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GLOBAL INTUITIONISTIC LOGIC AND ITS SEMANTIC COMPLETENESS

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GI, Global Intuitionistic logic, is an intuitionistic modal predicate logic which was first studied in the form of a sequent calculus in Takeuti-Titani[2]. Later another version of GI was studied in Titani[3]. The goal of this paper is to prove the semantic completeness of Titani's GI with respect to complete Heyting algebras with a unary operation $\Box$ called a "globalization."

We note here that Ono[1] contains completeness theorems for several propositioinal sequent calculi similar to the propositional part of Titani's GI.

1 Syntax of GI

1.1 Language L of GI

1.1.1 Symbols of L

(1) Individual constants: $c_0, c_1, c_2, \ldots$

(2) Free variables: $a_0, a_1, a_2, \ldots$

(3) Bound variables: $x_0, x_1, x_2, \ldots$

(4) Predicate constants with n argument places (n=1,2,3,...) : $R^n{}_0, R^n{}_1, R^n{}_2, \ldots$

(5) Logical symbols: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\forall$, $\exists$, $\Box$

(6) Punctuation symbols: (,), ,(comma)

1.1.2 Well-formed formulas (wffs) of L

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Individual constants and free variables are called "terms."

(1) If $t_1,\ldots t_n$ are terms and R^n is a predicate constant with n argument places, then $R^n(t_1,\ldots t_n)$ is a wff.

(2) If A and B are wffs, so are $(A\wedge B)$, $(A\vee B)$, $(A\rightarrow B)$, $\neg A$, and $\Box A$.

(3) If A(t) is a wff with a term t and x is a bound variable, then $\forall xA(x)$ and $\exists xA(x)$ are wffs, where A(x) is obtained from A(t) by replacing each occurrence of t in A(t) with x.

(4) Wffs are obtained only by the above (1)−(3).

As usual, sentences are those wffs with no free variables. In what follows, we will consider only sentences.

1.1.3 $\Box$-closed sentences of L

(1) If A is a sentence, then $\Box A$ is a $\Box$-closed sentence.

(2) If A and B are $\Box$-closed sentences, so are $(A\wedge B)$, $(A\vee B)$, $(A\rightarrow B)$, $\neg A$.

(3) If A(c) is a $\Box$-closed sentence with an individual constant c, then $\forall xA(x)$ and $\exists xA(x)$ are $\Box$-closed sentences, where $\forall xA(x)$ and $\exists xA(x)$ are formed as in 1.1.2,(3).

(4) $\Box$-closed sentences are obtained only by the above (1)-(3)

1.1.4 Sequents of L

If $A_1, A_2,\ldots, A_m, B_1, B_2,\ldots, B_n$ are sentences, then

$$A_1, A_2,\ldots, A_m \Rightarrow B_1, B_2,\ldots, B_n \quad (m,n\geqq 0)$$

is a sequent of L.

We use Greek capital letters $\Gamma, \Delta, \Pi, \Lambda, \Gamma_0, \Gamma_1,\ldots$ to denote finite sequences of sentences separated by commas. We also use $\bar{\Gamma}$, $\bar{\Delta}$,... to denote finite sequences of $\Box$-closed sentences separated by commas.

1.2 Formal proofs in GI

The system GI contains axioms and a group of rules of inference, which consists of (1) structural rules and (2) logical rules.

1.2.1 Axioms of GI: any sequents of the form : $A \Rightarrow A$, where A is a sentence.

1.2.2 The structural rules of GI

Thinning: $\dfrac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$, $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A}$ Contraction: $\dfrac{A, A, \Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta}$, $\dfrac{\Gamma \Rightarrow \Delta, A, A}{\Gamma \Rightarrow \Delta, A}$

Interchange: $\dfrac{\Gamma, A, B, \Pi \Rightarrow \Delta}{\Gamma, B, A, \Pi \Rightarrow \Delta}$, $\dfrac{\Gamma \Rightarrow \Delta, A, B, \Lambda}{\Gamma \Rightarrow \Delta, B, A, \Lambda}$ Cut: $\dfrac{\Gamma \Rightarrow \Delta, A \quad A, \Pi \Rightarrow \Lambda}{\Gamma, \Pi \Rightarrow \Delta, \Lambda}$

1.2.3 The logical rules of GI

$\wedge\Rightarrow$: $\dfrac{A, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta}$, $\dfrac{A, \Gamma \Rightarrow \Delta}{B \wedge A, \Gamma \Rightarrow \Delta}$ $\Rightarrow\wedge$: $\dfrac{\Gamma \Rightarrow \Delta, A \quad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$

$\vee\Rightarrow$: $\dfrac{A, \Gamma \Rightarrow \Delta \quad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta}$ $\Rightarrow\vee$: $\dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, A \vee B}$, $\dfrac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta, B \vee A}$

$\rightarrow\Rightarrow$: $\dfrac{\Gamma \Rightarrow \Delta, A \quad B, \Pi \Rightarrow \Lambda}{A \rightarrow B, \Gamma, \Pi \Rightarrow \Delta, \Lambda}$ $\Rightarrow\rightarrow$: $\dfrac{A, \Gamma \Rightarrow \overline{\Delta}, B}{\Gamma \Rightarrow \overline{\Delta}, A \rightarrow B}$

$\neg\Rightarrow$: $\dfrac{\Gamma \Rightarrow \Delta, A}{\neg A, \Gamma \Rightarrow \Delta}$ $\Rightarrow\neg$: $\dfrac{A, \Gamma \Rightarrow \overline{\Delta}}{\Gamma \Rightarrow \overline{\Delta}, \neg A}$

$\forall\Rightarrow$: $\dfrac{A(c), \Gamma \Rightarrow \Delta}{\forall x A(x), \Gamma \Rightarrow \Delta}$, where c is an arbitrary individual constant.

$\Rightarrow\forall$: $\dfrac{\Gamma \Rightarrow \overline{\Delta}, A(c)}{\Gamma \Rightarrow \overline{\Delta}, \forall x A(x)}$, where c is an individual constant not occurring in the lower sequent.

$\exists\Rightarrow$: $\dfrac{A(c), \Gamma \Rightarrow \Delta}{\exists x A(x), \Gamma \Rightarrow \Delta}$, where c is an individual constant not occurring in the lower sequent.

$\Rightarrow\exists$: $\dfrac{\Gamma \Rightarrow \Delta, A(c)}{\Gamma \Rightarrow \Delta, \exists x A(x)}$, where c is an arbitrary individual constant.

$$\Box\Rightarrow: \quad \frac{A, \Gamma \Rightarrow \Delta}{\Box A, \Gamma \Rightarrow \Delta} \qquad \Rightarrow\Box: \quad \frac{\overline{\Gamma} \Rightarrow \overline{\Delta}, A}{\overline{\Gamma} \Rightarrow \overline{\Delta}, \Box A}$$

When a sequent $\Gamma \Rightarrow \Delta$ is provable in GI, we write $\vdash \Gamma \Rightarrow \Delta$.

1.3 Theorems (i.e., Provable sequents) in GI

(1) $\Rightarrow \Box A \vee \neg \Box A$

(2) $\Box A \Rightarrow A$

(3) $\Box(A \rightarrow B) \Rightarrow (\Box A \rightarrow \Box B)$

(4) $\Box \neg A \Rightarrow \neg \Box A$

(5) $\Box(A \wedge B) \Rightarrow (\Box A \wedge \Box B)$

(6) $(\Box A \wedge \Box B) \Rightarrow \Box(A \wedge B)$

(7) $\Box A \vee \Box B \Rightarrow \Box(A \vee B)$

(8) $\overline{A} \Rightarrow \Box \overline{A}$, for any $\Box$-closed sentence $\overline{A}$

(9) $\neg\neg \overline{A} \Rightarrow \overline{A}$, for any $\Box$-closed sentence $\overline{A}$

(10) $\neg \overline{A} \rightarrow B \Rightarrow \overline{A} \vee B$, for any $\Box$-closed sentence $\overline{A}$

(11) $\Rightarrow \overline{A} \vee \neg \overline{A}$, for any $\Box$-closed sentence $\overline{A}$

(12) $\Box(A \rightarrow B) \wedge \Box(B \rightarrow C) \Rightarrow \Box(A \rightarrow C)$

(12) $(\Box A \rightarrow \Box B) \Rightarrow \Box(\Box A \rightarrow B)$

(13) $\Box(\Box A \rightarrow B) \Rightarrow \Box(\Box A \rightarrow \Box B)$

(14) $\Box \forall x(A \rightarrow B(x)) \Rightarrow \Box(A \rightarrow \forall x B(x))$

(15) $\Box \forall x(A(x) \rightarrow B) \Rightarrow \Box(\exists x A(x) \rightarrow B)$

(16) $\forall x \Box A(x) \Rightarrow \Box \forall x A(x)$.

2 Semantics of GI

We now introduce structures for the language L, which we will call "complete Heyting algebras with a globalization (cHags, for short)."

2.1 cHag interpretations

Let Đ be a nonempty set and L(Đ) be the extended language obtained from L by adding a new individual constant $\bar{d}$ for each member d of Đ. By a <u>cHag interpretation for L(Đ)</u>, we mean a triple < Đ,H, $[\![\ ]\!]$ > such that :

(1) H is a complete Heyting algebra with a globalization $\Box$:

$$H = < H, \wedge, \vee, \to, \neg, \Box, 0, 1, \bigwedge, \bigvee >,$$

where $\Box$ is a unary operation on H satisfying the following conditions: for each a,b $\in$ H and for each indexed set $\{a_i\}_i \subseteq H$,

G1 $\Box a \leq a$

G2 $(\Box a \to \Box b) \leq \Box(\Box a \to b)$

G3 $\bigwedge_i \Box a_i \leq \Box \bigwedge_i a_i$

G4 If $\Box a \leq b$, then $\Box a \leq \Box b$

G5 $\Box a \vee \neg \Box a = 1$.

(2) $[\![\ ]\!]$ is a map from the constants of L(Đ) such that

(i) $[\![c]\!] \in$ Đ for each individual constant c of L

(ii) $[\![\bar{d}]\!] = d \in$ Đ for each d $\in$ Đ

(iii) $[\![R^n]\!]$ is a function: $Đ^n \to H$ for each predicate constant R^n with n argument places.

(3) The symbol $[\![\ ]\!]$ is also used to denote the truth value of a sentence of L(Đ):

(i) Let R^n be a predicate constant with n argument-places and let $t_1,\dots, t_n$ be individual constants of L(Đ). Then

$$[\![R^n(t_1,\dots,t_n)]\!] = [\![R^n]\!]([\![t_1]\!],\dots, [\![t_n]\!]) \in H.$$

(ii) For sentences of L(Đ) containing logical symbols, their truth values are determined by:

$$[\![A \wedge B]\!] \triangleq [\![A]\!] \wedge [\![B]\!]$$

$$[\![A \vee B]\!] \triangleq [\![A]\!] \vee [\![B]\!]$$
$$[\![A \rightarrow B]\!] \triangleq [\![A]\!] \rightarrow [\![B]\!]$$
$$[\![\neg A]\!] \triangleq \neg [\![A]\!]$$
$$[\![\forall x A(x)]\!] \triangleq \bigwedge_{d \in Đ} [\![A(\bar{d})]\!]$$
$$[\![\exists x A(x)]\!] \triangleq \bigvee_{d \in Đ} [\![A(\bar{d})]\!]$$
$$[\![\Box A]\!] \triangleq \Box [\![A]\!],$$

where $\wedge, \vee, \rightarrow, \neg, \bigwedge, \bigvee$, and $\Box$ in the right-hand side of $\triangleq$ are the operations on H.

Note: When c is an individual constant of L and $[\![c]\!] = d \in Đ$, we have $[\![A(c)]\!] = [\![A(\bar{d})]\!]$.

2.2 Validity

(1) A sentence A of L(Đ) is <u>valid in a cHag interpretation $<Đ, H, [\![\ ]\!]>$</u>, if $[\![A]\!] = 1$ for every $[\![\ ]\!]$.

(2) The truth value of a sequent of L(Đ) is defined as follows:

$$[\![A_1, A_2, \ldots, A_m \Rightarrow B_1, B_2, \ldots, B_n]\!] \triangleq [\![A_1 \wedge A_2 \wedge \ldots \wedge A_m \rightarrow B_1 \vee B_2 \vee \ldots \vee B_n]\!]$$
$$[\![A_1, A_2, \ldots, A_m \Rightarrow \ ]\!] \triangleq [\![\neg (A_1 \wedge A_2 \wedge \ldots \wedge A_m)]\!]$$
$$[\![\ \Rightarrow B_1, B_2, \ldots, B_n]\!] \triangleq [\![B_1 \vee B_2 \vee \ldots \vee B_n]\!].$$
$$[\![\ \Rightarrow \ ]\!] \triangleq [\![A \wedge \neg A]\!] \text{ for any sentence A.}$$

Let $A_1, A_2, \ldots, A_m \Rightarrow B_1, B_2, \ldots, B_n$ be a sequent of L(Đ). Then it is <u>valid in a cHag interpretation $<Đ, H, [\![\ ]\!]>$</u>, if $[\![A_1, A_2, \ldots, A_m \Rightarrow B_1, B_2, \ldots, B_n]\!] = 1$ for every $[\![\ ]\!]$.

Also, sequent $A_1, A_2, \ldots, A_m \Rightarrow B_1, B_2, \ldots, B_n$ of L is <u>valid</u>, in symbol, $\models A_1, A_2, \ldots, A_m \Rightarrow B_1, B_2, \ldots, B_n$, if $A_1, A_2, \ldots, A_m \Rightarrow B_1, B_2, \ldots, B_n$ is valid in every cHag interpretation.

Now the following two propositions are immediate:

Proposition 2.2.1. Let $H = <H, \wedge, \vee, \rightarrow, \neg, \Box, 0, 1, \bigwedge, \bigvee>$ be a cHag.

Then the following hold: for each $a,b \in H$ and each indexed set $\{a_i\}_i \subseteq H$,

(1) If $a \leq b$, then $\Box a \leq \Box b$

(2) $\Box a = \Box\Box a$

(3) $\Box a \wedge \Box b = \Box(\Box a \wedge \Box b)$

(4) $\Box(a \wedge b) = \Box a \wedge \Box b$

(5) $\Box a \vee \Box b = \Box(\Box a \vee \Box b)$

(6) $\Box a \vee \Box b \leq \Box(a \vee b)$

(7) $\Box a \to \Box b = \Box(\Box a \to \Box b)$

(8) $\Box(a \to b) \leq (\Box a \to \Box b)$

(9) $\neg\Box a = \Box\neg\Box a$

(10) $\bigwedge_i \Box a_i = \Box \bigwedge_i \Box a_i$

(11) $\bigvee_i \Box a_i = \Box \bigvee_i \Box a_i$

(12) $\Box 0 = 0$ and $\Box 1 = 1$.

Proposition 2.2.2. For each cHag interpretation and for each $\Box$-closed sentence $\bar{A}$ of L(Đ),

(1) $\Box[\![\bar{A}]\!] = [\![\bar{A}]\!]$

(2) $[\![\bar{A}]\!] \vee \neg [\![\bar{A}]\!] = 1$

(3) If $[\![\bar{A}]\!] \leq [\![B]\!]$, then $[\![\bar{A}]\!] \leq \Box [\![B]\!]$, where B is a sentence of L(Đ).

Theorem 2.2.3.(The Soundness Theorem for GI) Let $\Gamma \Rightarrow \Delta$ be a sequent of L such that $\vdash \Gamma \Rightarrow \Delta$. Then $\models \Gamma \Rightarrow \Delta$.

Proof: Induction on the length of the proof $\vdash \Gamma \Rightarrow \Delta$.

Theorem 2.2.4.(The Completeness Theorem for GI) Let $\Rightarrow \Gamma$ be a sequent of L such that $\models \ \Rightarrow \Gamma$. Then $\vdash \ \Rightarrow \Gamma$.

Proof: We prove that $\nvdash \bar{\Gamma}_1 \Rightarrow \bar{\Delta}_1$ implies $\not\models \bar{\Gamma}_1 \Rightarrow \bar{\Delta}_1$. Then this shows as a special case that $\nvdash\ \Rightarrow\Box A$ implies $\nvDash\ \Rightarrow\Box A$, where A is the disjunction of all the sentences in Γ. Since ($\vdash\ \Rightarrow\Box A$ iff $\vdash\ \Rightarrow A$) and ($\vDash\ \Rightarrow\Box A$ iff $\vDash\ \Rightarrow A$), we can obtain : $\nvdash\ \Rightarrow A$ implies $\nvDash\ \Rightarrow A$, i.e., $\nvdash\ \Rightarrow\Gamma$ implies $\nvDash\ \Rightarrow\Gamma$.

We now show in three steps that $\nvdash \overline{P} \Rightarrow \overline{Q}$ implies $\nvDash \overline{P} \Rightarrow \overline{Q}$, where $\overline{P}$ and $\overline{Q}$ are respectively the conjunction of all the sentences in $\overline{\Gamma}_1$ and the disjunction of all the sentences in $\overline{\Delta}_1$. Let Đ be the set of all individual constants of L and L(Đ) be the same as L. We sometimes regard L(Đ) as the set of sentences of L(Đ).

Step 1: The construction of a Ha (Heyting algebra)

Definition 1: Let A,B$\in$L(Đ). Set

(1) $A\leq B \overset{\wedge}{\Leftrightarrow} \vdash A, \overline{P}, \neg\overline{Q} \Rightarrow B$

(2) $A\equiv B \overset{\wedge}{\Leftrightarrow} (A\leq B$ and $B\leq A)$

(3) $[\![A]\!] \triangleq \{B\in L(Đ) : A\equiv B\}$

(4) $H \triangleq \{ [\![A]\!] : A\in L(Đ) \}$

(5) $[\![A]\!] \leq [\![B]\!] \overset{\wedge}{\Leftrightarrow} A\leq B$.

Then the relation $\equiv$ is an equivalence relation on L(Đ) and the relation $\leq$ on H is well-defined. The following three lemmas are immediate:

Lemma 2: For each A,B$\in$L(Đ),

(1) $A\in [\![A]\!]$

(2) $A\equiv B$ iff $[\![A]\!] = [\![B]\!]$

(3) $A\not\equiv B$ iff $[\![A]\!] \cap [\![B]\!] = \phi$

(4) $[\![B]\!] \leq [\![A\rightarrow A]\!] = [\![\overline{P}]\!]$

(5) $[\![A\wedge\neg A]\!] = [\![\overline{Q}]\!] \leq [\![B]\!]$

(6) $[\![\overline{P}]\!] \neq [\![\overline{Q}]\!]$.

Lemma 3: Let $[\![A]\!], [\![B]\!] \in H$. Then the g.l.b of $[\![A]\!]$ and $[\![B]\!]$, i.e. $[\![A]\!] \wedge [\![B]\!]$ exists and equals $[\![A \wedge B]\!]$. The l.u.b. of $[\![A]\!]$ and $[\![B]\!]$, i.e. $[\![A]\!] \vee [\![B]\!]$ exists and equals $[\![A \vee B]\!]$. The pseudo-complement of $[\![A]\!]$ relative to $[\![B]\!]$, i.e. $[\![A]\!] \to [\![B]\!]$ exists and equals $[\![A \to B]\!]$. Also $0 = [\![\bar{Q}]\!] = [\![A \wedge \neg A]\!]$ and $1 = [\![\bar{P}]\!] = [\![A \to A]\!]$ for any sentence A of L(Đ). Thus $< H, \wedge, \vee, \to, \neg, 0, 1 >$ is a Ha, where $\neg [\![A]\!] \triangleq [\![A]\!] \to 0$, which means $\neg [\![A]\!] = [\![A \to A \wedge \neg A]\!] = [\![\neg A]\!]$.

Lemma 4: For each $\forall x A(x), \exists x A(x) \in L(Đ)$,

$$[\![\forall x A(x)]\!] = \bigwedge_{c \in Đ} [\![A(c)]\!] \text{ and } [\![\exists x A(x)]\!] = \bigvee_{c \in Đ} [\![A(c)]\!].$$

Definition 5: Set $\Box [\![A]\!] \triangleq [\![\Box A]\!]$ for each $[\![A]\!] \in H$.

From this definition we can obtain

Lemma 6: For every A, B, A(c), $\bar{A}$ ($\Box$-closed) in L(Đ), the following hold:

(1) $\Box [\![\bar{A}]\!] = [\![\bar{A}]\!]$

(2) $\Box 1 = 1$ and $\Box 0 = 0$

(3) $[\![\bar{A}]\!] \wedge [\![\neg \bar{A}]\!] = 0$ and $[\![\bar{A}]\!] \vee [\![\neg \bar{A}]\!] = 1$

(4) $G1_H : \Box [\![A]\!] \leq [\![A]\!]$

$G2_H : \Box [\![A]\!] \to \Box [\![B]\!] \leq \Box(\Box [\![A]\!] \to [\![B]\!])$

$G3_H : \bigwedge_{c \in Đ} \Box [\![A(c)]\!] \leq \Box \bigwedge_{c \in Đ} [\![A(c)]\!]$,

i.e., $[\![\forall x \Box A(x)]\!] \leq [\![\Box \forall x A(x)]\!]$

$G4_H$: If $\Box [\![A]\!] \leq [\![B]\!]$, then $\Box [\![A]\!] \leq \Box [\![B]\!]$.

$G5_H : \Box [\![A]\!] \vee \neg \Box [\![A]\!] = 1$.

Thus $< H, \wedge, \vee, \to, \neg, \Box, 0, 1 >$ is a Ha with a globalization in the sense that G3 of a cHag holds in the form of $G3_H$.

Step 2: The construction of a new Ha

Definition 7: Let $\Box H \triangleq \{ [\![\bar{A}]\!] : \bar{A}$ is a $\Box$-closed sentence of L(Đ) $\}$.

Then $< \Box H, \wedge^{H}, \vee^{H}, \rightarrow^{H}, \neg^{H}, \Box^{H}, 0^{H}, 1^{H} >$, or simply $\Box H$, is a sublattice of H and a Ba (Boolean algebra) since $\Box H$ is a distributive lattice with 0 and 1 and for each $[\![\bar{A}]\!] \in \Box H$, $[\![\bar{A}]\!] \wedge \neg [\![\bar{A}]\!] = 0$ and $[\![\bar{A}]\!] \vee \neg [\![\bar{A}]\!] = 1$. It also holds that

$$\bigwedge^{\Box H}{}_{c \in \text{Đ}} [\![\bar{A}(c)]\!] = \bigwedge^{H}{}_{c \in \text{Đ}} [\![\bar{A}(c)]\!] = [\![\forall x \bar{A}(x)]\!] \text{ and}$$

$$\bigvee^{\Box H}{}_{c \in \text{Đ}} [\![\bar{A}(c)]\!] = \bigvee^{H}{}_{c \in \text{Đ}} [\![\bar{A}(c)]\!] = [\![\exists x \bar{A}(x)]\!].$$

Definition 8: Let $\boldsymbol{B}$ be a Ba and let (Q) be a set of infinite joins and meets in $\boldsymbol{B}$ as follows:

$$a_s = \bigvee^{\boldsymbol{B}}{}_{t \in Ts'} a_{s,t} \quad (s \in S') \text{ and}$$

$$b_s = \bigwedge^{\boldsymbol{B}}{}_{t \in Ts''} b_{s,t} \quad (s \in S''),$$

where two sets S' and S'' are at most countable.

Lemma 9 (Rasiowa & Sikorski's Theorem): Let $\boldsymbol{B}$ and (Q) be as in Definition 8. Then there exists a maximal filter ∇ of $\boldsymbol{B}$ such that

$$\forall s \in S' \; (a_s \in \nabla \Rightarrow \exists t \in Ts' \;\; a_{s,t} \in \nabla) \text{ and}$$

$$\forall s \in S'' \; ((\forall t \in Ts'' \;\; b_{s,t} \in \nabla) \Rightarrow b_s \in \nabla).$$

Such a filter is called a "Q-filter." From now on, we will use ∇ to denote the Q-filter in $\Box H$, where (Q) is the set of all infinite meets of the form $\bigwedge^{\Box H}{}_{c \in \text{Đ}} [\![\bar{A}(c)]\!]$ and all infinite joins of the form $\bigvee^{\Box H}{}_{c \in \text{Đ}} [\![\bar{A}(c)]\!]$.

Definition 10: For each $[\![A]\!], [\![B]\!] \in H$, set

(1) $[\![A]\!] \leqq [\![B]\!]$ iff $([\![A]\!] \mathrel{\Box\!\!\rightarrow} [\![B]\!]) \in \nabla$,

where $[\![A]\!] \mathrel{\Box\!\!\rightarrow} [\![B]\!] \triangleq \Box([\![A]\!] \rightarrow [\![B]\!])$

(2) $[\![A]\!] \sim [\![B]\!]$ iff $([\![A]\!] \leqq [\![B]\!]$ and $[\![B]\!] \leqq [\![A]\!])$.

Then the following lemma is immediate:

Lemma 11: For each $[\![A]\!], [\![B]\!], [\![C]\!] \in H$,

(1) $[\![A]\!] \sim [\![B]\!]$ iff $([\![A]\!] \mathrel{\Box\!\!\rightarrow} [\![B]\!] \wedge [\![B]\!] \mathrel{\Box\!\!\rightarrow} [\![A]\!]) \in \nabla$

(2) $[\![A]\!] \leqq [\![A]\!]$

(3) $[\![A]\!] \leqq [\![B]\!]$ and $[\![B]\!] \leqq [\![C]\!]$ implies $[\![A]\!] \leqq [\![C]\!]$

(4) $\sim$ is an equivalence relation on H.

Definition 12: For each $[\![A]\!] \in H$, let

$$|[\![A]\!]| \triangleq \{[\![B]\!] \in H : [\![A]\!] \sim [\![B]\!]\} \text{ and}$$

$$H^* \triangleq H/\sim \triangleq \{ |[\![A]\!]| : [\![A]\!] \in H\}.$$

Then for each $|[\![A]\!]|, |[\![B]\!]| \in H^*$, set

$$|[\![A]\!]| \lesssim |[\![B]\!]| \Leftrightarrow [\![A]\!] \leqq [\![B]\!].$$

Note that $|[\![A]\!]| = |[\![B]\!]|$ iff $[\![A]\!] \sim [\![B]\!]$ and that $\lesssim$ is well-defined and is a partial order on H^*. We now list two easy lemmas.

Lemma 13: Let $|[\![A]\!]|, |[\![B]\!]| \in H^*$. Then the g.l.b. of $|[\![A]\!]|$ and $|[\![B]\!]|$, i.e. $|[\![A]\!]| \wedge^{H^*} |[\![B]\!]|$ exists and equals $|[\![A]\!] \wedge [\![B]\!]|$. The l.u.b. of $|[\![A]\!]|$ and $|[\![B]\!]|$, i.e. $|[\![A]\!]| \vee^{H^*} |[\![B]\!]|$ exists and equals $|[\![A]\!] \vee [\![B]\!]|$. The pseudo-complement of $|[\![A]\!]|$ relative to $|[\![B]\!]|$, i.e. $|[\![A]\!]| \rightarrow^{H^*} |[\![B]\!]|$ exists and equals $|[\![A]\!] \rightarrow [\![B]\!]|$. Also $0^{H^*} = |0^H|$ and $1^{H^*} = |1^H|$. Thus $< H^*, \wedge^{H^*}, \vee^{H^*}, \rightarrow^{H^*}, \neg^{H^*}, 0^{H^*}, 1^{H^*} >$ is a Ha, where $\neg^{H^*} |[\![A]\!]| \triangleq |[\![A]\!]| \rightarrow^{H^*} 0^{H^*}$, which means $\neg^{H^*} |[\![A]\!]| = |[\![A]\!]| \rightarrow^{H^*} |0^H| = |[\![A]\!] \rightarrow 0^H| = |\neg[\![A]\!]|$.

Lemma 14: Let $\bar{A}$ be a $\Box$-closed sentence of L(Đ). Then

(1) $[\![\neg\bar{A}]\!] \in \nabla$ iff $[\![\bar{A}]\!] \notin \nabla$

(2) $|[\![\bar{A}]\!]| = 1^{H^*}$ iff $[\![\bar{A}]\!] \in \nabla$

(3) $|[\![\bar{A}]\!]| = 0^{H^*}$ iff $[\![\bar{A}]\!] \notin \nabla$.

(4) $[\![\bar{A}]\!] \in \nabla$ or $\neg[\![\bar{A}]\!] \in \nabla$, but not both.

Lemma 15: For each $\bigwedge^H_{c \in Đ} [\![A(c)]\!]$ and $\bigvee^H_{c \in Đ} [\![A(c)]\!] \in H$,

(1) $|\bigwedge^H_{c \in Đ} [\![A(c)]\!]| = \bigwedge^{H^*}_{c \in Đ} |[\![A(c)]\!]|$

(2) $|\bigvee^H_{c \in Đ} [\![A(c)]\!]| = \bigvee^{H^*}_{c \in Đ} |[\![A(c)]\!]|$.

Proof: Since $\vdash \forall xA(x) \Rightarrow A(c)$, we have $\vdash \ \Rightarrow \Box(\forall xA(x) \rightarrow A(c))$. Then $[\![\forall xA(x)]\!] \mathrel{\Box\!\!\to} [\![A(c)]\!] \in \nabla$, i.e. $|[\![\forall xA(x)]\!]| \lesssim |[\![A(c)]\!]|$ for each $c \in \text{Đ}$.

Now suppose $|[\![B]\!]| \lesssim |[\![A(c)]\!]|$, i.e. $[\![B]\!] \mathrel{\Box\!\!\to} [\![A(c)]\!] \in \nabla$ for each $c \in \text{Đ}$. Then $\bigwedge^{\Box H}{}_{c\in \text{Đ}} ([\![B]\!] \mathrel{\Box\!\!\to} [\![A(c)]\!]) \in \nabla$, since ∇ is a Q-filter. Now since $\bigwedge^{\Box H}{}_{c\in \text{Đ}} ([\![B]\!] \mathrel{\Box\!\!\to} [\![A(c)]\!]) = \bigwedge^{H}{}_{c\in \text{Đ}} ([\![B]\!] \mathrel{\Box\!\!\to} [\![A(c)]\!])$, we can obtain $\bigwedge^{H}{}_{c\in \text{Đ}} ([\![B]\!] \mathrel{\Box\!\!\to} [\![A(c)]\!]) \in \nabla$, from which we can also obtain $\Box\bigwedge^{H}{}_{c\in \text{Đ}}([\![B]\!] \rightarrow [\![A(c)]\!]) \in \nabla$ by $G3_H$. Since $\vdash \Box\forall x(B \rightarrow A(x)) \Rightarrow \Box(B \rightarrow \forall x A(x))$, we obtain $\Box([\![B]\!] \rightarrow \bigwedge^{H}{}_{c\in \text{Đ}} [\![A(c)]\!]) \in \nabla$. This means $|[\![B]\!]| \lesssim |[\![\forall xA(x)]\!]|$. The proof of (2) is similar.

Definition 16: Set $\Box^{H^*}|[\![A]\!]| \triangleq |\Box[\![A]\!]|$.

Now we can obtain the following three lemmas:

Lemma 17: For each $|[\![A]\!]|, |[\![B]\!]|, |[\![A(c)]\!]| \in H^*$,

$G1_{H^*}: \Box^{H^*}|[\![A]\!]| \lesssim |[\![A]\!]|$

$G2_{H^*}: \Box^{H^*}|[\![A]\!]| \rightarrow^{H^*} \Box^{H^*}|[\![B]\!]| \lesssim \Box^{H^*}(\Box^{H^*}|[\![A]\!]| \rightarrow^{H^*} |[\![B]\!]|)$

$G3_{H^*}: \bigwedge^{H^*}{}_{c\in \text{Đ}} \Box^{H^*}|[\![A(c)]\!]| \lesssim \Box^{H^*}\bigwedge^{H^*}{}_{c\in \text{Đ}} |[\![A(c)]\!]|$,

i.e. $|[\![\forall x\Box A(x)]\!]| \lesssim |[\![\Box\forall xA(x)]\!]|$

$G4_{H^*}$: If $\Box^{H^*}|[\![A]\!]| \lesssim |[\![B]\!]|$, then $\Box^{H^*}|[\![A]\!]| \lesssim \Box^{H^*}|[\![B]\!]|$

$G5_{H^*}: \Box^{H^*}|[\![A]\!]| \vee^{H^*} \neg^{H^*}\Box^{H^*}|[\![A]\!]| = 1^{H^*}$.

Thus $< H^*, \wedge^{H^*}, \vee^{H^*}, \rightarrow^{H^*}, \neg^{H^*}, \Box^{H^*}, 0^{H^*}, 1^{H^*} >$ is a Ha with a globalization in the sense that G3 of a cHag holds in the form of $G3_{H^*}$.

Lemma 18: The function g: $H \longrightarrow H^*$ defined by $[\![A]\!] \longmapsto |[\![A]\!]|$ is a natural homomorphism from H onto H^* and preserves not only $\Box$ but also infinite meets and joins of the form $\bigwedge^{H}{}_{c\in \text{Đ}} [\![A(c)]\!]$ and $\bigvee^{H}{}_{c\in \text{Đ}} [\![A(c)]\!]$, i.e.

$g(\Box[\![A]\!]) = \Box^{H^*}g([\![A]\!])$

$g(\bigwedge^{H}{}_{c\in \text{Đ}} [\![A(c)]\!]) = \bigwedge^{H^*}{}_{c\in \text{Đ}}\, g([\![A(c)]\!])$ and

$$g(\bigvee^{H}{}_{c\in Đ} [\![A(c)]\!]) = \bigvee^{H^*}{}_{c\in Đ}\, g([\![A(c)]\!]) .$$

Lemma 19: For each $|[\![A]\!]| \in H^*$,

$$\Box^{H^*}|[\![A]\!]| = \bigvee^{H^*}\{ \Box^{H^*}|[\![B]\!]| \in H^* : \Box^{H^*}|[\![B]\!]| \lesssim |[\![A]\!]| \}.$$

Step 3: The construction of a cHag

We now construct a cHag from H^*.

Lemma 20 (Rasiowa & Sikorski's Embedding Lemma): Let H^* be a Ha. Then there exist a cHa H^{**} and an isomorphism from H^* into H^{**}, preserving all infinite meets and joins.

By this lemma, we can obtain a cHa H^{**} from the Ha H^* in Step 2 and an isomorphism h: $H^* \longrightarrow H^{**}$ such that for each indexed set $\{a_i\}_i \subseteq H^*$,

$$h(\bigwedge^{H^*}{}_i\ a_i) = \bigwedge^{H^{**}}{}_i\, h(a_i) \quad \text{and} \quad h(\bigvee^{H^*}{}_i\ a_i) = \bigvee^{H^{**}}{}_i\, h(a_i) .$$

We denote this cHa $<H^{**}, \wedge^{H^{**}}, \vee^{H^{**}}, \to^{H^{**}}, \neg^{H^{**}}, 0^{H^{**}}, 1^{H^{**}}, \bigwedge^{H^{**}}, \bigvee^{H^{**}}>$ by "H^{**}."

Definition 21: Define a globalization $\Box^{H^{**}}$ as follows: for each $a \in H^{**}$,

$$\Box^{H^{**}}a = \bigvee^{H^{**}}\{ h(|\Box[\![A]\!]|) \in H^{**} : h(|\Box[\![A]\!]|) \leq a \},$$

where $\leq$ is the partial order on H^{**}.

Lemma 22: For each $a \in H^{**}$,

$$\Box^{H^{**}}a = \begin{cases} 1^{H^{**}} & \text{if } a = 1^{H^{**}}, \\ 0^{H^{**}} & \text{if } a \neq 1^{H^{**}}. \end{cases}$$

Proof: $h(0^{H^*}) = 0^{H^{**}}$ and $h(1^{H^*}) = 1^{H^{**}}$. By Lemma 14, each $|\Box[\![A]\!]| \in H^*$ is either 0^{H^*} or 1^{H^*}. So for each $h(|\Box[\![A]\!]|) \in H^{**}$,

$$h(|\Box[\![A]\!]|) = \begin{cases} 1^{H^{**}} & \text{if } |\Box[\![A]\!]| = 1^{H^*}, \\ 0^{H^{**}} & \text{if } |\Box[\![A]\!]| \neq 1^{H^*}. \end{cases}$$

$$\text{Then } \Box^{H^{**}}a = \bigvee^{H^{**}}\{ h(|\Box[\![A]\!]|) \in H^{**}: h(|\Box[\![A]\!]|) \leq a \}$$

$$= \begin{cases} 1^{H^{**}} & \text{if } a = 1^{H^{**}}, \\ 0^{H^{**}} & \text{if } a \neq 1^{H^{**}}. \end{cases}$$

Lemma 23: For each $a,b \in H^{**}$ and each indexed set $\{a_i\}_i \subseteq H^{**}$,

$G1_{H^{**}}$: $\Box^{H^{**}}a \leq a$

$G2_{H^{**}}$: $\Box^{H^{**}}a \to^{H^{**}} \Box^{H^{**}}b \leq \Box^{H^{**}}(\Box^{H^{**}}a \to^{H^{**}} b)$

$G3_{H^{**}}$: $\bigwedge^{H^{**}}{}_i \Box^{H^{**}}a_i \leq \Box^{H^{**}}\bigwedge^{H^{**}}{}_i\ a_i$

$G4_{H^{**}}$: If $\Box^{H^{**}}a \leq b$, then $\Box^{H^{**}}a \leq \Box^{H^{**}}b$

$G5_{H^{**}}$: $\Box^{H^{**}}a \vee^{H^{**}} \neg^{H^{**}}\Box^{H^{**}}a = 1^{H^{**}}$.

Thus $< H^{**}, \wedge^{H^{**}}, \vee^{H^{**}}, \to^{H^{**}}, \neg^{H^{**}}, \Box^{H^{**}}, 0^{H^{**}}, 1^{H^{**}}, \bigwedge^{H^{**}}, \bigvee^{H^{**}} >$ is a cHag and denoted by " H^{**} ."

Proof: Using Lemma 22, the proof is straightforward.

Lemma 24: The isomorphism h: $H^* \longrightarrow H^{**}$ preserves $\Box$, i.e.

$$h(\Box^{H^*}|[\![A]\!]|) = \Box^{H^{**}} h(|[\![A]\!]|) .$$

Proof: $h(\Box^{H^*}|[\![A]\!]|) = h(\bigvee^{H^*}\{\Box^{H^*}|[\![B]\!]| \in H^* : \Box^{H^*}|[\![B]\!]| \lesssim |[\![A]\!]|\})$,

by Lemma 19

$= \bigvee^{H^{**}}\{h(\Box^{H^*}|[\![B]\!]|) \in H^{**} : h(\Box^{H^*}|[\![B]\!]|) \lesssim h(|[\![A]\!]|)\}$,

since h preserves infinite joins

$= \Box^{H^{**}} h(|[\![A]\!]|)$ by the definition of $\Box^{H^{**}}$ in H^{**}.

Therefore the map $h \circ g: H \longrightarrow H^{**}$ is a homomorphism and preserves not only infinite meets and joins but also $\Box$. The definition of a map $[\![\ ]\!]^{**}$ in H^{**} goes as follows:

Definition 25:

(1) For the constants of L(Đ), set

$[\![c]\!]^{**} \triangleq c \in Đ$ for each individual constant c of Đ, and

$[\![R^n]\!]^{**} : Đ^n \longrightarrow H^{**}$ is defined by : for each $c_{i_1}, \ldots, c_{i_n} \in Đ$,

$[\![R^n]\!]^{**}([\![c_{i_1}]\!]^{**}, \ldots, [\![c_{i_n}]\!]^{**}) \triangleq h \circ g([\![R^n(c_{i_1}, \ldots, c_{i_n})]\!]) \in H^{**}$.

(2) For the sentences of L(Đ), set $[\![A]\!]^{**} \triangleq h \circ g([\![A]\!]) \in H^{**}$.

Lemma 26: For the map $[\![\ ]\!]^{**}$, we have

(1) $[\![A \wedge B]\!]^{**} = [\![A]\!]^{**} \wedge^{H^{**}} [\![B]\!]^{**}$

(2) $[\![A \vee B]\!]^{**} = [\![A]\!]^{**} \vee^{H^{**}} [\![B]\!]^{**}$

(3) $[\![A \rightarrow B]\!]^{**} = [\![A]\!]^{**} \rightarrow^{H^{**}} [\![B]\!]^{**}$

(4) $[\![\neg A]\!]^{**} = \neg^{H^{**}} [\![A]\!]^{**}$

(5) $[\![\forall x A(x)]\!]^{**} = \bigwedge^{H^{**}}{}_{c \in Ð} [\![A(c)]\!]^{**}$

(6) $[\![\exists x A(x)]\!]^{**} = \bigvee^{H^{**}}{}_{c \in Ð} [\![A(c)]\!]^{**}$

(7) $[\![\Box A]\!]^{**} = \Box^{H^{**}} [\![A]\!]^{**}$

Proof: For (5) we have $[\![\forall x A(x)]\!]^{**} = h \circ g([\![\forall x A(x)]\!]) = h \circ g(\bigwedge H_{c \in Ð} [\![A(c)]\!]) = \bigwedge^{H^{**}}{}_{c \in Ð}\, h \circ g([\![A(c)]\!]) = \bigwedge^{H^{**}}{}_{c \in Ð} [\![A(c)]\!]^{**}$. The rest are similar.

Therefore $< Ð, H^{**}, [\![\ ]\!]^{**} >$ is a cHag interpretation in which $[\![\bar{P}]\!]^{**} = h \circ g([\![\bar{P}]\!]) = 1^{H^{**}}$ and $[\![\bar{Q}]\!]^{**} = h \circ g([\![\bar{Q}]\!]) = 0^{H^{**}}$. Thus $\not\models \bar{P} \Rightarrow \bar{Q}$.
This completes the proof of the completeness theorem.

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