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ON THE CLASSIFICATION OF SMOOTH CURVES OF GENUS $g = 3,4,5,6$ WITH ONE PLACE AT INFINITY.

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§1. Introduction.

We consider a smooth affine curve $C = \{f(x,y) = 0\} \subset \mathbf{C}^2$ of degree n with one place at infinity, say at $\rho = (1;0;0)$ and let g be the genus of the smooth compactification of C. By the assumption, $f(x,y)$ is written as

$$f(x,y) = (y^{a_1} + \xi_1 x^{c_1})^{A_2} + (\text{lower terms}), \quad \xi_1 \in \mathbf{C}^*, \ c_1 < a_1, n = a_1 A_2 \tag{1.1}$$

where a_1, c_1, A_2 are integers and $\gcd(a_1, c_1) = 1$.
The purpose of this note is classify the possible normal forms for a given genus g, $g \leq 6$. We use the following result of A'Campo-Oka [AO]. Let $\bar{C}$ be the projective compactification of C.

Theorem (1.2). *There is a canonical factorization $A_i = a_i a_{i+1} \cdots a_k$ and a resolution tower of $(\bar{C}, \rho)$,$\mathcal{T}$,of toric modifications*

$$\mathcal{T} = \{ X_k \xrightarrow{p_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{p_1} X_0 = \mathbf{C}^2 \}$$

with the corresponding weight vectors $P_i = {}^t(a_i, b_i)$ for $i = 1, \ldots, k$ ($b_1 = a_1 - c_1$) which satisfies the following conditions. Let $h_i(x,y)$ be the A_{i+1}-th Tschirnhausen approximate polynomial of $f(x,y)$ as a polynomial of y and let $C_i = \{(x,y) \in \mathbf{C}^2; h_i(x,y) = 0\}$ for $i = 1, \ldots, k$.
Note that $\deg C_i = a_1 \cdots a_i$, $h_k = f$ and $C_k = C$.
(1) For each $i = 1, \ldots, k$,$\bar{C}_i$ passes through ρ and $(\bar{C}_i, \rho)$ is irreducible at ρ and $\Phi_i = p_1 \circ \cdots \circ p_i : X_i \to X_0$ gives a minimal resolution tower of $(\bar{C}_i, \rho)$.
(2) Milnor number $\mu(\bar{C}_i, \rho)$ is given by

$$\mu(\bar{C}_i, \rho) = 1 - A_1 + \sum_{s=1}^{k} (A_s - 1)\, b_s A_{s+1} \tag{1.2.1}$$

(3) The local intersection multiplicity $I(\bar{C}_i, \bar{C}; \rho)$ is given by

$$I(\bar{C}_i, \bar{C}; \rho) = \sum_{s=1}^{i+1} a_s b_s A_{s+1}^2 / A_{i+1}, \ i \leq k-1 \tag{1.2.2}$$

Using the modified Plücker formula and (1.2.1), we have the equality ((a_g), §8, [AO])

$$\sum_{i=1}^{k} (A_i - 1)\, b_i A_{i+1} = (A_1 - 1)^2 - 2g \tag{1.3}$$

By Bezout theorem and (1.2.2),we have the inequality $((b), \S 8, [AO])$

$$\sum_{i=1}^{k} a_i b_i A_{i+1}^2 \leq A_1^2 \tag{1.4}$$

§2. Main result.

Theorem (2.1). *C: a smooth curve in $\mathbf{C}^2$, homeomorphic to a surface with one puncture of genus $g = 3,4,5,6$. Then there exist an automorphism of C^2 moving the curve C to a curve which is one of the following models.*

g=3: a) n=4,$P_1 = (4,1)$, smooth at infinity, tangent to the line at infinity at a single point. An example is given by $\{y^4 + x^3 + 1 = 0\}$.
b) n=7,$P_1 = (7,5)$. The curve has a non-degenerate cusp singularity at infinity. An example is given by $\{y^7 + x^2 + 1 = 0\}$.
c) k=2,n=6,$P_1 = (3,1), P_2 = (2,9)$. An example is given by $\{(y^3 + x^2)^2 + x = 0\}$.

g=4: a) k=1,n=5,$P_1 = (5,2)$. $\{y^5 + x^3 + 1 = 0\}$.
b) k=1,n=9,$P_1 = (9,7)$. $\{y^9 + x^2 + 1 = 0\}$.
c) k=2,n=6,$P_1 = (3,1), P_2 = (2,7)$. $\{(y^3 + x^2)^2 + xy + 1 = 0\}$.
d) k=2,n=9,$P_1 = (3,1), P_2 = (3,16)$. $\{(y^3 + x^2)^3 + y = 0\}$.

g=5: a) k=1,n=11,$P_1 = (11,9)$. $\{y^{11} + x^2 + 1 = 0\}$.
d) k=2,n=6,$P_1 = (3,1), P_2 = (2,5)$. $\{(y^3 + x^2)^2 + xy^2 + 1 = 0\}$.

g=6: a) k=1,n=5,$P_1 = (5,1)$. $\{y^5 + x^4 + 1 = 0\}$.
b) k=1,n=7,$P_1 = (7,4)$. $\{y^7 + x^3 + 1 = 0\}$.
c) k=1,n=13,$P_1 = (13,11)$. $\{y^{13} + x^2 + 1 = 0\}$.
d) k=2,n=6,$P_1 = (3,1), P_2 = (2,3)$. $\{(y^3 + x^2)^2 + x^3 + 1 = 0\}$.
e) k=2,n=10,$P_1 = (5,3), P_2 = (2,15)$. $\{(y^5 + x^2)^2 + x = 0\}$.
f) k=2,n=9,$P_1 = (3,1), P_2 = (3,14)$. $\{(y^3 + x^2)^3 + y^2 + 1 = 0\}$.

Proof. If necessary,applying the Jung automorphisms:

$$\phi : \mathbf{C}^2 \to \mathbf{C}^2,\ \phi(x,y) = (y^{a_1} + \xi_1 x, y),$$

we can assume that $a_1 > c_1 \geq 2$. If k=1,then we have $(a_1 - 1)(c_1 - 1) = 2g$ by (1.3), hence,

$$a_1 > c_1 = 1 + \frac{2g}{a_1 - 1}.$$

Using the above inequality and $\gcd(a_1, b_1) = 1$, we can get the preceding results in the case of $k = 1$. So,we consider the case $k \geq 2$. In this case, using that $(1 - A_2) \times (1.4) + A_2 \times (1.3)$, we can get the following inequality $((\star), \S 8, [AO])$:

$$A_2 \leq \frac{2g - 1}{(a_1 - 1)(c_1 - 1) - 1} \leq 2g - 1 \tag{2.2}$$

$g = 3$: (The result of this case is given in [AO] without proof.) By (2.2), $A_2 = 2,3,4,5$. ($A_2 = 1$ if and only if $k = 1$)

If $A_2 = 5$, $(a_1 - 1)(c_1 - 1) - 1 = 1$ by (2.2). Hence, $k = 2, a_1 = 3, c_1 = 2, b_1 = 1, a_2 = A_2 = 5, n = A_1 = 15$. By (1.3), we have $b_2 = 30$. This contradicts $\gcd(a_2, b_2) = 1$.
If $A_2 = 4$, then $a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2).

(i) $k = 2, a_2 = A_2 = 4, n = A_1 = 12$. By (1.3), $b_2 = 71/3$. This contradicts the fact that b_2 is a integer.

(ii) $k = 3, n = A_1 = 12, a_2 = 2, a_3 = A_3 = 2$. By (1.3),

$$6b_2 + b_3 = 71, \tag{1}$$

hence,

$$b_2 = \frac{71 - b_3}{6} < \frac{71}{6} < 12. \tag{2}$$

By (b),

$$4b_2 + b_3 \leq 48. \tag{3}$$

Using (1) and (3) , we get $2b_2 \geq 23$, hence, $b_2 \geq 12$. This contradicts (2).

If $A_2 = 3$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2). and $a_2 = A_2 = 3, n = A_1 = 9$. By (1.3), $b_2 = 17$. Thus the tower has the weight vectors $P_1 = (3, 1), P_2 = (3, 17)$. We shall show that there is no polynomial $f(u, v)$ of degree 9 with the weight vectors above. Let

$$f(u, v) = (v^3 + u)^3 + \sum_{\alpha,\beta} c_{\alpha,\beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 9,\ 9 < 3\alpha + \beta. \tag{4}$$

We consider an admissible toric modification $p : X_1 \to \mathbf{C}^2$. We may assume that $\sigma = (E_1, P_1), E_1 = (1, 0)$, is the left toric cone of the divisor $E(P_1)$ and let (s, t) be the toric coordinates. Then $u = st^3, v = t$. Hence,

$$\begin{aligned}\pi_\sigma^* f(s,t) &= t^9(1+s)^3 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta} \\ &= t^9 \left\{(1+s)^3 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta-9}\right\}.\end{aligned}$$

By (4), there is no (α, β) such that $3\alpha + \beta - 9 = 17$. Therefore $P_2 = (3, 17)$ is not the second weight vector for $f(u, v)$. Thus this case does not occur.
If $A_2 = 2$, then $(a_1 - 1)(c_1 - 1) - 1 = 1$ or 2 by (2.2).

(i) If $(a_1 - 1)(c_1 - 1) - 1 = 2$, then $a_1 = 4, c_1 = 2, b_1 = 2$. This contradicts $\gcd(a_1, b_1) = 1$.

(ii) If $(a_1 - 1)(c_1 - 1) - 1 = 1$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$, and $a_2 = A_2 = 2, n = A_1 = 6$. By (a_g), $b_2 = 9$. Thus the tower has the weight vectors $P_1 = (3, 1), P_2 = (2, 9)$. Let

$$f(u, v) = (v^3 + u)^2 + \sum_{\alpha,\beta} c_{\alpha,\beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 6,\ 6 < 3\alpha + \beta.$$

Using the preceding admissible toric modification,

$$\begin{aligned}\pi_\sigma^* f(s,t) &= t^6(1+s)^2 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta} \\ &= t^6 \left\{(1+s)^2 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta-6}\right\}.\end{aligned}$$

If $\alpha = 5, \beta = 0$, then $3\alpha + \beta - 6 = 9$. So if $c_{5,0} \neq 0$, the second weight vector for $f(u, v)$ can be $P_2 = (2, 9)$. For example, let $f(u, v) = (v^3 + u)^2 + u^5$. Then $F(x, y) = (y^3 + x^2)^2 + x$, which is non-singular in $\mathbf{C}^2$.

$g=4$: By (2.2), $A_2 = 2,3,4,5,6,7$. If $A_2 = 7$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2), and $a_2 = A_2 = 7, n = A_1 = 21$. By (1.3), $b_2 = 42$. This contradicts $\gcd(a_2, b_2) = 1$.
If $A_2 = 6$, then $a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2).

(i) $k = 2, a_2 = A_2 = 6, n = A_1 = 18$. By (1.3), $b_2 = 179/5$. This contradicts the fact that b_2 is a integer.

(ii) $k = 3, n = A_1 = 18, a_2 = 2$ or 3.
If $a_2 = 2, then a_3 = A_3 = 3$. By (1.3),

$$15b_2 + 2b_3 = 179, \tag{5}$$

hence,

$$b_2 = \frac{179 - 2b_3}{15} < \frac{179}{15} < 12. \tag{6}$$

By (b),

$$6b_2 + b_3 \leq 72. \tag{7}$$

Using (5) and (7) , $3b_2 \geq 35$, hence, $b_2 \geq 12$. This contradicts (6).
If $a_2 = 3, a_3 = A_3 = 2$. By (1.3),

$$10b_2 + b_3 = 179, \tag{8}$$

hence,

$$b_2 = \frac{179 - b_3}{10} < 18. \tag{9}$$

By (b),

$$6b_2 + b_3 \leq 108. \tag{10}$$

Using (8) and (10) , $4b_2 \geq 71$, hence, $b_2 \geq 18$. This contradicts (9).

If $A_2 = 5$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2), and $a_2 = A_2 = 5, n = A_1 = 15$. By (1.3), $b_2 = 59/2$. This contradicts the fact that b_2 is a integer.
If $A_2 = 4$, then $a_1 = 3, c_1 = 2, b_1 = 1$ by (2.2).

(i) $k = 2, a_2 = A_2 = 4, n = A_1 = 12$. By (1.3),$b_2 = 23$. Thus the tower hasf the weight vectors $P_1 = (3,1), P_2 = (4,23)$. Let

$$f(u,v) = (v^3 + u)^4 + \sum_{\alpha,\beta} c_{\alpha,\beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 12,\ 12 < 3\alpha + \beta. \tag{11}$$

Using the preceding admissible toric modification:$u = st^3, v = t$,

$$\begin{aligned} \pi_\sigma^* f(s,t) &= t^{12}(1+s)^4 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta} \\ &= t^{12}\left\{(1+s)^4 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta-12}\right\}. \end{aligned}$$

By (11), there is no (α, β) such that $3\alpha + \beta - 12 = 23$. Therefore $P_2 = (4,23)$ is not the second weight vector for $f(u,v)$. Thus this case does not occur.

(ii) $k = 3, n = A_1 = 12, a_2 = 2, a_3 = A_3 = 2$. By (1.3),

$$6b_2 + b_3 = 69, \tag{12}$$

hence,

$$b_2 = \frac{69 - b_3}{6} < \frac{69}{6} < 12. \tag{13}$$

By (b),

$$4b_2 + b_3 \leq 48. \tag{14}$$

Using (12) and (14) , $2b_2 \geq 21$, hence, $b_2 \geq 11$. By this inequality and (13), we can conclude that $b_2 = 11$. And $b_3 = 3$ by (12). Thus the tower has the weight vectors $P_1 = (3,1), P_2 = (2,11), P_3 = (2,3)$. Let

$$f(u,v) = (v^3 + u)^4 + (\text{higher terms}).$$

Then

$$h_1(u,v) = (v^3 + u) + (\text{higher terms}),$$

$$h_2(u,v) = (v^3 + u)^2 + (\text{higher terms}),$$

where h_i is A_{i+1}-th Tschirnhausen approximate polynomial of f. Since h_1 is the 2-th Tschirnhausen approximate polynomial of h_2, $h_2(u,v)$ is written as

$$h_2(u,v) = h_1(u,v)^2 + \sum_{\alpha,\beta} c_{\alpha,\beta} u^\alpha v^\beta$$

$$\beta \leq 2,\ \alpha + \beta \leq 6,\ 6 < 3\alpha + \beta. \tag{15}$$

Using the preceding admissible toric modification:$u = st^3, v = t$,

$$\pi_\sigma^* h_1(s,t) = t^3\{(1+s) + \dots\}.$$

Hence

$$p_1^* h_1(u_1, v_1) = {u_1}^3 v_1,$$

$$p_1^* h_2(u_1, v_1) = {u_1}^6 {v_1}^2 + p_1^*(\sum c_{\alpha,\beta} u^\alpha v^\beta).$$

And now, by $P_2 = (2,11)$ we have

$$p_1^* h_2(u_1, v_1) = {u_1}^6({v_1}^2 + {u_1}^{11}) + (\text{higher terms}).$$

Therefore,the monomial ${u_1}^{17}$ must exist in $p_1^*(\sum c_{\alpha,\beta} u^\alpha v^\beta)$. Though

$$\pi_\sigma^*(\sum c_{\alpha,\beta} u^\alpha v^\beta) = \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta}),$$

by (15) there is no (α, β) such that $3\alpha + \beta = 17$. Therefore we find that $P_2 = (2,11)$ is not the second weight vector for $f(u,v)$. Thus this case does not occur.

If $A_2 = 3$, then $(a_1 - 1)(c_1 - 1) - 1 = 1$ or 2 by (2.2). Since $\gcd(a_1, b_1) = 1$, $(a_1 - 1)(c_1 - 1) - 1 \neq 2$. Therefore $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$, and $a_2 = A_2 = 3, n = A_1 = 9$, By (1.3), $b_2 = 16$. Thus the tower has the weight vectors $P_1 = (3,1), P_2 = (3,16)$. Let

$$f(u,v) = (v^3 + u)^3 + \sum_{\alpha,\beta} c_{\alpha,\beta} u^\alpha v^\beta$$

$$\alpha + \beta \leq 9,\ 9 < 3\alpha + \beta.$$

Using the preceding admissible toric modification,

$$\begin{aligned}\pi_\sigma^* f(s,t) &= t^9(1+s)^3 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta} \\ &= t^9 \left\{(1+s)^3 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta-9}\right\}.\end{aligned}$$

If $\alpha = 8, \beta = 1$, then $3\alpha + \beta - 9 = 16$. So if $c_{8,1} \neq 0$, the second weight vector for $f(u,v)$ can be $P_2 = (3,16)$. For example, let $f(u,v) = (v^3+u)^3 + u^8 v$. Then $F(x,y) = (y^3+x^2)^3 + y$, which is non-singular in $\mathbf{C}^2$.

If $A_2 = 2$, then $(a_1-1)(c_1-1)-1 = 1,2,3$ by (2.2). Since $\gcd(a_1,b_1) = 1$, $(a_1-1)(c_1-1)-1 \neq 2$.

(i) If $(a_1-1)(c_1-1)-1 = 1$, then $k = 2, a_1 = 3, c_1 = 2, b_1 = 1$, and $a_2 = A_2 = 2, n = A_1 = 6$. By (1.3), $b_2 = 7$. Thus the tower has the weight vectors $P_1 = (3,1), P_2 = (2,7)$. Let

$$f(u,v) = (v^3+u)^2 + \sum_{\alpha,\beta} c_{\alpha,\beta} u^\alpha v^\beta$$

$$\alpha+\beta \leq 6, \ 6 < 3\alpha+\beta.$$

Using the preceding admissible toric modification:$u = st^3, v = t$,

$$\begin{aligned}\pi_\sigma^* f(s,t) &= t^6(1+s)^2 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta} \\ &= t^6 \left\{(1+s)^2 + \sum c_{\alpha,\beta} s^\alpha t^{3\alpha+\beta-6}\right\}.\end{aligned}$$

If $\alpha = 4, \beta = 1$, then $3\alpha+\beta-6 = 7$. So if $c_{4,1} \neq 0$, the second weight vector for $f(u,v)$ can be $P_2 = (2,7)$. For example, let $f(u,v) = (v^3+u)^2+u^4v+u^6$. Then $F(x,y) = (y^3+x^2)^2+xy+1$, which is non-singular in $\mathbf{C}^2$.

(ii) If $(a_1-1)(c_1-1)-1 = 3$, then $k = 2, a_1 = 5, c_1 = 2, b_1 = 3$, and $a_2 = A_2 = 2, n = A_1 = 10$. By (1.3), $b_2 = 19$. Thus the tower has the weight vectors $P_1 = (5,3), P_2 = (2,19)$. Let

$$f(u,v) = (v^5+u^3)^2 + \sum_{\alpha,\beta} c_{\alpha,\beta} u^\alpha v^\beta$$

$$\alpha+\beta \leq 10, \ 30 < 5\alpha+3\beta. \tag{16}$$

We may assume that $\sigma = (Q_1, P_1), Q_1 = (2,1)$, is the left toric cone of the divisor $E(P_1)$ and let (s,t) be the toric coordinates. Then $u = s^2t^5, v = st^3$. Hence

$$\begin{aligned}\pi_\sigma^* f(s,t) &= s^{10}t^{30}(1+s)^2 + \sum c_{\alpha,\beta} s^{2\alpha+\beta} t^{5\alpha+3\beta} \\ &= t^{30} \left\{s^{10}(1+s)^2 + \sum c_{\alpha,\beta} s^{2\alpha+\beta} t^{5\alpha+3\beta-30}\right\}.\end{aligned}$$

By (16), there is no (α,β) such that $5\alpha+3\beta-30 = 19$. Therefore $P_2 = (2,19)$ is not the second weight vector for $f(u,v)$. Thus this case does not occur.

The cases $g = 5,6$ are proved likewise. □

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