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On finite modified Nash V-determinacy of polynomial map-germs

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1 Introduction

We consider triviality of a family of zero-sets of real polynomial mappings. In [9], the author introduced the notion of modified Nash triviality as a natural and desirable one for it, and gave a modified Nash triviality theorem in the weighted homogeneous polynomial case. Following [9], T. Fukui, S. Koike and M. Shiota gave some fundamental results to construct local theory on modified Nash triviality. See [10] for the survey of this field. In this note, we discuss the problem of finite modied Nash V-determinacy. In particular, we give a characterization of the finite determinacy (Theorem I) and a sufficient condition for an r-jet $w \in J^r(2,1)$ to be modified Nash V-sufficient (Theorem II).

2 Results

We first recall the definitions of a Nash manifold and a Nash mapping. A semialgebraic set of $\mathbf{R}^n$ is called a *Nash manifold* if it is a C^ω submanifold of $\mathbf{R}^n$. Let $M \subset \mathbf{R}^m$ and $N \subset \mathbf{R}^n$ be Nash manifolds. A C^ω mapping $f : M \to N$ is called a *Nash mapping* if the graph of f is semialgebraic in $\mathbf{R}^m \times \mathbf{R}^n$. See M. Shiota [16] for properties of Nash manifolds.

Let $M \subset \mathbf{R}^m$ be a Nash manifold possibly with boundary, and let $N_1, \ldots, N_q$ be Nash submanifolds of M possibly with boundary, which together with $N_0 = \partial M$ are normal crossing. Assume that $\partial N_i \subset N_0$, $i = 1, \ldots, q$. Then the following Nash Isotopy Lemma is given in [6]:

Theorem (2.1) ([6] Theorem I). *Let $\varpi : M \to \mathbf{R}^k$, $k > 0$, be a proper onto Nash submersion such that for every $0 \leq i_1 < \cdots < i_s \leq q$, $\varpi \mid N_{i_1} \cap \cdots \cap N_{i_s} : N_{i_1} \cap \cdots \cap N_{i_s} \to \mathbf{R}^k$ is a proper onto submersion. Then there exists a Nash diffeomorphism*

$$\phi = (\phi^*, \varpi) : (M; N_1, \ldots, N_q) \to (M^*; N_1^*, \ldots, N_q^*) \times \mathbf{R}^k$$

such that $\phi \mid M^* = id$, where Z^* denotes $Z \cap \varpi^{-1}(0)$ for a subset Z of M.

Furthermore, if previously given are Nash diffeomorphisms $\phi_{i_j} : N_{i_j} \to N_{i_j}^* \times \mathbf{R}^k$, $0 \leq i_1 < \cdots < i_a \leq q$, such that $\varpi \circ \phi_{i_j}^{-1}$ is the natural projection, and $\phi_{i_s} = \phi_{i_t}$ on $N_{i_s} \cap N_{i_t}$, then we can choose the above Nash diffeomorphism ϕ which satisfies $\phi \mid N_{i_j} = \phi_{i_j}$, $j = 1, \ldots, a$.

Remark (2.2). *In Theorem* (2.1), *we can replace* $\mathbf{R}^k$ *by one of the followings*:
(1) *an open cuboid* $\prod_{i=1}^{k}(a_i, b_i)$,
(2) *a closed cuboid* $\prod_{i=1}^{k}[a_i, b_i]$,
(3) *a Nash manifold which is Nash diffeomorphic to an open simplex.*

The above theorem is an effective tool to show modified Nash triviality for a family of zero-sets of polynomial (or Nash) mappings with isolated singularities. Therefore this is also useful to show finite modified Nash V-determinacy.

Let $\mathcal{N}(n,p)$ denote the set of Nash map-germs : $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, and let $\mathcal{A}(n,p)$ denote the set of analytic map-germs : $(\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$.

Definition (2.3). We say that two map-germs f_1,$f_2 \in \mathcal{N}(n,p)$ are *modified Nash V-equivalent* (resp. *modified Nash equivalent*), if there are two Nash modifications $\pi_1 : M_1 \to \mathbf{R}^n$, $\pi_2 : M_2 \to \mathbf{R}^n$, and a Nash diffeomorphism $\Phi : (W_1, \pi_1^{-1}(0)) \to (W_2, \pi_2^{-1}(0))$ which induces a homeomorphism $\phi : (U_1, 0) \to (U_2, 0)$ such that $\phi((U_1, f_1^{-1}(0) \cap U_1)) = (U_2, f_2^{-1}(0) \cap U_2)$ (resp. $f_1 = f_2 \circ \phi$), where W_1, W_2 are semialgebraic neighborhoods of $\pi_1^{-1}(0)$ in M_1 and $\pi_2^{-1}(0)$ in M_2, respectively, and U_1, U_2 are neighborhoods of 0 in $\mathbf{R}^n$.

Remark (2.4). *Similarly, we can define the notions of modified* C^k V*-equivalence and modified* C^k *equivalence for elements of* $\mathcal{A}(n,p)$.

Definition (2.5). We say that $f \in \mathcal{N}(n,p)$ is *finitely modified Nash V-determined*, if there is a positive integer r such that any $g \in \mathcal{N}(n,p)$ with $j^r g(0) = j^r f(0)$ is modified Nash V-equivalent to f.

We have a criterion for finite modified Nash V-determinacy of polynomial map-germs. We don't give the proof here.

Theorem I. *For a polynomial map-germ* $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^p, 0)$, *the following conditions are equivalent.*
(1) f *is finitely modified Nash* V*-determined.*
(2) $f^{-1}(0) \cap S(f) \subset \{0\}$ *as germs at* $0 \in \mathbf{R}^n$.

Remark (2.6). *Let* $f \in \mathcal{A}(n,p)$. *Originally, J. Bochnak and T. C. Kuo* ([1]) *proved that condition* (2) *in Theorem* I *is equivalent to finite* V*-determinacy of* f. *This was improved by M. Buchner and W. Kucharz* ([4]). *Precisely, they proved that condition* (2) *is euivalent to finite modified* C^k V*-determinacy of* f *where* k *is a positive integer.*

Next we consider the problem of sufficiency of jets. We don't distinguish r-jets from their polynomial representatives of degree not exceeding r. Recently, T. Fukui has introduced some invariant for modified C^ω equivalence in the function case ([5]). Let N_∞ denote the set of positive integers and infinity, and let $\Lambda(\mathbf{R}^n, 0)$ denote the set of analytic arcs at $0 \in \mathbf{R}^n$, namely, the set of analytic maps $\lambda : [0, \epsilon) \to \mathbf{R}^n$ with $\lambda(0) = 0$, $\lambda(s) \neq 0$, $s > 0$.

Fukui's Invariant. *For* $f \in \mathcal{A}(n,1)$, *we define the following subset of* N_∞:

$$A(f) = \{O(f \circ \lambda) \mid \lambda \in \Lambda(\mathbf{R}^n, 0)\},$$

where $O(\phi)$ *denotes the order of* ϕ *for* $\phi \in \mathcal{A}(1,1)$.

(2.7) *If* f, $g \in \mathcal{A}(n,1)$ *are modified* C^ω *equivalent, then* $A(f) = A(g)$.

Example (2.8). *Let $w = x^3 + 3xy^6 \in J^7(2,1)$. By the Kuiper-Kuo theorem, w is C^0-sufficient (see Theorem (2.11) and Remark (2.13) below). Let $f(x,y) = x^3 + 3xy^6 + y^8$. It is easy to see that $8 \notin A(w)$. On the other hand, $8 \in A(f)$. By (2.7), f is not modified C^ω-equivalent to w. Therefore w is not modified C^ω-sufficient.*

Remark that in the above example, w satisfies the Kuiper-Kuo condition as a real 7-jet, but w does not do as a complex 7-jet. Let v be a complex r-jet, and let $h : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ be a holomorphic function such that $j^r h(0) = v$. Define $F : (\mathbf{C}^n \times \Omega, \{0\} \times \Omega) \to (\mathbf{C}, 0)$ by $F(z;t) = (1-t)v(z) + th(z)$, where Ω is an open ball in $\mathbf{C}$ containing the interval $[0,1]$. For $t \in \Omega$, let $f_t : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ denote the function defined by $f_t(z) = F(z;t)$, and set $V_t = f_t^{-1}(0)$. Suppose that v satisfies the Kuiper-Kuo condition. Then $F^{-1}(0)$ satisfies the Kuo's Ratio Test ([13]) along $\{0\} \times \Omega$. This condition is equivalent to the condition that $\mu^*(V_t)$ is constant (see J. Briançon-J. P. Speder [3], J. P. Henry-M. Merle [7], B. Teissier [17]). Therefore it follows from the main result in H. B. Laufer [15] that in the case of surface singularities, $F^{-1}(0)$ admits a *strong simultaneous resolution* over Ω in the sense of B. Teissier [18]. Then we have the following question:

Question (2.9). *Let $w \in J^r(n,1)$ be a real r-jet. Suppose that w satisfies the Kuiper-Kuo condition as a complex jet. Then, is w modified C^ω V-sufficient (or modified C^ω-sufficient)?*

Let $w \in J^r(n,1)$, and let $f \in \mathcal{A}(n,1)$ (or $\mathcal{N}(n,1)$) with $j^r f(0) = w$. After this, let $f_t : (\mathbf{R}^n, 0) \to (\mathbf{R}, 0)$ denote the function defined by $f_t(x) = (1-t)w(x) + tf(x)$ for $t \in I = [0,1]$. Concerning the above question, T. C. Kuo has proved

Theorem (2.10) ([14]). *Let $w \in J^r(2,1)$ be a real r-jet. Suppose that w satisfies the Kuiper-Kuo condition as a complex jet. Then w is modified C^ω sufficient in $\mathcal{A}(2,1)$. Furthermore, $\{f_t\}_{0 \le t \le 1}$ is modified C^ω trivial along I.*

The author and T. C. kuo have proved the following fact on Lojasiewicz inequalities.

Theorem (2.11). *Let r be a positive integer. For $f \in \mathcal{A}(n,1)$, the following conditions are equivalent.*

(1) (The Kuiper-Kuo condition.) *There are $c, \alpha > 0$ such that*

$$|grad\ f(x)| \ge c|x|^{r-1} \text{ for } |x| < \alpha.$$

(2) (The Thom condition.) *There are $K, A > 0$ such that*

$$\sum_{i<j} |x_i \frac{\partial f}{\partial x_j} - x_j \frac{\partial f}{\partial x_i}|^2 + |f(x)|^2 \ge K|x|^{2r} \text{ for } |x| < A.$$

Remark (2.12). *It is easy to see that Theorem (2.11) holds for a C^r function f.*

Remark (2.13). *Suppose that $w \in J^r(n,1)$ satisfies the Kuiper-Kuo condition. Then w is C^0-sufficient in C^r functions. This is well-known as the Kuiper-Kuo theorem ([11],[12]). The converse is also true (J. Bochnak and S. Lojasiewicz [2]). At the almost same time as Kuiper and Kuo, R. Thom showed if $w \in J^r(n,1)$ satisfies the Thom condition, then w is C^0-sufficient. The Kuiper-Kuo condition implies no coalescing of critical points of*

$\{f_t\}_{0\leq t\leq 1}$ in the sense of King ([8]) for any realization of w. On the other hand, the Thom condition implies that the Milnor radii of $\{f_t^{-1}(0)\}_{0\leq t\leq 1}$ are uniformly positive. Therefore it seems that the Thom condition is stronger than the Kuiper-Kuo condition on the surface. But it follows from the above fact that Thom's result is equivalent to the Kuiper-Kuo theorem.

By using Theorems (2.1), (2.10) and (2.11), we can show

Theorem II. *Let $w \in J^r(2,1)$ be a real r-jet. Suppose that w satisfies the Kuiper-Kuo condition as a complex jet. Then w is modified Nash V-sufficient in $\mathcal{N}(2,1)$.*

Question (2.14). *Does Theorem* II *hold for general variables case?*

In the same way as Theorem II, we can reduce the above question to the following

Question (2.15). *Let $w \in J^r(n,1)$ be a real r-jet. Suppose that w satisfies the Kuiper-Kuo condition as a comlex jet. Then is w modified C^1 V-sufficient in $\mathcal{N}(n,1)$? Furthermore, is $\{f_t^{-1}(0)\}_{0\leq t\leq 1}$ modified C^1 V-trivial along I as set-germs, not as embedded varieties?*

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