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On the number of pyramids of a generic space curve

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Abstract. We prove that for a generic closed space curve, the number of pyramids (triple points of the tangent developable) is congruent modulo 2 to the sum of the indices of the torsion zero points. This index is defined as the number of trisecant lines of the curve passing through the torsion zero point. The result is deduced from the study of the singularities of the tangent developable surface of the curve.

1. Introduction

In this paper we study the number of triple points of the tangent developable of a space curve. The tangent developable of a space curve $\alpha : S^1 \to \mathbf{R}^3$ is the surface $\chi(\alpha)$ in $\mathbf{R}^3$ defined by the tangent lines of α. The local form of this surface was first studied by Cleave [**C**], and recently the first author [**N**] proved that it is a topologically stable surface when the curve is generic (for a definition of a topologically stable surface, see [**IM**]). He obtained as a consequence that if the curve has no torsion zero points (topological cross caps of $\chi(\alpha)$), then the number of pyramids of the curve (triple points of $\chi(\alpha)$) is even.

Here, we give one step more in this direction. The above mentioned result can be easily generalized by applying a theorem by Szücs [**Sz**], which gives the following congruence:

$$T(\alpha) \equiv \sum_{i=1}^{k} n(x_i, \alpha) \mod 2.$$

The number $T(\alpha)$ is the number of pyramids of α, $x_1, \cdots, x_k$ are the torsion zero points of α and $n(x_i, \alpha)$ is the index of each torsion zero point conveniently defined (see also [**NS**]). The problem is that this index, $n(x_i, \alpha)$, does not give a priori any information on the geometry of the curve. We will show, by using a new proof of the Szücs theorem given by the authors in [**NS**], that the index $n(x_i, \alpha)$ can be interpreted geometrically as the number of trisecant lines of α passing through x_i.

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Note that this result can be considered dual to the congruence obtained by Banchoff, Gaffney and McCrory in [**BGM**], where they show that for a generic space curve $\alpha : S^1 \to \mathbf{R}^3$ we have

$$\widetilde{T}(\alpha) \equiv \sum_{i=1}^{k} \widetilde{n}(x_i, \alpha) \mod 2,$$

where now $\widetilde{T}(\alpha)$ is the number of tritangent planes of α and the index $\widetilde{n}(x_i, \alpha)$ is one half of the number of points in the intersection off the point x_i of the curve with the osculating plane at the torsion zero point x_i (see also [**O**]). Although their paper [**BGM**] is previous to the Szücs one, they implicitly use the Szücs result for the dual surface of the curve. Remember that the dual surface of a regular space curve is the surface in $(\mathbf{R}P^3)^*$ defined by the tangent planes to α. When the curve is generic, the dual surface is again a topologically stable surface, the triple points corresponding to the tritangent planes to α and the cross caps to the torsion zero points (see [**BGM**] for details and [**Sch**] for the duality between the dual surface and the tangent developable of a generic space curve).

2. The number of pyramids of a generic space curve

Suppose that $\alpha : S^1 \to \mathbf{R}^3$ is a smooth space curve satisfying the general position conditions (GP) given in [**N**]. We also assume the conditions (1) and (2) in [**N**, Lemma 7]. Furthermore, we assume the additional conditions as follows:

(i) If $\tau(s) = 0$, then there are no quadrisecants of α passing through $\alpha(s)$, where τ is the torsion of α.

(ii) If there is a trisecant to α at $\alpha(s_0), \alpha(s_1), \alpha(s_2)$ with $\tau(s_0) = 0$, then the vectors $\alpha(s_1) - \alpha(s_0), \alpha'(s_1)$ and $\alpha'(s_2)$ are linearly independent.

(iii) The tritangent planes of α do not osculate at the tangency points.

(iv) The number of osculating planes of α containing a trisecant is finite.

(v) The subset $\mathcal{T}$ of $S^1 \times S^1 \times S^1 - \Delta$ defined by

$$\mathcal{T} = \{(s_1, s_2, s_3) : (\alpha(s_2) - \alpha(s_1)) \times (\alpha(s_3) - \alpha(s_1)) = 0\},$$

is a closed 1-dimensional submanifold of $S^1 \times S^1 \times S^1 - \Delta$, where

$$\Delta = \{(s_1, s_2, s_3) : s_i = s_j \text{ for some } i \neq j\}.$$

Lemma 2.1. *The set of smooth curves α satisfying the above general position conditions is residual in the space $C^\infty(S^1, \mathbf{R}^3)$, with the Whitney C^∞-topology.*

Proof: We consider the multijet space ${}_4J^3(S^1, \mathbf{R}^3)$ and take the coordinates

$$(s_1, s_2, s_3, s_4, r_1^0, r_2^0, r_3^0, r_4^0, \ldots, r_1^3, r_2^3, r_3^3, r_4^3),$$

where $s_i \in S^1$ and $r^i_j \in \mathbf{R}^3$. Then we define the subset $W_1 \subset {}_4J^3(S^1, \mathbf{R}^3)$ by the equations:

$$(r^0_2 - r^0_1) \times (r^0_3 - r^0_1) = (r^0_2 - r^0_1) \times (r^0_4 - r^0_1) = 0,$$
$$\det(r^1_1, r^2_1, r^3_1) = 0.$$

Clearly, W_1 is an algebraic subset of ${}_4J^3(S^1, \mathbf{R}^3)$ of codimension 5. By the multi-jet version of the Thom transversality theorem, the set of curves $\alpha \in C^\infty(S^1, \mathbf{R}^3)$ such that ${}_4j^3\alpha : (S^1)^{(4)} \to {}_4J^3(S^1, \mathbf{R}^3)$ is transversal to W_1 is residual in $C^\infty(S^1, \mathbf{R}^3)$. But since $(S^1)^{(4)}$ has dimension 4, it is obvious that the transversality condition is equivalent to condition (i).

Analogously, we see that the rest of conditions (ii),...,(v) give residual subsets in $C^\infty(S^1, \mathbf{R}^3)$. ∎

LEMMA 2.2. *Let $p = \alpha(s_0)$ be a torsion zero point. If α satisfies the above general position conditions, then the number of trisecants of α passing through p is finite.*

The proof of Lemma 2.2 will be given later (see the paragraph just after the proof of Proposition 2.8).

DEFINITION 2.3. For a torsion zero point $p = \alpha(s_0)$ of α, we define the *index* $n(p, \alpha)$ ($\in \mathbf{Z}$) to be the number of trisecants of α passing through p.

Recall that if α satisfies the condition (GP) as in [N] (condition 3), then the number of torsion zero points is finite.

The main purpose of this section is to prove the following.

THEOREM 2.4. *Let $\alpha : S^1 \to \mathbf{R}^3$ be a smooth space curve satisfying the general position conditions as stated above. Then we have*

$$T(\alpha) \equiv \sum_{i=1}^{k} n(x_i, \alpha) \mod 2,$$

where $T(\alpha)$ is the number of pyramids of α and $x_1, \cdots, x_k$ are the torsion zero points of α.

LEMMA 2.5. *Let $f : (s_0 - \epsilon, s_0 + \epsilon) \to \mathbf{R}$ be a C^∞ function. Then there exists another C^∞ function $f_* : (s_0 - \epsilon, s_0 + \epsilon) \to \mathbf{R}$ such that if $s \neq s_0$, then*

$$f_*(s) = \frac{f(s) - f(s_0)}{s - s_0},$$

$$\frac{df_*}{ds}(s) = \frac{f'(s)(s - s_0) - (f(s) - f(s_0))}{(s - s_0)^2},$$

and if $s = s_0$,

$$f_*(s_0) = f'(s_0), \quad \frac{df_*}{ds}(s_0) = \frac{1}{2}f''(s_0).$$

PROOF: Just apply a classical argument of analysis. We see easily that

$$\begin{aligned} f(s) - f(s_0) &= \int_0^1 \frac{d}{dt}\{f(s_0 + t(s - s_0))\}dt \\ &= \int_0^1 f'(s_0 + t(s - s_0))(s - s_0)dt \\ &= (s - s_0)f_*(s), \end{aligned}$$

where $f_*(s) = \int_0^1 f'(s_0 + t(s - s_0))dt$. The rest is easy to check. ∎

Now let $\alpha : S^1 \to \mathbf{R}^3$ be a C^∞ curve which is regular and simple. Given a point $p \in \mathbf{R}^3$ we define $t_p : S^1 \to \mathbf{R}P^2$ by $t_p(s) = [\alpha(s) - p]$ if $\alpha(s) \neq p$, and $t_p(s) = [\alpha'(s)]$ if $\alpha(s) = p$. Note that t_p is a smooth map.

LEMMA 2.6. *Suppose that* $p = \alpha(s_0)$. *Then* t_p *is an immersion at* s_0 *if and only if* $\kappa(s_0) \neq 0$, *where* κ *is the curvature of* α.

PROOF: Suppose for instance that $\alpha_3'(s_0) \neq 0$, where $\alpha(t) = (\alpha_1(t), \alpha_2(t), \alpha_3(t))$. We take coordinates in $\mathbf{R}P^2$ such that the homogeneous coordinate $[X, Y, Z]$ corresponds to $(X/Z, Y/Z)$. Then t_p gives the map $\tilde{t}_p$ in a neighbourhood of s_0 given by

$$\tilde{t}_p(s) = \left(\frac{\alpha_1(s) - \alpha_1(s_0)}{\alpha_3(s) - \alpha_3(s_0)}, \frac{\alpha_2(s) - \alpha_2(s_0)}{\alpha_3(s) - \alpha_3(s_0)}\right),$$

when $s \neq s_0$ and

$$\tilde{t}_p(s) = \left(\frac{\alpha_1'(s_0)}{\alpha_3'(s_0)}, \frac{\alpha_2'(s_0)}{\alpha_3'(s_0)}\right),$$

when $s = s_0$. By Lemma 2.5, this map is

$$\tilde{t}_p(s) = \left(\frac{\alpha_{1*}(s)}{\alpha_{3*}(s)}, \frac{\alpha_{2*}(s)}{\alpha_{3*}(s)}\right),$$

for s in a neighbourhood of s_0. Then $\tilde{t}_p$ is differentiable at s_0 and

$$\tilde{t}_p'(s_0) = \frac{1}{2\alpha_3'(s_0)^2}(\alpha_1''(s_0)\alpha_3'(s_0) - \alpha_1'(s_0)\alpha_3''(s_0), \alpha_2''(s_0)\alpha_3'(s_0) - \alpha_2'(s_0)\alpha_3''(s_0)).$$

Therefore $\tilde{t}_p'(s_0) = 0$ if and only if $\alpha'(s_0) \times \alpha''(s_0) = 0$; i.e., if and only if $\kappa(s_0) = 0$. ∎

LEMMA 2.7. *Suppose that $p = \alpha(s_0)$. Then t_p is an immersion at $s \neq s_0$ if and only if $\alpha'(s), \alpha(s) - \alpha(s_0)$ are not collinear.*

PROOF: Taking coordinates as in the proof of Lemma 2.6, we have

$$\tilde{t}_p(s) = \left(\frac{\alpha_1(s) - \alpha_1(s_0)}{\alpha_3(s) - \alpha_3(s_0)}, \frac{\alpha_2(s) - \alpha_2(s_0)}{\alpha_3(s) - \alpha_3(s_0)} \right).$$

Hence we have that $\tilde{t}'_p(s) = 0$ if and only if $\alpha'(s) \times (\alpha(s) - \alpha(s_0)) = 0$. ∎

PROPOSITION 2.8. *If α is a smooth curve satisfying the general position conditions and $\tau(s_0) = 0$, then t_p is an immersion with normal crossings for $p = \alpha(s_0)$.*

PROOF: Since $\kappa(s_0) > 0$ and there are no cross tangents passing through $\alpha(s_0)$ (see the condition 7 of [N]), t_p is an immersion.

On the other hand, a self-intersection of t_p happens when $t_p(s_1) = t_p(s_2)$ for $s_1 \neq s_2$. Again the fact that there are no cross tangents passing through $\alpha(s_0)$ implies that $s_1, s_2 \neq s_0$; therefore $[\alpha(s_1) - \alpha(s_0)] = [\alpha(s_2) - \alpha(s_0)]$, i.e., there is a trisecant to α passing through $\alpha(s_0), \alpha(s_1), \alpha(s_2)$. By our condition (i) t_p has no triple points. We prove that condition (ii) implies the normal crossing condition at a double point, that is, that $t'_p(s_1)$ and $t'_p(s_2)$ are not collinear.

Since $\alpha(s_1) - \alpha(s_0), \alpha(s_2) - \alpha(s_0)$ are collinear, we can choose a coordinate which is not zero for both vectors, for instance, $\alpha_3(s_i) - \alpha_3(s_0) \neq 0$, $i = 1, 2$. Then taking coordinates as in the proof of Lemma 2.6, t_p gives the map

$$\tilde{t}_p(s) = \left(\frac{\alpha_1(s) - \alpha_1(s_0)}{\alpha_3(s) - \alpha_3(s_0)}, \frac{\alpha_2(s) - \alpha_2(s_0)}{\alpha_3(s) - \alpha_3(s_0)} \right),$$

for $s \neq s_0$ in a neighbourhood of s_1, s_2 and hence

$$\tilde{t}'_p(s_i) = \frac{1}{(\alpha_3(s_i) - \alpha_3(s_0))^2} (\alpha'_1(s_i)(\alpha_3(s_i) - \alpha_3(s_0)) - (\alpha_1(s_i) - \alpha_1(s_0))(\alpha'_3(s_i)),$$

$$\alpha'_2(s_i)(\alpha_3(s_i) - \alpha_3(s_0)) - (\alpha_2(s_i) - \alpha_2(s_0))(\alpha'_3(s_i))).$$

Now, if we set

$$\alpha(s_1) - \alpha(s_0) = (a_{11}, a_{12}, a_{13}),$$
$$\alpha(s_2) - \alpha(s_0) = \lambda(\alpha(s_1) - \alpha(s_0)) = \lambda(a_{11}, a_{12}, a_{13}),$$
$$\alpha'(s_1) = (a_{21}, a_{22}, a_{23}),$$
$$\alpha'(s_2) = (a_{31}, a_{32}, a_{33}),$$

then an easy (but tedious) computation gives

$$\det(\tilde{t}'_p(s_1), \tilde{t}'_p(s_2)) = \frac{1}{a_{13}^3 \lambda} \det(a_{ij}) \neq 0.$$

This completes the proof. ∎

Note that Lemma 2.2 is a direct consequence of Proposition 2.8, since $n(\alpha(s_0), \alpha)$ is equal to the number of double points of the immersion with normal crossings $t_p : S^1 \to \mathbf{R}P^2$.

Let $\alpha : S^1 \to \mathbf{R}^3$ be a smooth space curve which satisfies our general position conditions as above. We define $\chi_\alpha : S^1 \times \mathbf{R} \to \mathbf{R}^3$ by $\chi_\alpha(s,t) = \alpha(s) + t\alpha'(s)$. We denote by $\chi(\alpha)$ the tangent developable of α; i.e., $\chi(\alpha) = \chi_\alpha(S^1 \times \mathbf{R})$. Furthermore, we define the smooth map $\Phi : S^1 \times S^1 \times \mathbf{R} \to \mathbf{R}^3$ by

$$\Phi(s_1, s_2, t) = \frac{\alpha(s_1) + \alpha(s_2)}{2} + t\int_0^1 \alpha'(s_1 + \tau(s_2 - s_1))d\tau.$$

Note that, if $s_1 \neq s_2$, we have

$$\Phi(s_1, s_2, t) = \frac{\alpha(s_1) + \alpha(s_2)}{2} + t\frac{\alpha(s_2) - \alpha(s_1)}{s_2 - s_1}$$

and that, if $s_1 = s_2$, we have

$$\Phi(s_1, s_1, t) = \alpha(s_1) + t\alpha'(s_1).$$

Thus, for a point $q \in \mathbf{R}^3 - \chi(\alpha)$, t_q has a double point at s_1 and s_2 if and only if $\Phi(s_1, s_2, t) = q$ for some t.

Set $\mathcal{B}(\subset S^1 \times S^1)$ to be the bitangency set of α defined in [**NR**]. We know that $\mathcal{B}$ is a closed 1-dimensional submanifold of $S^1 \times S^1$. Define Ω to be $\Phi(\mathcal{B} \times \mathbf{R})$. Furthermore we define $T(\subset \mathbf{R}^3)$ to be the union of the trisecants of α. We see that T is of dimension 2 by our assumption (v). Note that Ω is also 2-dimensional and that the complements $\mathbf{R}^3 - \Omega$ and $\mathbf{R}^3 - T$ are open and dense in $\mathbf{R}^3$.

The following lemma can be proved by the same argument as in the proof of Proposition 2.8.

LEMMA 2.9. *For $p \in \mathbf{R}^3 - (\chi(\alpha) \cup \Omega \cup T)$ the map $t_p : S^1 \to \mathbf{R}P^2$ is an immersion with normal crossings.*

LEMMA 2.10. *Suppose that $p, p' \in \mathbf{R}^3 - (\chi(\alpha) \cup \Omega \cup T)$ and that they are in the same connected component of $\mathbf{R}^3 - \chi(\alpha)$. Then the number of double points of t_p has the same parity as that of $t_{p'}$.*

The above lemma is obvious, since t_p and $t_{p'} : S^1 \to \mathbf{R}P^2$ are regularly homotopic and the parity of the number of double points is an invariant of the regular homotopy class.

Remark. There are exactly four regular homotopy classes of immersions $f : S^1 \to \mathbf{R}P^2$, which are characterized by:

1) the $\mathbf{Z}_2$ class of f in $\pi_1(\mathbf{R}P^2)$;
2) the parity of the number of double points of $\tilde{f}$, where $\tilde{f}$ is an immersion with normal crossings that approximates f. Note that, for $p \in \alpha(S^1)$, t_p represents the nontrivial class of $\pi_1(\mathbf{R}P^2)$, while for $p \notin \alpha(S^1)$, the trivial class of $\pi_1(\mathbf{R}P^2)$.

DEFINITION 2.11. Let C be a connected component of $\mathbf{R}^3 - \chi(\alpha)$. Take a point $p \in C - (\Omega \cup T)$, which is non-vacuous. We say that C is a *blue region* (resp. *red region*) if the number of double points of t_p is odd (resp. even). Note that this does not depend on the choice of the point p by Lemma 2.10. Furthermore, define B (resp. R) to be the union of all blue (resp. red) regions of $\mathbf{R}^3 - \chi(\alpha)$.

LEMMA 2.12. $\bar{B} \cap \bar{R} = \partial B = \partial R = \chi(\alpha)$.

For the proof of Lemma 2.12, we need the following lemmas.

LEMMA 2.13. *$\chi(\alpha) - \Omega$ is dense in $\chi(\alpha)$.*

PROOF: Take a point $p \in \chi(\alpha) \cap \Omega$. We have only to show that there exists a point $q \in \chi(\alpha) - \Omega$ arbitrarily close to p. We may assume that p is a simple regular point of $\chi(\alpha)$. Suppose $p \in \chi(\alpha) \cap \Omega$. Note that $p = \Phi(s, s; t)$ for some $s \in S^1$ and $t \in \mathbf{R}^3$; in other words, $p = \chi_\alpha(s, t)$. First suppose that $(s, s) \in \mathcal{B}$. Then s is a torsion-zero point of α. Since α has only finitely many torsion-zero points, such a point p should lie in a 1-dimensional subspace of $\chi(\alpha)$. Thus we may assume that $p = \Phi(s_1, s_2; t')$ for some $(s_1, s_2) \in \mathcal{B}$ $(s_1 \neq s_2)$ and some $t' \in \mathbf{R}$. It is easily checked that the tangent space of Ω at p is spanned by $\{\alpha(s_2) - \alpha(s_1), \alpha'(s_i)\}$, where $i = 1$ or 2. On the other hand, the tangent space of $\chi(\alpha)$ at p is spanned by $\{\alpha'(s), \alpha''(s)\}$. Thus, if $\Phi|(\mathcal{B} \times \mathbf{R})$ and χ_α are not transverse at p, then there exists a plane P tangent to α at s_1, s_2 and s which osculates at $\alpha(s)$. Suppose that $s = s_i$. Then P is a bitangent osculating plane of α. Since there are only finitely many pairs $(t_1, t_2) \in \mathcal{B}$ $(t_1 \neq t_2)$ corresponding to bitangent osculating planes ([**NR**]), the point $p = \Phi(s_1, s_2; t')$ is in a 1-dimensional set. Hence, we may assume that s, s_1 and s_2 are all distinct. This contradicts to the condition (iii). Thus $\Phi|(\mathcal{B} \times \mathbf{R})$ and χ_α are transverse at p. Hence the intersection of Ω and $\chi(\alpha)$ at p is of 1-dimension. This completes the proof. ∎

LEMMA 2.14. *$\chi(\alpha) - T$ is dense in $\chi(\alpha)$.*

PROOF: Take a point p in $\chi(\alpha) \cap T$. We will find a point $q \in \chi(\alpha) - T$ which is arbitrarily close to p. We may assume that p is a simple regular point of $\chi(\alpha)$. Define

$$\Theta : T \times \mathbf{R} \to \mathbf{R}^3$$

by

$$\Theta(s_1, s_2, s_3; t) = t\frac{\alpha(s_2) - \alpha(s_1)}{s_2 - s_1} \left(= t \int_0^1 \alpha'(s_1 + \sigma(s_2 - s_1)) d\sigma \right).$$

Note that Θ is a smooth map of a 2-dimensional manifold (see our condition (v)) and that $\Theta(\mathcal{T}) = T$.

Let V be the union of the tangent lines of α at the points where the osculating plane contains a trisecant. By our condition (iv), it is a finite union of lines. Thus we may assume that $p \in \chi(\alpha) - V$.

Now suppose that, for a point $(s_1, s_2, s_3; t) \in \mathcal{T} \times \mathbf{R}$, $p = \Theta(s_1, s_2, s_3; t)$ and that Θ is *not* transverse to $\chi(\alpha)$ at $(s_1, s_2, s_3; t)$. We assume that $p = \chi_\alpha(s', t')$ $(s' \in S^1, t' \neq 0)$. Then we see that the osculating plane of α at $\alpha(s')$ contains a trisecant. This contradicts to the fact that $p \notin V$. Hence, Θ is transverse to χ_α at p. Thus the intersection of $\Theta(\mathcal{T})$ and $\chi(\alpha)$ is of 1-dimension at p. This completes the proof. ∎

Proof of Lemma 2.12: Take a simple regular point $p \in \chi(\alpha)$ of the tangent developable. We have only to show that $p \in \bar{B} \cap \bar{R}$. By Lemmas 2.13 and 2.14, we may assume that $p \in \chi(\alpha) - (\Omega \cup T)$. Suppose $p = \alpha(s_0) + r_0\alpha'(s_0)$ $(s_0 \in S^1, r_0 \in \mathbf{R} - \{0\})$. Note that s_0 is not a torsion-zero point, since $p \notin \Omega$. By the proof of Proposition 2.8, the map $t_p : S^1 \to \mathbf{R}P^2$ is an immersion with normal crossings off s_0. If t_p has a double point at s_0, there exists a cross tangent passing through $\alpha(s_0)$. Since there are only finitely many cross tangents (see [**NR**]), we may assume that $t_p(s_0)$ is not a double point of t_p, changing p if necessary.

We may assume that $\alpha(s_0) = (0,0,0)$, $\alpha'(s_0) = (1,0,0)$, that the (x_1, x_2)-plane osculates at $\alpha(s_0)$, that $(\alpha'(s_0) \times \alpha''(s_0)) \cdot (0,0,1) > 0$ and that the torsion of α at s_0 is positive. Then we have $\alpha_2''(s_0) > 0$ and $\alpha_3'''(s_0) > 0$. Hence, for some small positive number θ, $\alpha(s) \in \{x_1 < 0, x_3 < 0\}$ for $s_0 - \theta < s < s_0$ and $\alpha(s) \in \{x_1 > 0, x_3 > 0\}$ for $s_0 < s < s_0 + \theta$, where we identify a neighborhood of s_0 in S^1 with an interval in $\mathbf{R}$.

For simplicity, we assume $r_0 > 0$. Take a point $q = p + (0,0,q_3) = (r_0, 0, q_3)$ close to p with $q_3 \neq 0$. Note that t_q is an immersion with normal crossings, since $q \notin \chi(\alpha) \cup \Omega \cup T$. Furthermore, the number of double points of t_q is equal to that of t_p off a small neighborhood of s_0. Recall that t_p has no double points in a neighborhood of s_0. Now we consider the number of double points of t_q in the neighborhood of s_0. We will prove that if $q_3 < 0$, then t_q does not have any double points in the neighborhood of s_0, while if $q_3 > 0$, t_q has exactly one double point in the neighborhood of s_0.

For a small open disk neighborhood D of p with $D \cap (\Omega \cup T) = \emptyset$, $D - \chi(\alpha)$ has exactly two connected components D_+ and D_-, where D_+ is the region which contains a point q with $(q - p) \cdot \alpha'''(s_0) > 0$ (i.e., $q_3 > 0$).

Lemma 2.15. *If D is sufficiently small, t_q has exactly one double point in a neighborhood of s_0 if $q \in D_+$, and it has no double point in the neighborhood if $q \in D_-$.*

PROOF: We identify a neighborhood of $s_0 \in S^1$ with $(-\varepsilon, \varepsilon)$, s_0 being identified with 0, and we set $W = \{(s_1, s_2, t) \in (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon) \times (r_0 - \delta, r_0 + \delta) : s_1 > s_2\}$ $(\delta > 0)$. Recall that the image of $W' = \{(s_1, s_2, t) : s_1 = s_2\}$ by Φ is precisely the tangent developable of α. We have only to show that Φ maps W injectively onto D_+ for some ε and δ. For this, we calculate the differential of Φ at $u = (0, 0, r_0)$. As a basis of $T_u(S^1 \times S^1 \times \mathbf{R})$, we take $\{(\partial/\partial s_1) + (\partial/\partial s_2), (\partial/\partial s_1) - (\partial/\partial s_2), (\partial/\partial t)\}$. Note that $\{(\partial/\partial s_1) + (\partial/\partial s_2), (\partial/\partial t)\}$ constitutes a basis of $T_u W'$ and that $(\partial/\partial s_1) - (\partial/\partial s_2)$ is the direction normal to W' in $S^1 \times S^1 \times \mathbf{R}$ toward W. Then we have

$$\begin{aligned} d\Phi_u((\partial/\partial s_1) + (\partial/\partial s_2)) &= \frac{\partial}{\partial s}\Phi(s, s, t)\Big|_{(s,s,t)=u} \\ &= \frac{\partial}{\partial s}(\alpha(s) + t\alpha'(s))\Big|_{(s,s,t)=u} \\ &= \alpha'(s_0) + r_0\alpha''(s_0), \end{aligned}$$

$$\begin{aligned} d\Phi_u(\partial/\partial t) &= \frac{\partial}{\partial t}\Phi(s_1, s_2, t)\Big|_{(s_1,s_2,t)=u} \\ &= \left(\int_0^1 \alpha'(s_1 + \tau(s_2 - s_1))d\tau\right)\Big|_{(s_1,s_2,t)=u} = \alpha'(s_0), \end{aligned}$$

and

$$\begin{aligned} & d\Phi_u((\partial/\partial s_1) - (\partial/\partial s_2)) \\ &= \frac{\partial}{\partial s}\Phi(s, -s, t)\Big|_{(s,s,t)=u} \\ &= \frac{\partial}{\partial s}\left(\frac{\alpha(s) + \alpha(-s)}{2} + t\int_0^1 \alpha'(s - 2s\tau)d\tau\right)\Big|_{(s,-s,t)=u} \\ &= \left(\frac{\alpha'(s) - \alpha'(-s)}{2} + t\int_0^1 (1 - 2\tau)\alpha''(s - 2s\tau)d\tau\right)\Big|_{(s,-s,t)=u} = 0. \end{aligned}$$

Thus the rank of $d\Phi_u$ is equal to 2. Note also that $d\Phi_u(T_u(S^1 \times S^1 \times \mathbf{R})) = d(\Phi|W')_u(T_u W')$ and that $\Phi|W'$ is a local diffeomorphism around u onto an open neighborhood of p in $\chi(\alpha)$. Furthermore, we have

$$\begin{aligned} & \frac{\partial^2}{\partial s^2}\Phi(s, -s, t)\Big|_{(s,s,t)=u} \\ &= \frac{\partial}{\partial s}\left(\frac{\alpha'(s) - \alpha'(-s)}{2} + t\int_0^1 (1 - 2\tau)\alpha''(s - 2s\tau)d\tau\right)\Big|_{(s,-s,t)=u} \\ &= \left(\frac{\alpha''(s) + \alpha''(-s)}{2} + t\int_0^1 (1 - 2\tau)^2\alpha'''(s - 2s\tau)d\tau\right)\Big|_{(s,-s,t)=u} \\ &= \alpha''(s_0) + \frac{r_0}{3}\alpha'''(s_0). \end{aligned}$$

Thus, if we take the derivative of second order along $(\partial/\partial s_1) - (\partial/\partial s_2)$, it does not lie on the image of $d\Phi_u$. Hence we can change the system of coordinates of $\mathbf{R}^3$ around p in a homeomorphic way so that $d(\Phi|W \cup W')_u$ has rank 3 with respect to this new C^0 coordinates. Hence, Φ maps W injectively onto D_+ for some small W. (In other words, Φ has a fold singularity (or $\Sigma_{1,0}$-singularity) along W' in a neighborhood of u, the discriminant set being the tangent developable. See [**Mo**].) This completes the proof of Lemma 2.15. ▮

Thus we see that p is in the closure of both B and R. This completes the proof of Lemma 2.12. ▮

Remark. As a digression, it would be interesting to study the behavior of the map Φ. For a generic curve, is Φ stable? What are the singularities and the critical values? Is it related to the theory of self-translation surfaces [**MNR**]?

By the above lemma, the decomposition $\mathbf{R}^3 - \chi(\alpha) = B \cup R$ coincides with the decomposition guaranteed by the 2-color theorem ([**NS**, Lemma 2.1]) applied to the topologically stable map $\chi_\alpha : S^1 \times \mathbf{R} \to \mathbf{R}^3$. (More precisely, we have to compactify the map as is done in [**N**] in order to apply the result of [**NS**] and then we restrict to $S^1 \times \mathbf{R}$ and $\mathbf{R}^3$.) Recall that, in [**NS**], we have made the convention that the index $n(p, \chi(\alpha))$ of a cross cap point p of $\chi(\alpha)$ with respect to the 2-color theorem is defined to be 1 if the outside region of $\mathbf{R}^3 - \chi(\alpha)$ in a neighborhood of p is red and 0 if it is blue. Note also that the torsion-zero points of α coincides exactly to the cross cap points of $\chi(\alpha)$ ([**N**]).

PROPOSITION 2.16. *The two definitions of the index of a torsion-zero point p — the index with respect to the number of trisecants passing through p, $n(p, \alpha)$, and that with respect to the 2-color theorem, $n(p, \chi(\alpha))$ — coincide with each other modulo 2.*

PROOF: Let $p = \alpha(s_0)$ ($s_0 \in S^1$) be a torsion-zero point. Take a point $q \in \mathbf{R}^3 - \chi(\alpha)$ close to p. We may assume that $q \notin \chi(\alpha) \cup \Omega \cup T$. Then the map t_q and t_p are immersions with normal crossings. Although t_q and t_p are not even homotopic, yet they have the same number of double points off a neighborhood J of s_0. This is because the maps $t_p|(S^1 - J)$ and $t_q|(S^1 - J)$ are sufficiently "close" to each other as immersions into $\mathbf{R}P^2$ (note that t_p has no double points in J). By Lemma 2.10, the parity of the number of double points of t_q in the neighborhood of s_0 depends only on the region to which q belongs; more precisely it is odd when q is in the "outside region" of the cross-cap point p of $\chi(\alpha)$, and it is even when q is in the "inside region". This is because we can take an appropriate point q in the "outside region" such that t_q has exactly one double point in the neighborhood J of s_0. In fact, given a pair of bitangent points $\alpha(s_1), \alpha(s_2)$ in J, the segment joining $\alpha(s_1), \alpha(s_2)$ is included in the local convex hull of the curve, which lies in the "outside region" of the cross-cap point. Thus, any point q in this segment, t_q has exactly one double point in J (for example, see Figure 4 of [**C**]). Now consider the case where the index $n(p, \alpha)$

with respect to trisecants is odd. Then the outside region is the red region and the inner region is the blue region. Hence the index $n(p, \chi(\alpha))$ of p with respect to the 2-color theorem is equal to 1 and coincides with the index with respect to trisecants. The other case is similar. Hence the index with respect to the number of trisecants coincides with the index with respect to the 2-color theorem modulo 2. This completes the proof. ∎

Now our Theorem 2.4 follows from [**NS**] (or [**Sz**]) and Proposition 2.16, by using the compactification $\tilde{\chi}_\alpha : S^1 \times \mathbf{R}^* \to \mathbf{R}P^3$ of χ_α as in [**N**]. Note that $(\tilde{\chi}_\alpha)_*[S^1 \times \mathbf{R}^*] \in H_2(\mathbf{R}P^3; \mathbf{Z}_2)$ vanishes, since $S^1 \times \mathbf{R}^* \cong S^1 \times S^1$ is orientable, where $[S^1 \times \mathbf{R}^*] \in H_2(S^1 \times \mathbf{R}^*; \mathbf{Z}_2)$ is the fundamental class.

Remark. Theorem 2.4 implies that the number of pyramids of a generic space curve is congruent modulo 2 to the number of trisecants passing through the torsion zero points, where such trisecants are counted with multiplicities which are defined to be the number of torsion zero points they pass through. We note that we could add the condition that a trisecant passes through at most one torsion zero point in the general position conditions from the beginning. This condition is generic (i.e., even after adding this condition, we have Lemma 2.1), and then we have that the number of pyramids is congruent modulo 2 to the number (counted without multiplicities) of trisecants passing through the torsion zero points.

Corollary 2.17. *Let $\alpha : S^1 \to \mathbf{R}^3$ be a smooth curve satisfying the general position conditions, such that the number of pyramids $T(\alpha)$ is odd. Then α has at least two torsion zero points.*

A special case of Theorem 2.4 is when the curve α is convex. By using a result by Sedykh [**Se**], we can look at the structure of the convex envelope of a generic curve and deduce that a convex generic curve has no trisecants. Then we get the following immediate consequence.

Corollary 2.18. *Let $\alpha : S^1 \to \mathbf{R}^3$ be a convex curve satisfying our general position conditions and the general position conditions given in* [**Se**]. *Then the number of pyramids of α is even.*

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