完全加群.

これらの定理では完全加群であることの証明には次を用いている.

命題 ([HE, Proposition 18]). A を ネータ環, P, Q を共に grade k の A の完全イデアルとし, $P \not\subset Q$, $Q \not\subset P$ とする.

(i) $\mathrm{grade}(P+Q) \leq k+1$ ならば $P \cap Q$ は grade k の完全イデアル.

(ii) $\mathrm{grade}(P+Q) = k+1$ ならば, $P \cap Q$ は完全 $\Longleftrightarrow$ $P+Q$ は完全.

§14. Eagon-Northcott 複体 ([EN],[HE],[BV,2.C]). ネータ環 A の元を成分とする m 行 n 列の行列 $X = (x_{ji})$ を考える. $(m \leq n)$. S_q を $\{X_1^{j_1} \cdots X_m^{j_m} : j_1 + \cdots + j_m = q, j_k \geq 0\}$ を基底とする階数 $\binom{q+n-1}{q}$ の自由 A-加群 $S_q = \bigoplus_{j_1+\cdots+j_m=q} AX_1^{j_1} \cdots X_m^{j_m}$, K_p は $\{e_{i_1 \cdots i_p} : 1 \leq i_1 < \cdots < i_p \leq n\}$ を基底とする階数 $\binom{n}{p}$ の自由加群 $K_p = \bigoplus Ae_{i_1 \cdots i_p}$ とする. このとき 複体 $R.$ を次のように定義する. $R_{q+1} = K_{s+q} \otimes_A S_q$ $(q = 0, 1, ..., n-m)$, $R_0 = A$ とおく. 微分作用素 $d : R_{q+1} \to R_q$ $(q > 0)$ は

$$d(e_{i_1 \cdots i_{m+q}} \otimes X_1^{j_1} \cdots X_m^{j_m}) = \sum_{k : j_k > 0} \sum_{p=1}^{m+q} (-1)^{p+1} x_{k i_p} e_{i_1 \cdots \widehat{i_p} \cdots i_{m+q}} \otimes X_1^{j_1} \cdots X_k^{j_k - 1} \cdots X_m^{j_m}$$

で定義する. 微分作用素 $d : R_1 = K_m \otimes_A S_0 \to R_0 = A$ は

$$d(e_{i_1 \cdots i_m} \otimes 1) = \begin{vmatrix} x_{1i_1} & x_{1i_2} & \cdots & x_{1i_m} \\ x_{2i_1} & x_{2i_2} & \cdots & x_{2i_m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{mi_1} & x_{mi_2} & \cdots & x_{mi_m} \end{vmatrix}$$

で定義すると $R.$ は複体になる.

定理. $m \leq n$ とする. $X = (x_{ji})$ を m 行 n 列の行列, $I = I_m(X)$ とおく.

(i) $\mathrm{grade}(I) \leq n - m + 1$ で, 等号成立ならば, $R.$ は A/I の自由分解で A/I は完全加群.

(ii) A が局所環で上の等号成立, かつ各 $x_{ji} \in \mathbf{m}$ ならばこれは極小自由分解.

§15. Gulliksen-Negård 複体 ([BV, 2.D]). $X = (x_{ij})$ を A の元を成分とする n 次正方行列, $Y = (y_{ij})$ を X の余因子行列とする. $M_n(A)$ を A の元を成分とする n 次正方行列全体とする.

$$A \xrightarrow{\iota} M_n(A) \oplus M_n(A) \xrightarrow{\pi} A$$

を $\iota(a) = (aE, aE)$ (E は単位行列), $\pi(U, V) = \mathrm{tr}(U - V)$ で定義する. $F = \mathrm{Ker}\pi / \mathrm{Im}\iota$ とおくと F は階数 $2(n^2 - 1)$ の自由 A-加群である.

$$(\mathbf{L}(\mathrm{X})) \qquad 0 \to A \xrightarrow{d_4} M_n(A) \xrightarrow{d_3} F \xrightarrow{d_2} M_n(A) \xrightarrow{d_1} A \to A/I_{n-1}(X) \to 0$$

を $d_1(M \mod A_n(A)) = \mathrm{tr}(YM)$, $d_2((U, V)$ の class$) = UX - XV$, $d_3(W) = (XW, WX)$ の class, $d_4(a) = aY$ で定義すると $\mathbf{L}(\mathrm{X})$ は複体である.

定理. $X = (x_{ji})$ を n 次正方行列とし, $I = I_{n-1}(X)$ とおく.

(i) $\mathrm{grade}(I) \leq 4$ で, 等号成立ならば $\mathbf{L}(X)$ は A/I の自由分解で A/I は完全加群.

(ii) A が局所環で上の等号成立, かつ 各 $x_{ji} \in \mathbf{m}$ ならばこれは極小自由分解.

§16. **Józefiak 複体** ([J]). $X = (x_{ij})$ を A の元を成分とする n 次対称行列, $Y = (y_{ij})$ を X の余因子行列とする. $M_n(A)$ を A の元を成分とする n 次正方行列全体, $A_n(A)$ を A の元を成分とする n 次歪対称行列全体とする.

$$(\mathbf{L}(\mathrm{X}))\quad 0 \to A_n(A) \xrightarrow{d_3} \mathrm{Ker}\{M_n(A) \xrightarrow{tr} A\} \xrightarrow{d_2} M_n(A)/A_n(A) \xrightarrow{d_1} A \to A/I_{n-1}(X) \to 0$$

ここで $d_1(M \mod A_n(A)) = \mathrm{tr}(YM)$, $d_2(N) = XN \mod A_n(X)$, $d_3(S) = SX$ と定義するとこれは複体.

定理. $X = (x_{ji})$ を n 次対称行列とし, $I = I_{n-1}(X)$ とおく.

(i) $\mathrm{grade}(I) \leq 3$ で, 等号成立ならば $\mathbf{L}(X)$ は A/I の自由分解で A/I は完全加群.

(ii) A が局所環で上の等号成立, かつ 各 $x_{ji} \in \mathbf{m}$ ならばこれは極小自由分解.

§17. **補足**. §13 の 2 定理で grade について等号が成立する場合, [L],[PW],[JPW] で標数 0 の場合に一般線型群の表現を用いて $A/I_t(X)$ の極小自由分解を具体的に構成している旨, 名古屋大学の橋本光靖氏より御教示頂いた. 橋本氏に感謝の意を表したい.

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