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Zero sets of solutions of the heat equation

兵庫教育大 渡辺金治 Kinji Watanabe)

1. Introduction. The purpose of this note is to study the zero set :

$$Z(u) = \{(x,t) \in \Omega\times(-T,T) \ ; \ u(x,t) = 0 \}$$

of an analytic solution u of the following equation :

$$\frac{\partial u}{\partial t} = \sum_{j,k=1}^{2} a_{j,k} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{2} b_j \frac{\partial u}{\partial x_j} + cu \quad \text{in} \quad \Omega\times(-T,T), \qquad (1,1)$$

satisfying the Dirichlet condition :

$$u = 0 \quad \text{on} \quad \partial\Omega\times(-T,T). \qquad (1,2)$$

Here Ω is a bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega$ of analytic class, $a_{j,k}$, b_j and c are analytic functions and ($a_{j,k}$) is symmetric and positive definite in $\overline{\Omega}\times(-T,T)$.

When the space dimension is equal to 1, i.e. , $\Omega = (0,1)$, Angenent [1] proved , without assumption of analyticity , the finiteness and the nonincrease with respect to t of the number of

$$Z(u(t)) = \{ x \in \Omega \ ; \ u(x,t) = 0 \}$$

for classical solution u. Moreover Watanabe [2] showed that near each point (x_o,t_o) in the singular part $S(u)$ of $Z(u)$:

$$S(u) = \{(x,t) \in Z(u) \ ; \ \mathrm{grad}_x\, u(x,t) = 0 \}$$

$Z(u)$ is equal to the union of 2ℓ or $2\ell + 1$ continuous curves for some $\ell \geqq 1$ such that

$$x - x_o = \sqrt{t_o - t}\ A \{\pm \lambda_{2\ell,j} + o(1) \},\ 1 \leqq j \leqq \ell, \text{ as } t \uparrow t_o$$

or

$$x - x_o = \sqrt{t_o - t}\ A \{\pm \lambda_{2\ell+1,j} + o(1)\},\ 1 \leqq j \leqq \ell, \text{ as } t \uparrow t_o$$

$$x - x_o = o(\sqrt{|t-t_o|}) \quad \text{as } t \to t_o.$$

Here A is a non zero constant and $\lambda_{k,j}$, $1 \leqq j \leqq [k/2]$, are the positive roots of the k-th Hermite polynominal of one variable.

On the other hand,on account of complexities of zero sets of Hermite polynominals of several variables , which are defined as polynominal solutions of Hermite's equation :

$$2\,\Delta H - \sum_{j=1}^{n} x_j \frac{\partial H}{\partial x_j} + mH = 0 \text{ in } \mathbb{R}^n ,\ \Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$$

we encounter difficulties to see zero sets of solutions of equations similar to (1,1) in the case of the space dimension > 1.

One of important facts concerning Hermite polynominal H of two variables of the form :

$$H(x_1,x_2) = x_1^m + \sum_{j=1}^{m} h_j(x_2)\, x_1^{m-j}$$

is the following.

Lemma 1. The discriminant as polynominal in x_1 of H is not identically zero.

2. Nodal line. To analyse the zero set $Z(u(t))$ we use the following notation.

Definition 1. Let f be a non constant analytic function in Ω and U an open subset of Ω. A curve γ is said to be a nodal line of f in U if it satisfies the following two conditions. (i) It is a connected continuous curve in $Z(f) \cap U$ with arc length s such that for each $\gamma(s_o)$ in the singular part $S(f)$ of $Z(f)$ we have as germ of set at 0

$$\{ T(\gamma(s)-\gamma(s_o)); |s-s_o| \text{ small}\} = \{(\lambda(\zeta),\zeta^d); \zeta \in \mathbb{C}, |\zeta| \text{ small}\} \cap \mathbb{R}^2$$

for some rotation T around the origin , an integer $d \geq 1$, and a holomorphic function λ near $\zeta = 0$. (ii) It contains at least two points and maximal with respect to the inclusion relation in the set of all curves satisfying (i).

Example 1. Let p be a non constant homogeneous polynominal and set

$$W(x,t) = \Sigma \frac{1}{k!} t^k \Delta^k p(x), H(x) = W(x,-1), H^*(x) = W(x,1).$$

These mean that W is a solution of $W_t = \Delta W$ and H is a Hermite polynominal. When we put $Z(p) = \{ \theta \in [0,\pi) ; p(\cos\theta, \sin\theta) = 0 \}$ and denote by $\nu(\theta)$ its vanishing order at θ, $Z(H)$ is not empty and the number of all unbounded nodal lines of H (resp. H^*) in $\mathbb{R}^2$ is equal to

$$\sum_{\theta\in Z(p)} \nu(\theta) , \text{ (resp. } \# \{ \theta \in Z(p) ; \nu(\theta) \text{ is odd } \}).$$

This is a consequence of Lemma 2 in the next section. By the maximum principle H^* has no bounded nodal domain, i.e. , each connected component of $\mathbb{R}^2 \setminus Z(H^*)$ is unbounded and hence there is no compact nodal line of H^* in $\mathbb{R}^2$.

Example 2. Put $u(x,t) = x_1^2 - x_2^2(1 - x_2^2) + 12tx_2^2 + 12t^2$. Then u is a solution of $u_t = \Delta u$ and the number of its nodal lines in $\mathbb{R}^2$ is not nonincreasing with respect to t.

3. Results. Let u be a non trivial analytic solution of (1,1). We assume that (0,0) be in $S(u)$ and u can be written near (0,0)

$$u(x,t) = u_o \{ x_1^m + \sum_{j=1}^{m} u_j(x_2,t)\, x_1^{m-j} \} \tag{3,1}$$

for some $u_o \neq 0$, $u_j(0,0) = 0$, $m \geqq 2$.

Lemma 2. Assume that there is an analytic function ϕ near $x_2 = 0$ such that $\phi(0) = \phi'(0) = 0$, $u(x,0)(x_1 - \phi(x_2))^{-m} \neq 0$ for $|x|$ small enough. Then there is an open neighbourhood U of $x = 0$ and $\delta > 0$ such that the following holds.
(i) $Z(u) \cap U \times \{0 < t < \delta\}$ is empty when m is even , an analytic surface of the form :

$$x_1 - \phi(x_2) = o(\sqrt{t}) \text{ , as } t \downarrow 0$$

when m is odd... (ii) $Z(u) \cap U \times \{ 0 > t > -\delta\}$ is equal to the union of m analytic surfaces of the forms :

$$x_1 - \phi(x_2) = \sqrt{-t}\, A(x_2)\{ \pm \lambda_{\ell,j} + o(1)\}, 1 \leqq j \leqq \ell, \text{as } t\uparrow 0.$$

Here $\ell = [m/2]$, $A(x_2) \neq 0$ and we have to add an analytic surface of the form $x_1 - \phi(x_2) = o(\sqrt{-t})$, as $t \uparrow 0$, when m is odd.

Secondly we consider the case where $S(u) \cap \{0<t<\delta, |x|<\delta\}$ is not empty for any $\delta > 0$. Since it follows from Lemma 1 that the discriminant as polynominal in x_1 of (3,1) is not identically zero, we have, by using the Puiseux expansion, that $Z(u) \cap Z(\partial u/\partial x_1) \cap \{0 < t\}$ is, near (0,0), equal to the union of finitely many curves of the form :

$$(\lambda(t), \mu(t^a), t^b), \quad t > 0.$$

Here a and b are integers $\geqq 1$, λ and μ are analytic near $t = 0$. Take such a curve and put

$$v(y,t) = u(y_1 + \lambda(t), y_2 + \mu(t^a), t^b).$$

Lemma 3. Under the notation mentioned above, there are, for each $\bar{t} > 0$ small enough, an integer $d \geqq 1$ and holomorphic functions Φ_j near $(0, \bar{t})$ such that

$$v(y_1, \zeta^d, t) = v_o \Pi (y_1 - \Phi_j(\zeta, t)), \quad v_o \neq 0, \ \Phi_j(0, \bar{t}) = 0.$$

When $S(u) \cap \{0 > t > -\delta, |x| < \delta\}$ is not empty for any δ, an analogous result to Lemma 3 for $t < 0$ holds.

We are now ready to state the main results of this note. Let

$$u(x_1, \zeta^D, 0) = u_o \Pi (x_1 - \psi_j(\zeta))^{\nu_j} \qquad (3,2)$$

be the Puiseux expansion of $u(x,0)$ at $x = 0$ for some integer

$D \geqq 1$ and holomorphic functions ψ_j near $\zeta = 0$ satisfying $\psi_j(0) = 0$, $\psi_j(\zeta) - \psi_k(\zeta) \neq 0$ for $j \neq k$, $|\zeta| > 0$ small enough. We denote by Σ_1 the set of the index j appeared in (3,2) such that as germ of set at 0

$$\Gamma_j \equiv \{(\psi_j(\zeta), \zeta^D) \ ; \ \zeta \in C, \ |\zeta| \text{ small}\} \cap \mathbb{R}^2 \neq \{0\}$$

and choose a subset Σ_2 of Σ_1 such that

$$Z(u(0)) = \cup \{\Gamma_j \ ; \ j \in \Sigma_2\} \tag{3,3}$$

and that $\Gamma_j \neq \Gamma_k$ for $j \neq k \in \Sigma_2$. Put

$$J_o(-) = \sum_{j \in \Sigma_2} \nu_j \ , \ J_o(+) = \# \{ j \in \Sigma_2 \ ; \ \nu_j \text{ is odd} \} .$$

Proposition 1. Under the notation mentioned above, there are an open neighbourhood U of $x = 0$ and $\delta > 0$ such that for each $0 < t < \delta$, $Z(u(t)) \cap U$ is equal to the union of $J_o(+)$ noncompact nodal lines of $u(t)$ in U and that for each $0 > t > -\delta$ $Z(u(t)) \cap U$ is not empty and the number of noncompact nodal lines of $u(t)$ in U is equal to $J_o(-)$.

When u also satisfies the boundary condition (1,2), concerning the number $J(t)$ of nodal lines of $u(t)$ in Ω we have the following.

Proposition 2. J is a step function, i.e. , it is constant in $(-\rho,0)$ and also in $(0,\rho)$ for some $\rho > 0$.

4. Proofs. In this section we give brief proofs of Lemmas and Propositions.

Proof of Lemma 1. We use induction on m. Suppose that H can be written near some point (z_o,ζ_o) in C^2

$$H(z,\zeta) = (z - h(\zeta))^{\nu} H_o \quad , \quad H_o \neq 0$$

for some holomorphic function h near ζ_o and integer $\nu \geqq 2$. Since $\partial H/\partial x_1$ is also a Hermite polynominal , we have by assumption of induction and Hermite equation that $\nu = 2$ and

$$\frac{\partial h(\zeta)^2}{\partial \zeta} = -1 \qquad \text{near} \quad \zeta_o .$$

So h and H can be written

$$h(\zeta) = a + bi + \sigma\zeta i \ , \quad \sigma = \pm 1,$$
$$H(x_1 + a, x_2 - \sigma b) = |x|^4 H_1(x) \quad \text{in} \quad \mathbb{R}^2$$

for some real numbers a , b and polynominal H_1 . Let p be the initial form of $H(x_1 + a,x_2 - \sigma b)$ at $(0,0)$. Since p has a factor $|x|^2$, this gives a contradiction to the harmonicity of p which follows from Hermite's equation.

Proof of Lemma 2. We introduce a parameter τ with $|\tau|$ small. Put

$$W(\tau,x,t) = \sum_{2k+|\alpha|=m} \frac{t^k x^\alpha}{k!\alpha!}\left(\frac{\partial}{\partial s}\right)^k\left(\frac{\partial}{\partial y}\right)^\alpha u(y_1+\phi(y_2+\tau),y_2+\tau,s)\Big|_{\substack{y=0\\ s=0}} .$$

Then it follows from (1,1) that $W(\tau)$ satisfies

$$\frac{\partial W}{\partial t} = \sum_{j,k=1}^{2} A_{j,k}(\tau)\frac{\partial^2 W}{\partial x_j \partial x_k} ,$$

for some symmetric and positive definite matrix $(A_{j,k})$. Since the initial date of $W(\tau)$ is independent on x_2 , $W(\tau)$ is also independent on x_2 and hence it is determined by Hermite polynominal of one variable. So we can easly see $Z(u)$ by using the relation as $|x_1| + |t| \to 0$,

$$u(x_1 + \phi(\tau),\tau,t) = W(\tau,x,t) + O(\{|x_1|^2 + |t|\}^{(m+1)/2}).$$

Proof of Lemma 3. Take the maximum ν of k such that $\{(\lambda(t),\mu(t^a),t^b)\ ;\ t > 0$ small enough $\}$ is contained in the intersection of $Z((\frac{\partial}{\partial x_1})^j u)$, $j = 0,\cdots,k$. Since the discriminant as polynominal in y_1, obtained by the Weierstrass preparation theorem for v near $(0,0)$, does not vanish for $y_2 \neq 0$, $t > 0$, there are holomorphic functions ϕ_j, $1 \leq j \leq \nu$, in $\{ (\zeta,t) \in C \times \mathbb{R}\ ;\ 0 < \arg \zeta < 2\pi, |\zeta|>0, |t - \bar{t}|$ small enough $\}$ such that $v(\phi_j(\zeta,t),\zeta, t) = 0$. So we have for some permutation ρ of $\{1,\cdots,\nu\}$ and an integer d that ρ^d = identity and

$$\lim_{\theta \downarrow 0} \phi_j(re^{i(2\pi - \theta)},t) = \lim_{\theta \downarrow 0} \phi_{\rho(j)}(re^{i\theta},t).$$

When we put for $2k\pi \leqq d\theta < 2(k+1)\pi$, $0 \leqq k < d$,

$$\Phi_j(re^{i\theta},t) = \phi_{\rho^k(j)}(r^d e^{i(d\theta - 2k\pi)},t),$$

it is easy to see an expression of v.

Proof of Proposition 1. We verify the claim of this proposition for $t > 0$. Take $\varepsilon_j > 0, j=1,2,$ so small that the Puiseux expansion (3,2) of $u(x,0)$ and (3,3) hold in $\{ x ; |x_j| < 2\varepsilon_j \}$ and that $u(\pm\varepsilon_1,x_2,0) \neq 0$ for $|x_2| \leqq \varepsilon_2$. Set $U = \{ x ; |x_j| < \varepsilon_j \}$. When $J_o(+) = 0$, i.e. , $Z(u(0)) \cap U = \{0\}$, by the maximum principle we have that $Z(u) \cap U \times (0,\delta)$ is empty for small $\delta > 0$. Assume that $J_o(+) > 0$. For $\delta > 0$ small let

$$\{Z(u) \setminus Z(\partial u/\partial x_1)\} \cap U \times (0,\delta) = \cup \{S_j ; j = 1,\cdots,N\}$$

be the decomposition into its connected components and we denote by $\gamma_\Lambda[t^*]$ the union of closures in U of $S_j \cap \{t = t^*\}$, $j \in \Lambda$ for a subset Λ of $\{1,\cdots,N\}$. Then $\gamma_\Lambda[t^*]$ is not a closed curve for any Λ and hence there is no compact nodal line of $u(t^*)$ in U. On the other hand it follows from Lemma 2 that the number of non compact nodal lines of $u(t^*)$ in U is equal to $J_o(+)$.

Proof of Proposition 2. We show that nodal lines of $u(t)$ in U, which is the neighbourhood of $x = 0$ given in the proof of Proposition 1, depend " continuously " on $t > 0$. We use the notation in Lemma 3 and its proof. By the method of constraction of Φ_j we assume without lose of generality that d is even and that $\mathrm{Im}\, \phi_j(r,t) = 0$ in $\{(r,t) ; 0 < r , |t - \bar{t}|$ small $\}$. Then we claim that there is a complex number ω and $\sigma = \pm 1$ such that for each t with $|t - \bar{t}|$ small we have as germ of set at 0

$$\begin{aligned} &\{(\Phi_j(\zeta,t),\zeta^d);\zeta \in \mathbb{C},|\zeta| \text{ small } \} \cap \mathbb{R}^2 \\ &= \{ (\Phi_j(r,t),r^d);0 \leq r \text{ small}\} \cup \{(\Phi_j(\omega r,t),\sigma r^d);0 < r \text{ small}\}. \end{aligned} \tag{4,1}$$

For each t with $|t - \bar{t}|$ small,this equality (4,1) holds with some $\omega = \omega(t)$, $\sigma = \sigma(t)$, $\omega^d = \sigma$. Put

$$I(+) = \{t ; \sigma(t) = 1\} , \quad I(-) = \{t ; \sigma(t) = -1\} .$$

When $\bar{t}$ is an accumulation point of $I(+)$, we can replace $\omega(t)$ by a number $\omega = \exp(2k\pi i/d)$ independent on t so that (4,1) hold for any t with $0 < |t - \bar{t}|$ small enough. We have to check the case where $\Phi_j(r,\bar{t}) = \Phi_j(\omega r,\bar{t})$ for any $r > 0$. But we have by definition of Φ_j

$$\phi_j(r,\bar{t}) = \Phi_j(r,\bar{t}) = \Phi_j(\omega r,\bar{t}) = \phi_{\rho^k(j)}(r,\bar{t})$$

and hence $j = \rho^k(j)$, $\Phi_j(r,t) = \Phi_j(\omega r,t)$ for any (r,t) which imply a contradiction. When $\bar{t}$ is an accumulation point of $I(-)$, it is easy to verify (4,1).

By the assumption of analyticies of coefficients in (1,1) and $\partial\Omega$, we have that u is analytic in some neighbourhood of $\bar{\Omega} \times (-T,T)$ and hence we obtain the " continuous dependence " on $t \in (0,\delta)$,for δ small enough, of nodal lines of $u(t)$ in Ω .

Reference

[1] Angenent, S. B., The zero set of a solution of a parabolic equation, J.reine angew., 390 (1988),79-96.

[2] Watanabe, K., Remarques sur l'ensemble d'une solution d'une équation parabolique en dimension d'espace 1, preprint.