# Glueing of Algebras
# for Substructural Logics

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October 11, 1995

### Abstract

We will introduce the notion of the glueing of algebras for substructural logics and prove the disjunction and existence properties for them, by using the glueing.

First, we will introduce the glueing of algebras for intuitionistic substructural propositional logics and prove the disjunction property of them, which includes logics with $n$-contraction [11] and logics with knotted structural rules [3]. No cut-free systems for them are known and therefore usual syntactic arguments do not work for them.

Next we will introduce also the glueing for intuitionistic substructural predicate logics. More precisely, we will define the glueing of hyperdoctrinal semantics which was introduced in [12] and will prove both the disjunction and existence properties of substructural predicate logics. Lastly, we will give a sufficient condition for a given substructural predicate logic with the weakening to have these properties.

## Introduction

In order to study various logics in the uniform way, Ono [7] discussed substructural logics regarded as any extensions of the *full Lambek logic* (FL) which, roughly speaking, is the sequent calculus obtained from the sequent calculus LJ for intuitionistic logic by deleting all the structural rules (For similar discussion, see [1], [2], [9]) . In [12], we introduced po-hyperdoctrinal semantics for intuitionistic substructural predicate logics including FL. In this paper, we will discuss the disjunction and existence properties for them as an application of this semantics.

The glueing construction has been used in the categorical logic to prove the disjunction and existence properties for various logics. For example, glueing topoi (or the Freyd cover) has been used for intuitionistic higher-order theories (see [6]) and glueing hyperdoctrines for the first-order intuitionistic logic (see [10]). In this paper, we will extend it to the glueing of algebras for substructural logics. By using glueing, we will prove the disjunction and existence properties for many of them.

In section 1, we define intuitionistic substructural logics that will be discussed in this paper and glueing of algebras for them, and show basic results on glueing.

In section 2, we will prove the disjunction property of many intuitionistic substructural propositional logics, which includes logics with $n$-contraction [11] and with knotted structural rules [3]. No cut-free systems for them are known and therefore usual syntactic arguments do not work for them. Connectification operators for intuitionistic linear logic which was introduced in [11] to prove the disjunction property for linear logic and its extension with $n$-contraction, can be regarded as a special case of glueing.

In section 3, we will extend the glueing to that for predicate logics. More precisely, we will introduce the glueing of po-hyperdoctrinal semantics discussed in [12]. Then we can prove both the disjunction and existence properties for many substructural predicate logics.

In [4], Hosoi introduced the delta operation $\Delta$ on super-intuitionistic propositional logics. It is easy to see that the intuitionistic propositinal logic is the only fixed point of $\Delta$. Then Komori [5] and N.-Y. Suzuki [14] extended it to the operation for super-intuitionistic predicate logics and showed that every super-intuitionistic predicate logic which is a fixed point of $\Delta$ enjoys the disjunction and existence properties. In section 4, we will show that the delta operation can be extended to an operation for substructural predicate logics with the weakening and as an application of our glueing construction, will give a similar sufficient condition for them to have these properties.

# 1 Preliminaries

We will fix a first-order language, consisting of the logical constants $1, 0, \top$ and $\bot$, and the logical connectives $\vee, \wedge, *$ and $\supset$ (In this paper, we will follow the notation for logical operators of [8],[16]). We assume also that it contains neither constants nor function symbols.

When a logic is obtained from FL by adding some structural rules, we call it a *structural extension* of FL and denote it by $\mathrm{FL}_\sigma$ where $\sigma$ is a set of corresponding letters for structural rules. In this paper, we will discuss structural rules, exchange(e), weakening(w), contraction(c), $n$-contraction($n \geq 2$), and knotted structural rules($m \rightsquigarrow k$), where $n, m, k$ are natural numbers such that $n \geq 2, m \neq k$ and $k > 0$. Let $\mathrm{FL}_\sigma$ be a structural extension of FL. A *super-$\mathrm{FL}_\sigma$ logic* **L** is a set of formulas such that

1. $\mathrm{FL}_\sigma \subseteq \mathbf{L}$,

2. **L** is closed under two kinds of modus ponens, i.e.
   (m.p.1) if formulas $\alpha$ and $\alpha \supset \beta$ are in **L**, then $\beta$ is also in **L**,
   (m.p.2) if formulas $\alpha$ and $\beta \supset \alpha \supset \gamma$ are in **L**, then $\beta \supset \gamma$ is also in **L**,

3. **L** is closed under adjunction, i.e. if formulas $\alpha$ and $\beta$ are in **L**, then $\alpha \wedge \beta$ is also in **L**,

4. **L** is closed under generalization and substitution.

A super-$\mathrm{FL}_\sigma$ propositional logic is a set of only propositional formulas which includes a propositional $\mathrm{FL}_\sigma$ and is closed under (m.p.1), (m.p.2), adjunction and substitution. Throughout this paper, we sometimes call a super-FL logic only a logic.

A structure $\mathbf{A} = \langle A, \supset, \vee, \wedge, *, 1, 0, \top, \bot \rangle$ is an FL-*algebra* if

1. $\langle A, \vee, \wedge, \top, \bot \rangle$ is a lattice,

2. $\langle A, *, 1 \rangle$ is a monoid,

3. $z * (x \vee y) * w = (z * x * w) \vee (z * y * w)$ for every $x, y, z, w \in A$,

4. $x * y \leq z \Leftrightarrow x \leq y \supset z$ for every $x, y, z \in A$,

5. $0 \in A$.

We will sometimes introduce another implication and denote it by $\supset'$, which satisfies

4'. $y * x \leq z \Leftrightarrow x \leq y \supset' z$ for every $x, y, z \in A$.

Some knowledge of algebraic semantics and FL-algebras will be assumed. We refer the reader to [8],[16].

In [10], glueing of Heyting algebras was given and applied fibrewise to hyperdoctrines. We will extend it to glueing of FL-algebras.

**Definition 1.1** *A function* $\delta : \mathbf{A} \to \mathbf{B}$ *between FL-algebras is called a* fringe *morphism if it is a meet-semilattice morphism satisfying* $\delta 1 \geq 1, \delta 0 \geq 0$ *and* $\delta(\phi * \psi) \geq \delta\phi * \delta\psi$ *for all* $\phi, \psi \in \mathbf{A}$.

**Definition 1.2 (glueing)** *Let* $\mathbf{A}, \mathbf{B}$ *be FL-algebras,* $\gamma : \mathbf{A} \to \mathbf{B}$ *a fringe morphism. Define* $Gl(\gamma) = \langle |Gl(\gamma)|, \supset, \vee, \wedge, *, 1, 0, \top, \bot \rangle$ *by*

$$|Gl(\gamma)| = \{(b, a) \in \mathbf{B} \times \mathbf{A} \mid b \leq \gamma a\},$$

*the order relation* $(b, a) \leq (b', a')$ *iff* $b \leq b'$ *in* $\mathbf{B}$ *and* $a \leq a'$ *in* $\mathbf{A}$,
*implication* $(b, a) \supset (b', a') = ((b \supset b') \wedge \gamma(a \supset a'), a \supset a')$,
*and other FL-algebra operations pointwise*
*(for example,* $1 = (1, 1), (b, a) * (b', a') = (b * b', a * a'))$.

*This* $Gl(\gamma)$ *is called the* glueing *of* $\mathbf{A}$ *to* $\mathbf{B}$ *along* $\gamma$.

It is easy to see that $Gl(\gamma)$ is an FL-algebra and the second projection function $\rho : Gl(\gamma) \to \mathbf{A}$ is an FL-morphism so that the following diagram commutes.

$$Gl(\gamma) \xrightarrow{\rho} \mathbf{A}$$

with maps to $\mathbf{B} \times \mathbf{A}$ and $\pi_2$ to $\mathbf{A}$.

The following lemma shows a basic property of glueing of FL-algebras.

**Lemma 1.3** *Let* $\mathcal{V}$ *be a class of FL-algebras characterized by a set* $E$ *of inequalities without using* $\supset$. *If both FL-algebras* $\mathbf{A}$ *and* $\mathbf{B}$ *are included in* $\mathcal{V}$ *and* $\gamma : \mathbf{A} \to \mathbf{B}$ *is a fringe morphism between them, then* $Gl(\gamma)$ *is also included in* $\mathcal{V}$.

PROOF. We fix a countable set $X$ of variables and a signature $\Sigma = \langle 1, 0, \top, \bot, *, \wedge, \vee \rangle$. By our assumption, any inequality in $E$ is of the form $e_1 \leq e_2$ where $e_i (i = 1, 2)$ is a $\Sigma$-term. For each valuation $\rho : X \to \mathbf{A}$, we can define an interpretation $[\![e]\!]_\rho$ on $\mathbf{A}$ in a usual way by induction on $e$. We write $\mathbf{A} \models e_1 \leq e_2$ if $[\![e_1]\!]_\rho \leq [\![e_2]\!]_\rho$ in $\mathbf{A}$ for all $\rho : X \to \mathbf{A}$. We say that $\mathcal{V}$ is characterized by a set $E$ of inequalities when for any FL-algebra $\mathbf{A}$ $\mathbf{A} \in \mathcal{V}$ iff $\mathbf{A} \models e_1 \leq e_2$ for all $e_1 \leq e_2 \in E$.

When we use this notation, it suffices to show that $\mathbf{A} \models e_1 \leq e_2$ and $\mathbf{B} \models e_1 \leq e_2$ implies $Gl(\gamma) \models e_1 \leq e_2$. This follows from the fact that for a given $\rho : X \to Gl(\gamma)$,

$$[\![e_1]\!]_\rho = ([\![e_1]\!]_{\pi_1 \rho}, [\![e_1]\!]_{\pi_2 \rho}) \leq ([\![e_2]\!]_{\pi_1 \rho}, [\![e_2]\!]_{\pi_2 \rho}) = [\![e_2]\!]_\rho$$

since $\Sigma$ does not include $\supset$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

## 2 Glueing in propositional logics

In this section, we will consider only propositional logics and the corresponding FL-algebras. Results in this section will be extended to predicate logics in the next section. Let $\mathbf{L}(\mathbf{A})$ be the set of all formulas (in propositional logics) which are valid in a given FL-algebra $\mathbf{A}$. Then $\mathbf{L}(\mathbf{A})$ is a logic. Conversely, for any logic $\mathbf{L}$, it is easy to see by the standard argument that the Lindenbaum algebra $\mathcal{A}_{\mathbf{L}}$ of $\mathbf{L}$ is an FL-algebra which satisfies $\mathbf{L} = \mathbf{L}(\mathcal{A}_{\mathbf{L}})$. For a given logic $\mathbf{L}$, we define the class $\mathcal{V}(\mathbf{L})$ of FL-algebras by $\mathcal{V}(\mathbf{L}) = \{ \mathbf{A} : \text{FL-}algebra \mid \mathbf{L} \subseteq \mathbf{L}(\mathbf{A}) \}$. A logic $\mathbf{L}$ has the disjunction property, if $\alpha \vee \beta \in \mathbf{L}$ implies $\alpha \in \mathbf{L}$ or $\beta \in \mathbf{L}$ for each formula $\alpha$ and $\beta$.

**Definition 2.1** *An* FL-*algebra* $\mathbf{A}$ *has the property (DP) if* $1 \leq \phi * \psi$ *implies* $1 \leq \phi$ *or* $1 \leq \psi$ *for all* $\phi, \psi \in \mathbf{A}$.

We notice that the property (DP) for the Lindenbaum algebra of $\mathbf{L}$ is equivalent to the disjunction property for $\mathbf{L}$.

**Definition 2.2** *For an* FL-*algebra* $\mathbf{A}$, *define the function* $\gamma : \mathbf{A} \to \mathbf{2}$ *by*

$$\gamma : \phi \mapsto \begin{cases} \top & \textit{if } \phi \geq 1, \\ \bot & \textit{otherwise} \end{cases}$$

*The* FL-*algebra* $\mathbf{2}$ *has* $\{\top (= 1), \bot (= 0)\}$ *as an underlying set and satisfies* $* = \wedge$. *Then* $\gamma$ *is a fringe morphism. The* FL-*algebra* $\widehat{\mathbf{A}}$, *called the glued algebra of* $\mathbf{A}$, *is defined as the glueing* $Gl(\gamma)$ *of* $\mathbf{A}$ *to* $\mathbf{2}$ *along* $\gamma$. *We call that* $\mathcal{V}(\mathbf{L})$ *is closed under glueing if* $\widehat{\mathbf{A}} \in \mathcal{V}(\mathbf{L})$ *for all* $\mathbf{A} \in \mathcal{V}(\mathbf{L})$.

We will show two lemmas below in order to prove Proposition 2.5, which is the main result in this section.

**Lemma 2.3** $\widehat{\mathbf{A}}$ *satisfies (DP)*.

PROOF. Given $(a, \phi), (b, \psi) \in \widehat{\mathbf{A}}$, if $(a, \phi) \vee (b, \psi) \geq 1$, then $a \vee b = \top$, so either $a = \top$ and then $\phi \geq 1$, or $b = \top$ and then $\psi \geq 1$. □

**Lemma 2.4** *Suppose that* $\mathcal{V}(\mathbf{L})$ *is closed under glueing. If* $\mathbf{A}$ *is a free algebra in* $\mathcal{V}(\mathbf{L})$, *then* $\mathbf{A}$ *is a retract of* $\widehat{\mathbf{A}}$, *i.e. the second projection function* $\rho : \widehat{\mathbf{A}} \to \mathbf{A}$ *has a right inverse* $\iota$.

$$
\begin{array}{ccc}
\mathbf{A} & \xrightarrow{\quad \iota \quad} & \widehat{\mathbf{A}} \\
 & {}_{id_{\mathbf{A}}}\searrow & \downarrow {}^{\rho} \\
 & & \mathbf{A}
\end{array}
$$

PROOF. Suppose that $\mathbf{A}$ is freely generated by a set $X$. Take $\delta : X \to \widehat{\mathbf{A}}$ as $\delta(x) = (\gamma x, x)$. By the freeness of $\mathbf{A}$, $\rho \bar{\delta} = id_{\mathbf{A}}$. So we can take $\bar{\delta}$ as $\iota$. □

**Proposition 2.5** *If* $\mathcal{V}(\mathbf{L})$ *is closed under glueing, then* $\mathbf{L}$ *has the disjunction property.*

PROOF. Since the Lindenbaum algebra $\mathbf{A_L}$ of $\mathbf{L}$ is freely generated by the set of propositional variables, it suffices to show the following: If an $\mathrm{FL}_\sigma$-algebra $\mathbf{A}$ is free in $\mathcal{V}(\mathbf{L})$, then it satisfies (DP). Now, for any given $\phi, \psi \in \mathbf{A}$, if $\phi \vee \psi \geq 1$, then using $\iota$ from Lemma 2.4,

$$
1 = \iota 1 \leq \iota(\phi \vee \psi) = \iota\phi \vee \iota\psi \text{ in } \widehat{\mathbf{A}}.
$$

So by Lemma 2.3 either $1 \leq \iota\phi$ in which case $1 \leq \rho\iota\phi = \phi$, or else $1 \leq \iota\psi$ in which case $1 \leq \rho\iota\psi = \psi$. Thus $\mathbf{A}$ satisfies (DP). □

Using Lemma 1.3 and Proposition 2.5, we obtain the following corollary.

**Corollary 2.6** *A structural extension* $\mathrm{FL}_\sigma$ *of* $\mathrm{FL}$ *has the disjunction property, where* $\sigma \subseteq \{e, c, w, n, n \rightsquigarrow k\}$.

**Remark 2.7 (connectification)** When a given logic $\mathbf{L}$ has weakening, 1 becomes equal to $\top$. Then, it is easy to see that glueing is equivalent to connectification operators in propositional affine logics (see [11]).

# 3 Glueing in predicate logics

In this section, we will extend the results in previous section to those of predicate logics. We use po-hyperdoctrinal semantics for substructural predicate logics. This semantics can be regarded as an extension of algebraic semantics for them. Basic definition and properties are stated here, but we refer the reader to [12] for details.

Let $P^k$ be $k$-place predicate variable, $\alpha$ a formula, and $\vec{z} = (z_1, \ldots, z_m)$ variables not occurring in $\alpha$. Following [13], $\alpha^{[m]}$ denotes the result of replacing in $\alpha$ each atomic subformula of the form $P^k(\vec{x})$ by $P^{k+m}(\vec{x}, \vec{z})$. We may restrict our attention to the following FL-doctrines because our language contains neither constants nor function symbols.

**Definition 3.1** *Let $\Sigma_X$ be the free category with finite products on single object $X$, and* **FL**$_\sigma$ *the category of* FL$_\sigma$*-algebras with* $\supset'$*. An* FL$_\sigma$*-doctrine $\mathcal{A}$ over $\Sigma_X$ is a functor* $\mathcal{A} : \Sigma_X{}^{\mathrm{op}} \to$ **FL**$_\sigma$ *(as usual, we write $\alpha^*$ for $\mathcal{A}(\alpha) : \mathcal{A}(J) \to \mathcal{A}(I)$ where $\alpha : I \to J$ in $\Sigma_X$) such that*

1. *for all projection $\pi_2 : K \times I \to I$ in $\Sigma_X$, $\pi_2^* : \mathcal{A}(I) \to \mathcal{A}(K \times I)$ has a left and a right adjoint, donoted by $\exists_{K,I}$ and $\forall_{K,I}$ respectively.*

2. *for these adjoints, the following Beck-Chevally conditions hold: for all $\alpha : I \to J$ in $\Sigma_X$, $\alpha^* \exists_{K,J} = \exists_{K,I}(1 \times \alpha)^*$. Likewise for $\forall_{K,I}$.*



For any FL-doctrine $\mathcal{A}$, we define a *structure* $\mathcal{M}$ in $\mathcal{A}$ by an element $\mathcal{M}P^k \in \mathcal{A}(X^k)$ for each predicate symbol $P^k(k \geq 0)$. Given a formula $\alpha$ and a finite list $\vec{x}$ of variables $(FV(\alpha) \subseteq \vec{x}, |\vec{x}| = m)$, we define an *interpretation* $\mathcal{M}(\alpha; \vec{x}) \in \mathcal{A}(X^m)$ by induction on $\alpha$:

1. $\mathcal{M}(P^k(\vec{y}); \vec{x}) = (\pi_{m,k})^*(\mathcal{M}P^k)$ where $\vec{y} \subseteq \vec{x}$,

2. $\mathcal{M}(\Diamond; \vec{x}) = \Diamond$ in $\mathcal{A}(X^m)$ where $\Diamond \in \{\top, \bot, 1, 0\}$,

3. $\mathcal{M}(\alpha \circ \beta; \vec{x}) = \mathcal{M}(\alpha; \vec{x}) \circ \mathcal{M}(\beta; \vec{x})$ where $\circ \in \{*, \wedge, \vee, \supset, \supset'\}$,

4. $\mathcal{M}((\mathcal{Q}y)\alpha; \vec{x}) = \mathcal{Q}_{X,X^m}\mathcal{M}(\alpha; y\vec{x})$ where $\mathcal{Q} \in \{\exists, \forall\}$.

where $\pi_{m,k}$ denotes the unique morphism whose composition with each $\pi_i$ is $\pi_i$.



A formula $\alpha$ is said to be *valid* in an FL$_\sigma$-doctrine $\mathcal{A}$ if for every structure $\mathcal{M}$ in $\mathcal{A}$, $\mathcal{M}(\bar{\alpha}; ) \geq 1$ where $\bar{\alpha}$ is the universal closure of $\alpha$. Then we write $\mathcal{A} \models \alpha$. We define the following two sets of formulas determined by $\mathcal{A}$.

$$\mathbf{L}^\sim(\mathcal{A}) = \{\alpha \in FORM | \mathcal{A} \models \alpha\}$$
$$\mathbf{L}(\mathcal{A}) = \{\alpha \in FORM | \mathcal{A} \models \alpha^{[m]} \text{ for all } m \geq 0\}$$

$\mathbf{L}^{\sim}(\mathcal{A})$ is closed under two kinds of modus ponens, adjunction and generalization, but it is not always closed under substitution (see [12]). On the other hand, $\mathbf{L}(\mathcal{A})$ is closed under substitution, and hence it is a super-$\mathrm{FL}_\sigma$ logic.

For a given logic $\mathbf{L}$, if there exists an FL-doctrine $\mathcal{A}$ such that $\mathbf{L} = \mathbf{L}(\mathcal{A})$, then we say that $\mathbf{L}$ is characterized by $\mathcal{A}$ and also that $\mathbf{L}$ is complete with respect to hyperdoctrinal semantics. We can show that every logic is complete, by using the Lindenbaum FL-doctrine for the logic. We define a class $\mathcal{D}(\mathbf{L})$ of $\mathrm{FL}_\sigma$-doctrines by $\mathcal{D}(\mathbf{L}) = \{\mathcal{A} : \mathrm{FL}_\sigma\text{-doctrines}|\mathbf{L} \subseteq \mathbf{L}(\mathcal{A})\}$.

As the first step, we define fringe morphisms of FL-doctrines in order to introduce glued FL-doctrines in place of glued algebras.

**Definition 3.2** *Let* $\mathcal{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_\sigma, \mathcal{B} : \mathbf{D}^{\mathrm{op}} \to \mathbf{FL}_\sigma$ *be* $\mathrm{FL}_\sigma$*-doctrines. A fringe morphism* $\mathcal{A} \to \mathcal{B}$ *of* $\mathrm{FL}_\sigma$*-doctrines is a pair* $(\Delta, \delta)$ *of a cartesian functor* $\Delta : \mathbf{C} \to \mathbf{D}$ *and a natural transformation* $\delta : \mathcal{A} \to \mathcal{B}\Delta$ *such that*

1. *for each* $I \in Obj\mathbf{C}$, $\delta_I : \mathcal{A}(I) \to \mathcal{B}(\Delta I)$ *is a fringe morphism between FL-algebras.*

2. *for all projection* $\pi_2 : K \times I \to I$ *in* $\mathbf{C}$, *the following commutes:*

$$
\begin{array}{ccc}
K \times I & \mathcal{A}(K \times I) \xrightarrow{\delta_{K \times I}} \mathcal{B}(\Delta(K \times I)) \\
\pi_2 \Big\downarrow & \pi_2^* \Big\uparrow \vdash \Big\downarrow \exists_{K,I} \qquad \pi_2^* \Big\uparrow \vdash \Big\downarrow \exists_{\Delta K, \Delta I} \\
I & \mathcal{A}(I) \xrightarrow{\quad \delta_I \quad} \mathcal{B}(\Delta I)
\end{array}
$$

*Likewise for* $\forall_{K,I}$.

*To indicate that* $(\Delta, \delta) : \mathcal{A} \to \mathcal{B}$ *is a fringe morphism of* $\mathcal{A}$ *to* $\mathcal{B}$ *we sometimes write as the following 2-categorical diagram:*



**Definition 3.3** *Let* $\mathcal{P} : \mathcal{S}ets^{\mathrm{op}} \to \mathbf{FL}_\sigma$ *be an* $\mathrm{FL}_\sigma$*-doctrine of the powerset functor and* $\Gamma : \mathbf{C} \to \mathcal{S}ets$ *the global sections. For an* $\mathrm{FL}_\sigma$*-doctrine* $\mathcal{A} : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_\sigma$, *define a natural transformation* $\gamma : \mathcal{A} \to \mathcal{P}\Gamma$ *by*

- *for each* $I \in Obj\mathbf{C}$, $\gamma_I : \mathcal{A}(I) \to \mathcal{P}(\Gamma I)$ *as* $\gamma_I(\phi) = \{i \in \Gamma I | i^*\phi \geq 1\}$.

*It is easy to see that* $(\Gamma, \gamma)$ *is the fringe morphism* $\mathcal{A} \to \mathcal{P}$.



*Then a glued* $\mathrm{FL}_\sigma$*-doctrine* $\widehat{\mathcal{A}} : \mathbf{C}^{\mathrm{op}} \to \mathbf{FL}_\sigma$ *consists of*

1. $\widehat{\mathcal{A}}(I) = Gl(\gamma_I)$ for each $I \in Obj\mathbf{C}$,

2. for each $\alpha : I \to J$, $\alpha^* : \widehat{\mathcal{A}}(J) \to \widehat{\mathcal{A}}(I)$ is the restriction of $\alpha^*$ in $\mathcal{P}\Gamma \times \mathcal{A}$.

We call that $\mathcal{D}(\mathbf{L})$ is closed under glueing if $\widehat{\mathcal{A}} \in \mathcal{D}(\mathbf{L})$ for any $\mathcal{A} \in \mathcal{D}(\mathbf{L})$.

**Remark 3.4** For any projection $\pi_2 : K \times I \to I$ in $\mathbf{C}$, we can take $\forall_{K,I}$ and $\exists_{K,I}$ as follows: for $(X, \phi) \in \widehat{\mathcal{A}}(K \times I)$,

- $\forall_{K,I}(X, \phi) = (\forall_{K,I}X \cap \gamma_I(\forall_{K,I}\phi), \forall_{K,I}\phi)$ in $\widehat{\mathcal{A}}(I)$,

- $\exists_{K,I}(X, \phi) = (\exists_{K,I}X, \exists_{K,I}\phi)$ in $\widehat{\mathcal{A}}(I)$.

A predicate logic $\mathbf{L}$ has the disjunction property, if $\alpha \vee \beta \in \mathbf{L}$ implies $\alpha \in \mathbf{L}$ or $\beta \in \mathbf{L}$ for each formula $\alpha$ and $\beta$. A predicate logic $\mathbf{L}$ has the existence property, for each formula $\alpha(x)$ if $\exists x \alpha(x) \in \mathbf{L}$ implies $\alpha(t) \in \mathbf{L}$ for some term $t$. We will introduce the properties (DP),(EP) and free FL-doctrines. It is easily seen that (DP) and (EP) for the Lindenbaum FL-doctrine of $\mathbf{L}$ are equivalent to the disjunction and existence properties for $\mathbf{L}$, respectively.

**Definition 3.5** Let $\mathcal{A}$ be an $FL_\sigma$-doctrine over $\mathbf{C}$.
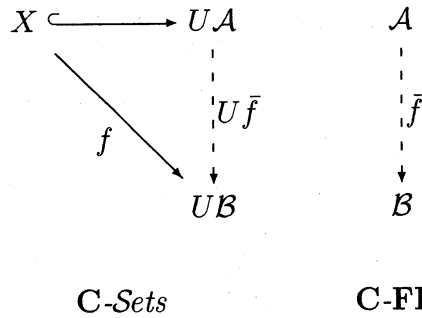
- $\mathcal{A}$ has the property (DP) if for any $\phi, \psi \in \mathcal{A}(1)$, if $1 \leq \phi \vee \psi$ then $1 \leq \phi$ or $1 \leq \psi$.

- $\mathcal{A}$ has the property (EP) if for any $K \in Obj\mathbf{C}$ and any $\phi \in \mathcal{A}(K \times 1)$, if $1 \leq \exists_{K,1}\phi$ then for some $k : 1 \to K \in \mathbf{C}$. $1 \leq \langle k, id_1 \rangle^* \phi$.

$$
\begin{array}{ccc}
\langle k, id_1 \rangle^* \phi & \longleftarrow\!\!\!\!\dashv\ \phi\ \vdash\!\!\!\!\longrightarrow & \exists_{K,1}\phi \\
\\
& & \overset{\exists_{K,1}}{\longrightarrow} \\
\mathcal{A}(1) \underset{\langle k, id_1 \rangle^*}{\longleftarrow} & \mathcal{A}(K \times 1) \underset{\pi_2^*}{\overset{\perp}{\rightleftarrows}} & \mathcal{A}(1) \\
\vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots \\
1 \underset{\langle k, id_1 \rangle}{\longrightarrow} & K \times 1 \underset{\pi_2}{\longrightarrow} & 1
\end{array}
$$

**Definition 3.6** For a given category $\mathbf{C}$, the category $\mathbf{C}$-Sets consists of

- families $\{X_I\}_{I \in Obj\mathbf{C}}$ of sets as objects,

- families $\{f_I : X_I \to Y_I\}_{I \in Obj\mathbf{C}}$ of functions as morphisms from $\{X_I\}_{I \in Obj\mathbf{C}}$ to $\{Y_I\}_{I \in Obj\mathbf{C}}$.

Let $\mathbf{C}\text{-}\mathbf{FL}_\sigma$ be a category of $FL_\sigma$-doctrines over $\mathbf{C}$ and $U : \mathbf{C}\text{-}\mathbf{FL}_\sigma \to \mathbf{C}$-Sets the underlying functor. An $FL_\sigma$-doctrine $\mathcal{A}$ over $\mathbf{C}$ is freely generated by $X = \{X_I\}_{I \in Obj\mathbf{C}}$ if it has the following universal property: for any $FL_\sigma$-doctrine $\mathcal{B}$ over $\mathbf{C}$ and $f : X \to U\mathcal{B}$ in $\mathbf{C}$-Sets, there exists a unique morphism $\bar{f} : \mathcal{A} \to \mathcal{B}$ in $\mathbf{C}\text{-}\mathbf{FL}_\sigma$ such that the following diagram commutes.

$$
\begin{array}{ccc}
X \hookrightarrow U\mathcal{A} & & \mathcal{A} \\
\quad \searrow_{f} \quad \downarrow U\bar{f} & & \downarrow \bar{f} \\
U\mathcal{B} & & \mathcal{B}
\end{array}
$$

$$\text{C-}\mathcal{S}\text{ets} \qquad\qquad \text{C-}\mathbf{FL}_\sigma$$

We notice that the Lindenbaum $\mathrm{FL}_\sigma$-doctrine $\mathcal{A}_\mathbf{L} : \mathbf{C_L}^{\mathrm{op}} \to \mathbf{FL}_\sigma$ of $\mathbf{L}$ is freely generated by $\{X_{\vec{x}}\}_{\vec{x} \in Obj\mathbf{C_L}}$ in $\mathcal{D}(\mathbf{L})$ where $X_{\vec{x}}$ is the set of all atomic formulas $R(\vec{x})$ for $\vec{x} \in Obj\mathbf{C_L}$.

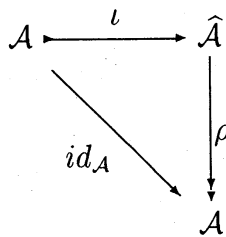**Lemma 3.7** $\widehat{\mathcal{A}}$ *satisfies (DP) and (EP).*

PROOF.

**(DP)** It suffices to replace the FL-algebra $\mathbf{A}$ by $\mathcal{A}(1)$ in Lemma 2.3.

**(EP)** Given $(X, \phi) \in \widehat{\mathcal{A}}(K \times 1)$, if

$$(\top, 1) = 1 \leq \exists_{K,1}(X, \phi) = (\{id_1 \in \Gamma1 | \langle k, id_1 \rangle \in X, \text{ some } k \in \Gamma K\}, \exists_{K,1}\phi),$$

then the first component is not empty, i.e. there is some $k : 1 \to K$ in $\mathbf{C}$ with $\langle k, id_1 \rangle \in X$, which means that $1 \leq \langle k, id_1 \rangle^* \phi$ since $(X, \phi) \in \widehat{\mathcal{A}}(K \times 1)$.
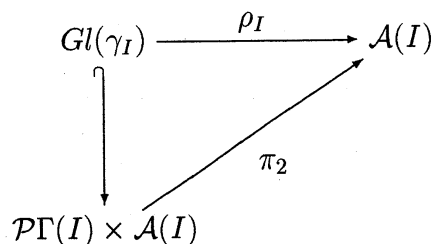
$\square$

**Lemma 3.8** *Suppose* $\mathcal{D}(\mathbf{L})$ *is closed under glueing. If an* $\mathrm{FL}_\sigma$-*doctrine* $\mathcal{A}$ *over* $\mathbf{C}$ *is free in* $\mathcal{D}(\mathbf{L})$, *then* $\mathcal{A}$ *is a retract of* $\widehat{\mathcal{A}}$. *That is,*

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\ \iota\ } & \widehat{\mathcal{A}} \\
 & {\scriptstyle id_\mathcal{A}} \searrow & \downarrow {\scriptstyle \rho} \\
 & & \mathcal{A}
\end{array}
$$

*where* $\rho : \widehat{\mathcal{A}} \to \mathcal{A}$ *is the morphism*

$$\mathbf{C}^{\mathrm{op}} \underset{\mathcal{A}}{\overset{\widehat{\mathcal{A}}}{\rightrightarrows}} \Downarrow \rho \ \ \mathrm{FL}_\sigma$$

*of* $\mathrm{FL}_\sigma$-*doctrines such that for all* $I \in Obj\mathbf{C}$ *the following diagram commutes.*

$$
\begin{array}{ccc}
Gl(\gamma_I) & \xrightarrow{\ \rho_I\ } & \mathcal{A}(I) \\
\downarrow & \nearrow_{\pi_2} & \\
\mathcal{P}\Gamma(I) \times \mathcal{A}(I) & &
\end{array}
$$

PROOF. Consider fibrewise. Then we can show our lemma in the same way as Lemma 2.4. □

**Proposition 3.9** *If $\mathcal{D}(\mathbf{L})$ is closed under glueing, then $\mathbf{L}$ has the disjunction and existence properties.*

PROOF. We show that if an FL-doctrine over $\mathbf{C}$ is free in $\mathcal{D}(\mathbf{L})$ then it satisfies (DP) and (EP).

**(DP)** It suffices to replace the FL-algebra $\mathbf{A}$ by $\mathcal{A}(1)$ in Proposition 2.5.

**(EP)** Given $\phi \in \mathcal{A}(K \times 1)$ (some $K \in Obj\mathbf{C}$), if $1 \leq \exists_{K,1}\phi$ in $\mathcal{A}(1)$, then using $\iota$ in Lemma 3.8,

$$1 = \iota_1 1 \leq \iota_1 \exists_{K,1}\phi = \exists_{K,1}\iota_{K\times 1}\phi \text{ in } \widehat{\mathcal{A}}(1).$$

So by Lemma 3.7, there exists $k : 1 \to K$ in $\mathbf{C}$ with $1 \leq \langle k, id_1\rangle^* \iota_{K\times 1}\phi$, hence

$$1 = \rho_1 1 \leq \rho_1 \langle k, id_1\rangle^* \iota_{K\times 1}\phi = \langle k, id_1\rangle^* \rho_{K\times 1}\iota_{K\times 1}\phi = \langle k, id_1\rangle^* \phi.$$

Then $\mathcal{A}$ satisfies (EP).

□

Let $\sigma \subseteq \{e, c, w, n, n \rightsquigarrow k\}$. Using Lemma 1.3 fibrewise, we can show that if $\mathcal{A}$ is an $FL_\sigma$-doctrine then so is $\widehat{\mathcal{A}}$, i.e. $\mathcal{D}(FL_\sigma)$ is closed under glueing. Therefore we obtain the following corollary.

**Corollary 3.10** *A structural extension $FL_\sigma$ of FL has the disjunction and existence properties, where $\sigma \subseteq \{e, c, w, n, n \rightsquigarrow k\}$.*

# 4 Delta operations for substructural logics

In this section, we will assume that $FL_\sigma$ is a structural extension of $FL_w$ and $\mathcal{D}(FL_\sigma)$ is closed under glueing. Then we will show that the delta operation can be extended to an operation on the set of super-$FL_\sigma$ logics and show that every super-$FL_\sigma$ logic $\mathbf{L}$ satisfying $\Delta_\sigma(\mathbf{L}) = \mathbf{L}$ has both the disjunction and existence properties.

**Definition 4.1** *For each formula $\alpha$, define $\Delta\alpha \equiv p \vee (p \supset \alpha)$ where $p$ is a propositional variable not occurring in $\alpha$. Let $\mathbf{L}$ be a super-$FL_\sigma$ logic. We define a super-$FL_\sigma$ logic $\Delta_\sigma(\mathbf{L})$ by $\Delta_\sigma(\mathbf{L}) = FL_\sigma + \{\Delta\alpha | \alpha \in \mathbf{L}\}$.*

**Lemma 4.2** *For every $FL_\sigma$-doctrine $\mathcal{A}$ and every formula $\alpha$, $\alpha \in \mathbf{L}(\mathcal{A})$ implies $\Delta\alpha \in \mathbf{L}(\widehat{\mathcal{A}})$*

PROOF. Suppose $\mathcal{M}(\alpha^{[n]}; \vec{x}) = \top$ for all $\mathcal{M}$ in $\mathcal{A}$ and $n \geq 0$. We will show that $\mathcal{N}(\Delta\alpha^{[m]}; \vec{x}) = \top = (\Gamma I, \top)$ in $\widehat{\mathcal{A}}(I)$ ($\mathcal{N}$ in $\widehat{\mathcal{A}}$) where

$$\Delta\alpha^{[m]} \equiv P^m(\vec{z}) \vee (P^m(\vec{z}) \supset \alpha^{[m]}).$$

Let $\mathcal{N}(P^m(\vec{z}); \vec{x}) = (A, \phi)$ and $\mathcal{N}(\alpha^{[m]}; \vec{x}) = (B, \psi)$. Then $\mathcal{N}(\Delta\alpha^{[m]}; \vec{x}) = (A \vee ((A \supset B) \wedge \gamma(\phi \supset \psi)), \phi \vee (\phi \supset \psi))$. Since $p$ does not occurr in $\alpha$, $P^m$ does not in $\alpha^{[m]}$.

- If $\mathcal{N}(P^m(\vec{z}); \vec{x}) = (\Gamma I, \top)$, then $\mathcal{N}(\Delta\alpha^{[m]}; \vec{x}) = (\Gamma I, \top) = \top$.

- If $\mathcal{N}(P^m(\vec{z}); \vec{x}) \neq (\Gamma I, \top)$, then $A \neq \Gamma I$. By assumption, $\mathcal{N}(\alpha^{[m]}; \vec{x}) = (B, \top)$ i.e. $\psi = \top$, then $(A \vee (A \supset B)) \wedge \gamma(\phi \supset \top) = A \vee (A \supset B) = \Gamma I$. Therefore $\mathcal{N}(\Delta\alpha^{[m]}; \vec{x}) = (\Gamma I, \top) = \top$.

$\square$

**Proposition 4.3** *If* $\Delta_\sigma(\mathbf{L}) = \mathbf{L}$, *then* $\mathcal{D}(\mathbf{L})$ *is closed under glueing.*

PROOF. Suppose $\mathbf{L} \subseteq \mathbf{L}(\mathcal{A})$. Then we prove $\mathbf{L} \subseteq \mathbf{L}(\widehat{\mathcal{A}})$. Since $\Delta_\sigma(\mathbf{L})$ is a logic, it suffice to show the following:

1. If $\alpha \in \mathrm{FL}_\sigma$, then $\alpha \in \mathbf{L}(\widehat{\mathcal{A}})$ because $\mathcal{D}(\mathrm{FL}_\sigma)$ is closed under glueing, i.e. $\mathrm{FL}_\sigma \subseteq \mathbf{L}(\mathcal{A})$ implies $\mathrm{FL}_\sigma \subseteq \mathbf{L}(\widehat{\mathcal{A}})$.

2. If $\alpha \in \{\Delta\beta | \beta \in \mathbf{L}\}$, then there exists a formula $\beta \in \mathbf{L}$ with $\alpha \equiv \Delta\beta$. By Lemma 4.2, $\Delta\beta \equiv \alpha \in \mathbf{L}(\widehat{\mathcal{A}})$.

$\square$

Using Proposition 3.9 and 4.3, we obtain the following corollary.

**Corollary 4.4** *If* $\Delta_\sigma(\mathbf{L}) = \mathbf{L}$, *then* $\mathbf{L}$ *has the disjunction and existence properties.*

# 5 Concluding Remarks

In super-intuitionistic propositional logic, the following fact is interesting in the sense of characterizing the intuitionistic propositional logic.

**Fact** For every propositional logic $\mathbf{L}$, $\Delta(\mathbf{L}) = \mathbf{L}$ if and only if $\mathbf{L}$ is the intuitionistic propositional logic. That is, the intuitionistic propositional logic is the unique fixed point of $\Delta$ (See [4]).

In super-$\mathrm{FL}_\sigma$ logics with the weakening, the corresponding result is not so clear. Does the delta operation $\Delta_\sigma$ characterize the logic $\mathrm{FL}_\sigma$? So we have the following question:

**Question** Is it true that for every super-$\mathrm{FL}_\sigma$ logic $\mathbf{L}$, $\Delta_\sigma(\mathbf{L}) = \mathbf{L}$ iff $\mathbf{L} = \mathrm{FL}_\sigma$?
Moreover when $\mathrm{FL}_\sigma$ has no weakeing, what operation on super-$\mathrm{FL}_\sigma$ logics does characterize $\mathrm{FL}_\sigma$ ?

**Acknowledgement**

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