数理解析研究所講究録
927 巻 1995 年 187-208

Lattices of Subframe Logics A Survey

Frank Wolter
School of Information Science
JAIST
Tatsunokuchi, Ishikawa 923-12
Japan

October 16, 1995

1 Introduction

Besides of investigating the intrinsic properties of a logic, like decidability or completeness, it is desirable to relate it to its neighbors. While in general it is far from clear what the neighbors of a logic are, modal logicians traditionally take the lattice of all modal logics or some principle filter within this lattice. Thus, for a modal logic Λ, relating it to its neighbors meant investigating the structure of the lattice of modal logics. However, in the late 70s it has become clear that the lattice of normal modal logics is extremely complex. Let us mention only the embedding of second order logic into modal logic due to Thomason [32] and the results of Blok (cf. [4], [5], [6], and below) about the structure of this lattice. Without any restriction as concerns the class of modal logics under consideration no positive result is available. And, as concerns the lattice structure, discrimination between (interesting) logics via there position within the lattice is not possible. (For instance, intuitively the theory of the reflexive frames, $\mathbf{T} = \mathbf{K} + \Box p \to p$, should have a specific position with in the lattice of normal modal logics, simply because it has such a simple geometric meaning. However, it just behaves like nearly all the others).

In this situation it is one of the basic questions to find and describe interesting proper sublattices of the lattice of modal logics, which allow a more detailed treatment and thus lead to finer tuned theory with more discriminative power. The object of this paper is to show that the lattice of subframe logics as defined here is such a lattice, and that the natural neighbors of a subframe logic can be found within the lattice of subframe logics. It will turn out that from the perspective of this lattice we observe many subtle and interesting differences between logics which seemed to behave similar if taken in the whole lattice.

This paper gives a survey of the results [33] and [37] on lattices of subframe logics. If a proposition is stated without proof and reference then the proof can always be found in [33] or [37].

Definition of subframe logics.
A structure $\mathcal{H} = \langle h, \vec{\lhd}, A\rangle$ is called a *n-frame* (or simply a frame) if $\vec{\lhd} = \langle \lhd_i : 1 \leq i \leq n\rangle$

is a sequence of binary relations on h and $A \subseteq 2^h$ is non-empty, closed under the boolean operations $\cap$ and $-$, and under

$$\Box_i a = \{x \in g : (\forall y \in g)(x \lhd_i y \Rightarrow y \in a)\},$$

for $1 \leq i \leq n$. n-frames form a natural semantics for normal modal logics in the propositional language $\mathcal{L}_n$ with n modal operations $\Box_i, 1 \leq i \leq n$. The logic $\mathsf{Th}\mathcal{H}$ of a frame $\mathcal{H}$ is the set of formulas which are valid in $\mathcal{H}$; we write $\mathcal{H} \models \phi$ if a formula ϕ is valid in $\mathcal{H}$. For a class $\mathbf{M}$ of frames put

$$\mathsf{Th}\mathbf{M} = \bigcap\{\mathsf{Th}\mathcal{H} : \mathcal{H} \in \mathbf{M}\}.$$

Conversely, for a modal logic Λ, call a frame $\mathcal{H}$ a Λ-*frame*, in symbols $\mathcal{H} \models \Lambda$, if all formulas $\phi \in \Lambda$ are valid in $\mathcal{H}$. The class of Λ-frames is denoted by $\mathsf{Gfr}\Lambda$. The mapping $\Lambda \mapsto \mathsf{Gfr}\Lambda$ is an anti-isomorphism (with respect to inclusion) between the lattice of modal logics (in the language $\mathcal{L}_n$) and classes of n-frames of the form $\mathsf{Gfr}\Lambda$ (cf. [29]).

For each n-frame $\mathcal{H}$ and each $b \in A$ with $b \neq \emptyset$, the structure

$$\mathcal{H}_b = \langle b, \langle \lhd_i \cap (b \times b) : 1 \leq i \leq n \rangle, \{a \cap b : a \in A\}\rangle$$

is a n-frame as well, and we call it a *subframe* of $\mathcal{H}$. A normal modal logic Λ is a *subframe logic* iff $\mathsf{Gfr}\Lambda$ is is closed under forming subframes. Let us introduce the operation Sf on the class of all frames Gfr by putting

$$\mathsf{Sf}\mathbf{M} = \text{ the class of isomorphic copies of subframes of frames in } \mathbf{M},$$

for $\mathbf{M} \subseteq \mathsf{Gfr}$. The notion of a subframe of a frame has at least two roots. Call a frame $\mathcal{H} = \langle h, \vec{\lhd}, A\rangle$ a *Kripke frame* if $A = 2^h$. In this case we shall write $\langle h, \vec{\lhd}\rangle$, or simply h instead of $\langle h, \vec{\lhd}, A\rangle$; the class of Kripke frames is denoted by Fr and we put $\mathsf{Fr}\Lambda = \mathsf{Gfr}\Lambda \cap \mathsf{Fr}$. Now, at the level of Kripke-frames $\langle h, \vec{\lhd}\rangle$ the set of subframes coincides with the set of substructures of the relational structure $\langle h, \vec{\lhd}\rangle$, in the sense of classical model theory. One can show (cf. [33]) that a *complete* logic Λ, i.e. a logic Λ with $\Lambda = \mathsf{Th}(\mathsf{Fr}\Lambda)$, is a subframe logic iff $\mathsf{Sf}(\mathsf{Fr}\Lambda) \subseteq \mathsf{Fr}\Lambda$. Hence, restricted to complete logics Λ, we are dealing precisely with those logics whose Kripke frames are closed under substructures. At the level of the Boolean algebra $\langle A, \cap, -, h\rangle$ forming the subframes is a natural extension of forming the relativization to an element $b \in A$, already discussed in the context of cylindric algebras (cf. [20]). Subframe logics containing $\mathbf{K4} = \mathbf{K} + \Box p \to \Box\Box p$ (the logic of the transitive frames) have been introduced by K. Fine in [15] by using splittings. Such a definition was available because of the following fundamental result in [15].

Theorem 1.1 *All subframe logics containing* $\mathbf{K4}$ *have the finite model property.*

A syntactic criterion.
Given a natural semantic definition the question arises whether we can describe subframe logics by means of syntactic closure conditions. Fortunately, this is the case. For a formula ϕ and a variable p, define $\phi \downarrow p$ inductively via

$$\begin{aligned}
q \downarrow p &= q \wedge p \\
(\phi \wedge \psi) \downarrow p &= (\phi \downarrow p) \wedge (\psi \downarrow p) \\
(\neg\phi) \downarrow p &= \neg(\phi \downarrow p) \wedge p \\
(\Box_i\phi) \downarrow p &= \Box_i(p \to \phi \downarrow p) \wedge p, \text{ for } 1 \leq i \leq n.
\end{aligned}$$

Put $\phi^{\mathsf{rel}} = \phi \downarrow p$, for a variable p not in ϕ, and put $\phi^{\mathsf{Sf}} = p \rightarrow \phi^{\mathsf{rel}}$. It is shown in [33] that a normal modal logic is a subframe logic if and only if it is closed under the rule

$$\phi/\phi^{\mathsf{Sf}}.$$

In a certain sense this characterization corresponds to the result of classical model theory that the models of a first order theory T are closed under substructures if and only if T is axiomatizable by universal sentences.

A complete sublattice.

The basic observation for a lattice theoretic treatment of subframe logics is that they form a complete sublattice of the lattice of modal logics (cf. [33]). For a subframe logic Λ denote by $\mathcal{S}\Lambda$ the lattice of subframe logics containing Λ. We can say a bit more. Consider a complete sublattice $\mathcal{D}$ of a complete lattice $\mathcal{F}$. Then, for $a \in \mathcal{F}$, the *upward projection* $a \uparrow_{\mathcal{D}}$ and *downward projection* $a \downarrow_{\mathcal{D}}$ of a in $\mathcal{D}$ are defined by

$$a \uparrow_{\mathcal{D}} = \bigwedge \langle b \in D : b \geq a \rangle \text{ and } a \downarrow_{\mathcal{D}} = \bigvee \langle b \in D : b \leq a \rangle,$$

respectively. Denote, for a modal logic Λ, by $\Lambda \uparrow$ the upward projection and by $\Lambda \downarrow$ the downward projection of Λ in $\mathcal{S}\mathbf{K}_n$. Here, $\mathbf{K}_n$ denotes the smallest normal modal logic in the language $\mathcal{L}_n$. By $\Lambda + \Gamma$ denote the smallest modal logic containing an n-modal logic Λ and $\Gamma \subseteq \mathcal{L}_n$. Assume that $\Lambda = \mathbf{K}_n + \Gamma$. Then

$$\Lambda \uparrow = \mathbf{K}_n + \Gamma^{\mathsf{Sf}}, \text{ where } \Gamma^{\mathsf{Sf}} = \{\phi^{\mathsf{Sf}} : \phi \in \Gamma\}.$$

It follows that the upward projections of effectively axiomatizable logics are effectively axiomatizable, as well. For the downward projection we have

$$\Lambda \downarrow = \mathsf{Th}(\mathsf{Sf}\mathbf{M}),$$

if $\Lambda = \mathsf{Th}(\mathbf{M})$, for a class of frames $\mathbf{M}$.

Examples.

Certainly the logics already introduced, $\mathbf{K}_n$, $\mathbf{T}$, and $\mathbf{K4}$, are subframe logics. The basic logic interpreting the provability predicate in arithmetic, namely

$$\mathbf{G} = \mathbf{K4} + \Box(\Box p \rightarrow p) \rightarrow \Box p,$$

is a subframe logic. This logic is characterized by the transitive Kripke frames without infinite ascending chains (cf. [8]). On the other hand, the most natural modal logics in which intuitionistic logic is embeddable via Gödel's translation, i.e.

$$\mathbf{S4} = \mathbf{K4} + \Box p \rightarrow p,$$

$$\mathbf{Grz} = \mathbf{S4} + \Box(\Box(p \rightarrow \Box p) \rightarrow p) \rightarrow p)),$$

are subframe logics. $\mathbf{Grz}$ is the logic characterized by the transitive and reflexive Kripke-frames without infinite strictly ascending chains (cf. [11]). The finite width logics $\mathbf{K.I}_n = \mathbf{K} + \mathbf{I}_n, n > 0$, where

$$\mathbf{I}_n = \bigwedge \langle \Diamond p_i | 1 \leq i \leq n+1 \rangle \rightarrow \bigvee \langle \Diamond (p_i \wedge (p_j \vee \Diamond p_j)) | i \neq j \; i,j \leq n+1 \rangle),$$

are subframe logics. They are, for $n > 0$, characterized by the Kripke frames in which no point has more than n incomparable successors (cf. [14]). Thus, $\mathbf{I}_1$ is also known as

.**3**, and corresponds to the condition known as right linearity. Hence **K4.3** and **S4.3** are well-known subframe logics (cf. [9]). The logics $\mathbf{K.Alt}_n = \mathbf{K} + \mathbf{alt}_n$, $n > 0$, are also subframe logics, where

$$\mathbf{alt}_n = \bigwedge \langle \Diamond p_i | 1 \leq i \leq n+1 \rangle \rightarrow \bigvee \langle \Diamond (p_i \wedge p_j) | i \neq j \rangle.$$

Those logics are, for $n > 0$, characterized by the Kripke frames in which no point has more than n successors (cf. [1] and [31]). Let us now turn to polymodal logics. Here we find the minimal modal logics with n operators so that all operators are *conjugated*, i.e., the logics $\mathbf{K}_n + \mathbf{cn}_\pi$, where $\pi : \{1, \ldots n\} \rightarrow \{1, \ldots, n\}$ with $\pi \circ \pi = Id$ and

$$\mathbf{cn}_\pi = \{p \rightarrow \Box_i \Diamond_{\pi(i)} p : 1 \leq i \leq n\} \tag{1.1}$$

$\mathbf{K}_n + \mathbf{cn}_\pi$ is characterized by the Kripke frames $\langle g, \vec{R} \rangle$ satisfying

$$R_i = R^{-1}_{\pi(i)}, 1 \leq i \leq n. \tag{1.2}$$

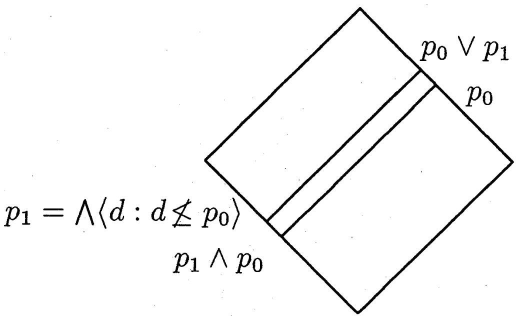
Examples are the selfconjugate logic $\mathbf{K.B}_1 = \mathbf{K} + \mathbf{cn}_{Id}$, which, if added to **S4** gives **S5**, and the *minimal tense logic* $\mathbf{K}.t = \mathbf{K}_2 + \mathbf{cn}_\pi$ with $\pi(1) = 2$ and $\pi(2) = 1$ (consult [10] and [17]). We denote this mapping π by t. Call a logic Λ a logic with conjugates if it contains a $\mathbf{K}_n + \mathbf{cn}_\pi$. The following simple construction from [23] and [16] gives us more subframe logics. Consider n monomodal logics $\Lambda_i, 1 \leq i \leq n$, and suppose that Λ_i is formulated in the language with $\Box_i, 1 \leq i \leq n$. Then the *fusion* of $\langle \Lambda_i : 1 \leq i \leq n \rangle$, $\bigotimes \langle \Lambda_i : 1 \leq i \leq n \rangle$, is the smallest modal logic in $\mathcal{L}_n$ containing $\bigcup \{\Lambda_i : 1 \leq i \leq n\}$. If all the Λ_i are subframe logics then the fusion is a subframe logic, as well. In fusions there is no connection between different modal operators. A number of interesting subframe logics is available by adding axioms to fusions, e.g., if Λ is a monomodal subframe logic, then

$$\Lambda.t = (\Lambda \otimes \mathbf{K}) + \mathbf{cn}_t$$

is a subframe logic, as well, known as the *minimal tense extension* of Λ (cf. [34] and [35]).

Motivation, the history, and tools.

We shall now discuss in which properties of lattices of subframe logics we are interested and why. The most important lattice theoretic concept we deal with is the notion of a *splitting* of a complete lattice of modal logics. Take a complete lattice $\mathcal{D} = \langle D, \vee, \wedge, 0, 1 \rangle$. Then we say that $p_0 \in D$ *splits* $\mathcal{D}$ if there exists a $p_1 \in D$ such that $\langle p_0, p_1 \rangle$ devides the lattice into two disjoint parts, the filter $\mathcal{E}p_1 = \{d \in D : d \geq p_1\}$ and the ideal $\mathcal{I}p_0 = \{d \in D : d \leq p_0\}$. In this case p_1 is uniquely determined by p_0 and we say that p_1 is the *splitting-companion* of p_0. p_1 is denoted by $\mathcal{D}/p_0$. The pair $\langle p_0, p_1 \rangle$ is called a *splitting-pair*. If p_1 is the splitting-companion of some p_0 then we simply say that p_1 is a *splitting* of $\mathcal{D}$. Another way to introduce splittings is the following. Call an element d (strongly) *prime* in $\mathcal{D}$ if, for $d \geq \bigwedge \langle d_i | i \in I \rangle$, there always is an $i \in I$ with $d \geq d_i$. McKenzie [25] shows that d is prime in $\mathcal{D}$ iff d splits $\mathcal{D}$. Hence splittings can be visualized as follows.



What is the use of this concept in modal logic? Let us consider a logic Λ in a complete sublattice $\mathcal{D}$ of the lattice of all normal modal logics. Natural questions about Λ in $\mathcal{D}$ are the following.

1. *Kripke-separation*
   A logic Λ has the Kripke separation property in $\mathcal{D}$ if there is no other logic $\Theta \in \mathcal{D}$ with $\mathsf{Fr}\Theta = \mathsf{Fr}\Lambda$. (Mostly this property is called *strict completeness*, e.g. in [4]).The problem is whether Λ has the Kripke separation property in $\mathcal{D}$.

2. *Finite Kripke separation*
   A logic Λ has the finite Kripke separation property in $\mathcal{D}$ if there is no other logic $\Theta \in \mathcal{D}$ with $\mathsf{fFr}\Theta = \mathsf{fFr}\Lambda$, where $\mathsf{fFr}\Theta$ denotes the set of finite Θ-frames. The question is whether Λ has the finite Kripke seperation property in $\mathcal{D}$.

3. *Lower covers*
   A logic $\Theta \neq \Lambda$ is a lower cover of Λ in $\mathcal{D}$ iff $\{\Theta_1 \in \mathcal{D} : \Theta \subseteq \Theta_1 \subseteq \Lambda\} = \{\Theta, \Lambda\}$. Which logics are the lower covers of Λ in $\mathcal{D}$, if there are any?

4. *Axiomatization problem*
   Let us assume that there is a recursive set $\mathcal{L}_\mathcal{D}$ of formulas which is *complete* for $\mathcal{D}$, i.e. (1) $\mathbf{K}_n + \phi \in \mathcal{D}$, for all $\phi \in \mathcal{L}_\mathcal{D}$, and (2) if $\Lambda \in \mathcal{D}$ then $\Lambda = \mathbf{K}_n + \Gamma$, for a set of formulas $\Gamma \subseteq \mathcal{L}_\mathcal{D}$. Is the axiomatization problem for Λ (relative to $\mathcal{D}$) decidable? In other words, is $\{\phi \in \mathcal{L}_\mathcal{D} : \mathbf{K}_n + \phi = \Lambda\}$ a recursive set?

Certainly the interest of the problems above depends on the lattice $\mathcal{D}$ and on Λ. We shall have a look at a simple example and summarize (a small part of) the research on splittings in modal logic in the 70s. Take for Λ the logic **S5**. Denote by $\mathcal{E}\Theta$ the lattice of normal logics containing a normal logic Θ. Then, tradionally one would take $\mathcal{D}$ to be one of the lattices $\mathcal{E}$**S4**, $\mathcal{E}$**K4**, or $\mathcal{E}$**K**. We first have a look at $\mathcal{E}$**S4**. Here all those problems are solved by the observation that **S5** is a splitting of $\mathcal{E}$**S4**. The pair

$$\langle \mathsf{Th}(\boxed{\bullet\!\rightarrow\!\bullet}), \mathbf{S5}\rangle$$

is the required splitting pair in $\mathcal{E}$**S4** (cf. [28]). (We draw frames $\langle g, S\rangle$ in such a way that $\times$ denotes an irreflexive point and $\bullet$ denotes a reflexive point). Using the figure on splittings this means

$$\mathbf{S5} = \bigcap\{\Lambda \in \mathcal{E}\mathbf{S4} : \boxed{\bullet\!\rightarrow\!\bullet} \not\models \Lambda\}.$$

Note that this splitting pair also shows in a nice way the geometrical meaning of **S5**. It just says that the frames validating **S5** are precisely those quasi ordered sets $\langle g, S\rangle$ in which S is an equivalence relations. The questions above are solved as follows. The figure on splittings shows that the only lower cover of **S5** in $\mathcal{E}$**S4** is **S5** $\cap$ $\mathsf{Th}(\boxed{\bullet \to \bullet})$. (3.) is solved. Now (1.) is solved by the consequence that $\boxed{\bullet \to \bullet} \models \Theta$, for each logic Θ with $\mathbf{S4} \subseteq \Theta$ and $\mathbf{S5} \not\subseteq \Theta$. So **S5** is the only logic containing **S4** whose Kripke frames are precisely the sets with an equivalence relation. (2.) is solved analogously (by using that **S5** has the finite model property). (4.) is translated into the problem whether $\{\phi : \mathbf{S4} + \phi = \mathbf{S5}\}$ is recursive. But,

$$\{\phi : \mathbf{S4} + \phi = \mathbf{S5}\} = \mathbf{S5} \cap \{\phi : \boxed{\bullet \to \bullet} \not\models \phi\},$$

and the set to the right is certainly recursive. Intuitively, the frame $\boxed{\bullet \to \bullet}$ is of more importance than its logic. So, we shall say that $\boxed{\bullet \to \bullet}$ splits **S4** and we shall write

$$\mathbf{S5} = \mathbf{S4}/\boxed{\bullet \to \bullet}$$

We come to $\mathcal{E}$**K4**. In this lattice **S5** is not a splitting. However, by extending the notion of a splitting to the notion of a *join-splitting* we can apply basically the same technique. An element $p_1 \in D$ is a *join-splitting* of $\mathcal{D}$ by $F \subseteq \mathcal{D}$ if all $p_0 \in F$ split $\mathcal{D}$ and $p_1 = \bigvee\langle \mathcal{D}/p : p \in F\rangle$. p_1 is denoted by $\mathcal{D}/F$. Again for an element $a \in D$ and $F \subseteq \mathcal{E}a$ we call a join-splitting $\mathcal{E}a/F$ a join-splitting of a and denote it by a/F. The following proposition states that join-splittings behave quite similar to splittings.

Proposition 1.2 *Suppose that* $p_1 = \mathcal{D}/F$. *Then, for all* $a \in D$, $a \geq p_1$ *if and only if* $a \not\leq p$, *for all* $p \in F$.

The following is shown in [28].

$$\mathbf{S5} = \mathbf{K4}/\{\boxed{\times}, \boxed{\times \to \bullet}, \boxed{\bullet \to \bullet}\}.$$

(We omit writing $\mathsf{Th}(-)$). In completely the same way as in $\mathcal{E}$**S4** one may now solve all the problems stated above for **S5** in $\mathcal{E}$**K4**. It should have become clear why splittings give us interesting information about logics. In [3], [27], and [28], a lot of other systems containing **K4** are shown to be join-splittings of $\mathcal{E}$**K4**. However, the basic question is whether we can apply the same technique in order to investigate **S5** in the lattice of all modal logics $\mathcal{E}$**K**. This is not the case. A frame g is called cyle free if there is no path of length > 0 from a point in g to itself, and a frame is rooted if there is a point x such that all the other points are endpoints of a path of length ≥ 0 from x. The following is proved in [4].

Theorem 1.3 *A logic* Θ *splits* $\mathcal{E}$**K** *if and only if* $\Theta = \mathsf{Th}(\mathcal{G})$, *for a finite and rooted and cycle free frame* $\mathcal{G}$.

This theorem means that there are nearly no interesting join-splittings in $\mathcal{E}$**K**. This can be seen by the observation that

$$\mathbf{D} = \mathbf{K} + \Diamond\top = \mathbf{K}/\boxed{\times} \tag{1.3}$$

is the largest join-splitting of $\mathcal{E}$**K** (cf. [4]). More important in the context of subframe logics is the following Corollary from [33].

Corollary 1.4 *No logic in* $\mathcal{S}$**K** *is a join-splitting of* $\mathcal{E}$**K**.

So we get none of the basic systems introduced above. But maybe the definition of a join-splitting is too weak! It might well be possible that there are splittings of $\mathcal{E}(\mathbf{K}+\Diamond\top)$ which we do not obtain as join-splittings of $\mathcal{E}\mathbf{K}$. So, we call an element $p_1 \in \mathcal{D}$ an *iterated splitting* of $\mathcal{D}$ by $(F_1, \ldots F_n)$, (here $F_i \subseteq D$, for all $1 \leq i \leq n$), if each $p_0 \in F_1$ splits $\mathcal{D}$ and, for $1 \leq i \leq n-1$, each $p_0 \in F_{i+1}$ splits $\mathcal{D}/F_1/F_2/\ldots/F_i$, and

$$p_1 = \mathcal{D}/F_1/F_2/\ldots/F_{n-1}/F_n.$$

The following proposition states that iterated splittings behave quite similar to join-splittings.

Proposition 1.5 *Suppose that $p_1 = \mathcal{D}/F_1/F_2\ldots/F_n$. Then, for all $a \in D$, $a \geq p_1$ if and only if $a \not\leq p$, for all $p \in F_1 \cup F_2 \ldots \cup F_n$.*

However, the following crucial result of [4] states that we get only one new (not very exciting) logic.

Theorem 1.6 *A logic $\Lambda \in \mathcal{E}\mathbf{K}$ is an iterated splitting of $\mathcal{E}\mathbf{K}$ if and only if it is a join-splitting of $\mathcal{E}\mathbf{K}$ or it is the inconsistent logic.*

The conclusion is that in the lattice $\mathcal{E}\mathbf{K}$ splittings are not the appropriate tool for studying interesting systems. One may ask whether some of the problems stated above have a positive solution without using splittings. Again, [4] gives a negative answer.

Theorem 1.7 *If a logic Λ is not an iterated splitting of $\mathcal{E}\mathbf{K}$ and not $= \mathbf{K}$ then there exist $2^{\aleph_0}$ logics Θ with $\mathsf{Fr}\Lambda = \mathsf{Fr}\Theta$. Moreover, Λ has $2^{\aleph_0}$ lower covers in $\mathcal{E}\mathbf{K}$.*

As concerns the axiomatization problem it is an old problem to show

Conjecture 1.8 *If a finitely axiomatizable logic Λ is not an iterated splitting of $\mathcal{E}\mathbf{K}$ and does not coincide with $\mathbf{K}$ then the axiomatization problem for Λ in $\mathcal{E}\mathbf{K}$ is undecidable.*

It is justified to conclude that only via sublattices there is hope to get positive results as concerns the lattice structure of the lattice of modal logics. So we shall now have a brief look at lattices of subframe logics. First note that we do not loose splitting pairs when we take a complete sublattice.

Proposition 1.9 *Suppose that $\mathcal{F}$ is a complete sublattice of $\mathcal{D}$ and that $\langle p_0, p_1\rangle$ is a splitting pair in $\mathcal{D}$. Then $\langle p_0 \downarrow_{\mathcal{F}}, p_1 \uparrow_{\mathcal{F}}\rangle$ is a splitting-pair in $\mathcal{F}$.*

Quite easily we obtain with (1.3) that

$$\langle \mathsf{Th}(\boxed{\mathsf{x}}), \mathbf{T}\rangle$$

is a splitting pair in the lattice of subframe logics. First, $\mathsf{Th}(\boxed{\mathsf{x}})\downarrow = \mathsf{Th}(\boxed{\mathsf{x}})$. Also, via simple syntactic manipulation,

$$(\mathbf{K}+\Diamond\top)\uparrow = \mathbf{K} + (\Diamond\top)^{\mathsf{Sf}} = \mathbf{K} + p \to \Diamond(p \wedge \top) = \mathbf{K} + p \to \Diamond p = \mathbf{T}.$$

Now we can solve for $\mathbf{T}$, in the lattice $\mathcal{S}\mathbf{K}$, all the problems stated above. For instance, the only lower cover of $\mathbf{T}$ in $\mathcal{S}\mathbf{K}$ is $\mathbf{T} \cap \mathsf{Th}(\boxed{\mathsf{x}})$, in contrast to the result that $\mathbf{T}$ has $2^{\aleph_0}$ lower covers in $\mathcal{E}\mathbf{K}$. Let us introduce some notation for the case of subframe logics. Let

Λ be a subframe logic. Then $\Theta \in \mathcal{S}\Lambda$ has the Sf-*separation property* (ssp) in $\mathcal{S}\Lambda$ iff it has the Kripke separation property in $\mathcal{S}\Lambda$. Θ has the Sf-*finite separation property* (fsp) in $\mathcal{S}\Lambda$ iff it has the finite Kripke separation property in $\mathcal{S}\Lambda$. The Sf-*axiomatization problem* is solvable for Θ in $\mathcal{S}\Lambda$ iff

$$\{\phi^{\mathsf{Sf}} : \Lambda + \phi^{\mathsf{Sf}} = \Theta\}$$

is a recursive set. Now $\mathbf{T}$ has ssp and fsp in $\mathcal{S}\mathbf{K}$ since $\mathbf{T}$ has the fmp and $\boxed{\times} \models \Theta$, for each subframe logic Θ with $\mathbf{T} \not\subseteq \Theta$. The Sf-axiomatization problem for $\mathbf{T}$ is decidable since

$$\{\phi^{\mathsf{Sf}} : \mathbf{K} + \phi^{\mathsf{Sf}} = \mathbf{T}\} = \mathbf{T} \cap \{\phi^{\mathsf{Sf}} : \boxed{\times} \not\models \phi^{\mathsf{Sf}}\}.$$

Certainly this example is not surprising. The main difference if compared with the lattice $\mathcal{E}\mathbf{K}$, is the fact that there are numerous interesting examples of iterated splittings in $\mathcal{S}\mathbf{K}$ which are not join-splittings. Thus, the result on $\mathbf{T}$ is useful, since now, in order to prove that a logic $\Lambda \supseteq \mathbf{T}$ is an iterated splitting of $\mathcal{S}\mathbf{K}$ it suffices to show that it is an iterated splitting of $\mathcal{S}\mathbf{T}$. At the moment we note only the following example.

$$\mathbf{S5} = \mathbf{K}/^{\mathsf{Sf}} \boxed{\times} /^{\mathsf{Sf}} \boxed{\bullet \longrightarrow \bullet} /^{\mathsf{Sf}} \boxed{\bullet\!<\!\substack{\bullet \\ \bullet}}.$$

(Here we omit writing $\mathsf{Th}(\mathsf{Sf}-)$ and $/^{\mathsf{Sf}}$ means splitting in the lattice of subframe logics). Thus, for $\mathbf{S5}$, all the problems stated above have a positive solution in the lattice of subframe logics.

2 On Correspondence

For subframe logics a number of concepts from completeness and correspondence theory turn out to be equivalent. First recall the following definitions of classes of frames. An n-frame $\langle h, \vec{\lhd}, A\rangle$ is *refined* if both

$$(\forall x, y \in h)(x = y \Leftrightarrow (\forall a \in A)(x \in a \Leftrightarrow y \in a)),$$

$$(\forall x, y \in h)(x \lhd_i y \Leftrightarrow (\forall a \in A)(x \in \Box_i a \Rightarrow y \in a)).$$

See [29] for an extensive study of refined frames. The class of refined frames is denoted by Rfr. A frame $\mathcal{H}$ is *descriptive* if it is refined and $\bigcap U \neq \emptyset$, for each ultrafilter U in the boolean reduct of $\mathcal{H}^+$ (consult [18]). We denote the class of finite Kripke frames by fFr and the class of finite and rooted Kripke frames by rFr. Also,

$$\mathsf{Rfr}\Lambda = \mathsf{Rfr} \cap \mathsf{Gfr}\Lambda,\ \mathsf{Fr}\Lambda = \mathsf{Fr} \cap \mathsf{Gfr}\Lambda,\ \mathsf{fFr}\Lambda = \mathsf{fFr} \cap \mathsf{Gfr}\Lambda,\ \mathsf{rFr}\Lambda = \mathsf{rFr} \cap \mathsf{Gfr}\Lambda,$$

for each logic Λ. Now we can define the concepts which will turn out to be equivalent. A logic Λ is *compact* (alias strongly complete) iff each set $\Gamma \subseteq \mathcal{L}$ which is consistent with Λ is satisfiable in a frame in $\mathsf{Fr}\Lambda$. A logic Λ is *r-persistent* if $\langle h, \vec{\lhd}\rangle \in \mathsf{Fr}\Lambda$ whenever $\langle h, \vec{\lhd}, A\rangle \in \mathsf{Rfr}\Lambda$. [13] calls r-persistent logics *natural* logics. Λ is *d-persistent* if $\langle h, \vec{\lhd}\rangle \in \mathsf{Fr}\Lambda$ whenever $\langle h, \vec{\lhd}, A\rangle$ is a descriptive Λ-frame. Following Goldblatt [19] we call a logic Λ *complex* if, for all $\mathcal{G} \in \mathsf{Gfr}\Lambda$, there exists $\mathcal{H} = \langle h, \vec{\lhd}, A\rangle$ with $\langle h, \vec{\lhd}\rangle \in \mathsf{Fr}\Lambda$ such that $\mathcal{H}^+ \simeq \mathcal{G}^+$.

Note that, in general, r-persistency does not imply d-persistency (cf. [13]). Also, in general, d-persistency does not imply elementarity (cf. [13]). Moreover there are compact logics which are not d-persistent (cf. [38]). A class of Kripke frames $\mathbf{F}$ is called *universal* iff it is definable by a set of universal first order sentences. $\mathbf{F}$ has the *finite embedding property* if $g \in \mathbf{F}$ if and only if each finite subframe f of g is in $\mathbf{F}$, for all $g \in \mathsf{Fr}$.

Theorem 2.1 *For a subframe logic Λ the following conditions are equivalent:*
(1) $\mathsf{Fr}\Lambda$ *is universal and Λ is complete.*
(2) Λ is elementary and complete.
(3) Λ is d-persistent.
(4) Λ is r-persistent.
(5) Λ is complex.
(6) Λ is compact.
(7) $\mathsf{Fr}\Lambda$ *has the finite embedding property and Λ is complete.*

Later we shall mostly work with r-persistency. However, the finite embedding property is closely related to splittings. Suppose that $\mathbf{F} \subseteq \mathsf{rFr}$ and Λ is a subframe logic. Define

$$\mathsf{Fr}\Lambda_{\mathbf{F}} = \{g \in \mathsf{Fr}\Lambda : (\forall f \in \mathbf{F}) f \notin \mathsf{Sf}g\}.$$

We write $\mathsf{Fr}_{\mathbf{F}}$ instead of $(\mathsf{Fr}\mathbf{K}_n)_{\mathbf{F}}$.

Corollary 2.2 *Suppose that a subframe logic Λ is complete and elementary. Then there is a set $\mathbf{F} \subseteq \mathsf{rFr}$ such that $\mathsf{Fr}\Lambda = \mathsf{Fr}_{\mathbf{F}}$. If Λ is finitely axiomatizable, then there exists a finite set $\mathbf{F} \subseteq \mathsf{rFr}$ with $\mathsf{Fr}\Lambda = \mathsf{Fr}_{\mathbf{F}}$.*

Intuitively, if we want to show that a complete and elementary subframe logic Λ is an iterated Sf-splitting, then we should take a set $\mathbf{F} \subseteq \mathsf{rFr}$ with $\mathsf{Fr}\Lambda = \mathsf{Fr}_{\mathbf{F}}$, and show that $\mathbf{F}$ defines an iterated Sf-splitting such that $\Lambda = \mathbf{K}_n/^{\mathsf{Sf}}\mathbf{F}$. (See the next chapter for a precise definition of the right hand side of this equation). This is indeed the simple idea behind many results to follow. Call a class of Kripke frames $\mathbf{F}$ *definable by modal formulas* if there exists a modal logic Λ with $\mathsf{Fr}\Lambda = \mathbf{F}$. Let us note the following characterizations.

Theorem 2.3 *A universal class of Kripke frames $\mathbf{F}$ is definable by modal formulas iff $\mathbf{F}$ is closed with respect to p-morphic images and disjoint unions.*

Corollary 2.4 *For each set $\mathbf{F} \subseteq \mathsf{rFr}$, the class $\mathsf{Fr}_{\mathbf{F}}$ is definable by modal formulas iff $\mathsf{Fr}_{\mathbf{F}}$ is closed under p-morphic images.*

3 A Splitting Lemma

From now on we are dealing with lattice theoretic properties of lattices of subframe logics. Suppose that Λ and Θ are subframe logics. Then $\Theta = \mathsf{Th}\mathcal{G}$, for a frame $\mathcal{G}$. Hence, $\Theta = \mathsf{Th}(\mathsf{Sf}\mathcal{G})$. Now suppose that Θ splits $\mathcal{S}\Lambda$. Then we shall say that $\mathcal{G}$ Sf-splits Λ and we shall denote the splitting $\mathcal{S}\Lambda/\Theta$ by $\Lambda/^{\mathsf{Sf}}\mathcal{G}$. For a set of frames $\mathbf{F}$ such that all frames in $\mathbf{F}$ Sf-split Λ we shall denote the join-splitting by the theories $\mathsf{Th}(\mathsf{Sf}\mathcal{G}), \mathcal{G} \in \mathbf{F}$, by $\Lambda/^{\mathsf{Sf}}\mathbf{F}$. By definition, $\Lambda/^{\mathsf{Sf}}\mathbf{F}$ is the smallest subframe logic Θ containing Λ with $\mathcal{G} \not\models \Theta$, for all $\mathcal{G} \in \mathbf{F}$.

The following Theorem provides a criterion for Sf-splittings. It is also important that we get an axiomatization of the Sf-splitting. For $n > 0$ we denote by $\mathcal{L}_n$ the propositional language of polymodal logic with n Boxes $\Box_1, \ldots, \Box_n$. For a formula $\phi \in \mathcal{L}_n$ and $m \in \omega$ the formula $\Box^m \phi$ is defined as follows.

$$\Box^0 \phi = \phi; \; \Box^{m+1}\phi = \bigwedge \langle \Box_i \Box^m \phi | 1 \leq i \leq n \rangle.$$

Let $\Box^{(m)}\phi = \bigwedge\langle \Box^i \phi | i \leq m\rangle$ and define for a set of formulas Γ Notice that the construction of $\Box^m\phi$ depends on the language. If we want to indicate that $\Box^m\phi$ is defined in the language $\mathcal{L}$ we write $\Box^m_{\mathcal{L}}\phi$. Consider a finite frame $\mathcal{G} = \langle g, \vec{S}\rangle$. Reserve a variable p_y for each $y \in g$. We define a formula $\nabla_{\mathcal{G}}$ by putting

$$\begin{aligned} \nabla_{\mathcal{G}} &= \bigwedge \langle p_y \rightarrow \Diamond_i p_z | y S_i z, 1 \leq i \leq n \rangle \\ &\wedge \bigwedge \langle p_y \rightarrow \Box_i \neg p_z | \neg(y S_i z), 1 \leq i \leq n \rangle \\ &\wedge \bigwedge \langle p_y \rightarrow \neg p_z | y \neq z \rangle. \end{aligned}$$

Theorem 3.1 *Suppose that $\mathcal{G} = \langle g, \vec{S}\rangle$ is a finite frame with root 0, and Λ is a subframe logic. Then (1) $\mathcal{G}$ Sf-splits Λ iff (2) there exists $m \in \omega$ such that for all $\mathcal{H} \in \mathsf{Gfr}\Lambda$*

$$\Box^{(m)}\nabla_{\mathcal{G}}, p_0 \textit{ is satisfiable in } \mathsf{Sf}\mathcal{H} \Rightarrow (\forall n \geq m)\Box^{(n)}\nabla_{\mathcal{G}}, p_0 \textit{ is satisfiable in } \mathsf{Sf}\mathcal{H}.$$

In this case $\Lambda/^{\mathsf{Sf}}\mathcal{G} = \Lambda + (\Box^{(m)}\nabla_{\mathcal{G}} \rightarrow \neg p_0)^{\mathsf{Sf}}$.

There exist more general versions of this result. [21] presents a characterization of splittings of lattices of type $\mathcal{E}\Lambda$ for finitely presentable algebras. In [33] this result is generalized to a characterization of splittings of arbitrary complete sublattices of the lattice of modal logics by arbitrary subdirectly irreducible algebras. However, here we shall not need those versions. For a large class of subframe logics Λ it can be deduced that all finite and rooted frames Sf-split $\mathcal{S}\Lambda$. An n-modal subframe logic Λ is *m-transitive*, $m > 0$, if the formula

$$\mathbf{tr}_m = (\Box^{(m)}_{\mathcal{L}_n} p \rightarrow \Box^{m+1}_{\mathcal{L}_n} p)^{\mathsf{Sf}}$$

belongs to Λ. For instance, $\mathbf{K4}$ is 1-transitive. Put $\mathbf{K}_n.\mathbf{Tr}_m = \mathbf{K}_n + \mathbf{tr}_m$. By definition, the logics $\mathbf{K}_n.\mathbf{Tr}_m$ are subframe logics. A tedious but straightforward proof shows that $\mathbf{K}_n.\mathbf{Tr}_m$ is d-persistent, hence complete and elementary. An n-frame g is a $\mathbf{K}_n.\mathbf{Tr}_m$-frame if and only if for each finite path from x to y in g there exists a subpath of length $\leq m$ from x to y in g. We get from the Theorem above

Corollary 3.2 *Suppose that Λ is a m-transitive subframe logic, for some $m > 0$. Then each $\mathcal{G} \in \mathsf{rFr}\Lambda$ Sf-splits Λ and $\Lambda/^{\mathsf{Sf}}\mathcal{G} = \Lambda + \Box^{(m)}\nabla_{\mathcal{G}} \rightarrow \neg p_0$.*

Note that in this case, for the axiomatization, we don't need $(\Box^{(m)}\nabla_{\mathcal{G}} \rightarrow \neg p_0)^{\mathsf{Sf}}$. We leave the proof to the reader since we shall not use this fact. That finite and rooted frames always Sf-split m-transitive logics Λ follows also, by Proposition 1.9, from the result of Rautenberg [28] that those frames already split $\mathcal{E}\Lambda$. The deeper reason is that the corresponding varieties of modal algebras have equational definable principle congruences (EDPC), consult [7].

We formulate the results as concerns the relation between the concepts we have introduced in the introduction. It will be said that a set $\mathbf{F}$ of finite and rooted frame defines an iterated Sf-splitting of a subframe logic Λ if there is a partition $\mathbf{F}_1 \cup \ldots \cup \mathbf{F}_n$ of $\mathbf{F}$ such that the frames in $\mathbf{F}_{i+1}$ Sf-split $\Lambda/^{\mathsf{Sf}}\mathbf{F}_1 \ldots /^{\mathsf{Sf}}\mathbf{F}_i$, for $0 \leq i < n$. The result is denoted by $\Lambda/^{\mathsf{Sf}}\mathbf{F}$.

Proposition 3.3 *Suppose that $\Theta \in \mathcal{S}\Lambda$, and Λ a subframe logic. Suppose that Θ is an iterated* Sf*-splitting of Λ by a set of finite and rooted frames* $\mathbf{F}$.

- *If $\mathbf{F}$ is finite and Θ is decidable, then the* Sf*-axiomatization problem for Θ in $\mathcal{S}\Lambda$ is decidable. We have*
$$\{\phi^{\mathsf{Sf}} : \Lambda + \phi^{\mathsf{Sf}} = \Theta\} = \{\phi^{\mathsf{Sf}} : \phi^{\mathsf{Sf}} \in \Theta, \ (\forall \mathcal{G} \in \mathbf{F})(\mathcal{G} \not\models \phi^{\mathsf{Sf}})\}$$
- *If Θ is complete, then Θ has ssp in $\mathcal{S}\Lambda$.*
- *If Θ has the fmp, then Θ has fsp in $\mathcal{S}\Lambda$.*
- *The lower covers of Θ in $\mathcal{S}\Lambda$ are*
$$\{\Theta \cap \mathsf{Th}(\mathsf{Sf}\mathcal{G}) : \mathcal{G} \in \mathbf{F}, \ \mathcal{G} \textit{ not subreducible onto another frame in } \mathbf{F}\}.$$

4 Characterizing Sf-splittings

In this section we characterize the Sf-splittings of basic lattices $\mathcal{S}\Lambda$. We know already from Proposition 1.3 and 1.9 that rooted and cycle free finite frames Sf-split $\mathbf{K}_n$. On the other hand, Blok has shown in [6] that only the reflexive point splits $\mathcal{E}\mathbf{T}$. The following result tells us that for $\mathcal{S}\mathbf{K}$ the situation is comparable to $\mathcal{E}\mathbf{K}$ while for $\mathcal{S}\mathbf{T}$ it is quite different from $\mathcal{E}\mathbf{T}$. Important for later applications is that we also obtain that those Sf-splittings are r-persistent. Let us recall the definitions.

Cycle free frames.
An n-frame $\mathcal{G} = \langle g, \vec{R}\rangle$ is *cycle free* if $x_0 \neq x_m$ for any path $\langle x_j | 0 \leq j \leq m\rangle$ in $\mathcal{G}$ with $0 \neq m$.

r-cycle free frames.
For an n-frame $\mathcal{G} = \langle g, \vec{S}\rangle$ define $\mathcal{G}^{\mathsf{x}} = \langle g, \langle S_i - \{(y,y) | y \in g\} | i \leq n\rangle\rangle$. In other words, we replace all reflexive points by irreflexive points. Then $\mathcal{G}$ is *r-cycle free* if $\mathcal{G}^{\mathsf{x}}$ is cycle free.

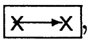
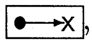
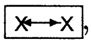
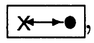
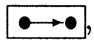
Theorem 4.1 *(1) A finite rooted n-frame $\mathcal{G}$* Sf*-splits $\mathbf{K}_n$ iff $\mathcal{G}$ is cycle free. In this case*
$$\mathbf{K}_n/^{\mathsf{Sf}}\mathcal{G} = \mathbf{K}_n + \Box^{(dp(\mathcal{G}))}\nabla_{\mathcal{G}} \to \neg p_0.$$
*All join-*Sf*-splittings of $\mathbf{K}_n$ are r-persistent.*
(2) A finite rooted frame $\mathcal{G} \in \mathsf{Fr}(\otimes^n\mathbf{T})$ Sf*-splits $\otimes^n\mathbf{T}$ iff $\mathcal{G}$ is r-cycle free. In this case*
$$\otimes^n\mathbf{T}/^{\mathsf{Sf}}\mathcal{G} = \otimes^n\mathbf{T} + \Box^{(dp(\mathcal{G}))}\nabla_{\mathcal{G}} \to \neg p_0.$$
*All join-*Sf*-splittings of $\otimes^n\mathbf{T}$ are r-persistent.*

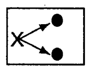
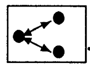
Hence splittings of $\mathcal{E}\Lambda$ are even more sensible to cycles than splittings in $\mathcal{S}\Lambda$. Let us note two more characterizations of splittings and Sf-splittings of important lattices which underline this interpretation. Recall that $\mathbf{K}.t$ is the minimal tense logic. Define for a 2-tree $\mathcal{T} = \langle t, <_1, <_2\rangle$ the frame $\mathcal{T}^{\mathbf{t}} = \langle t, <_1 \cup <_2^{-1}, <_2 \cup <_1^{-1}\rangle$. Obviously $\mathcal{T}^{\mathbf{t}} \models \mathbf{K}.t$. Also put $\mathcal{T}^{\mathbf{S5}} = \langle t, R_1, R_2\rangle$, where R_i is the reflexive, transitive and symmetric closure of $<_i$, for $i = 1, 2$. In the following theorem item (1) is shown in [22].

Theorem 4.2 *(1) A finite and rooted frame $\mathcal{G}$ splits $\mathbf{K}.t$ iff $\mathcal{G} = \boxed{\mathsf{x}}^{\,\mathbf{t}}$. (1') A finite and rooted frame $\mathcal{G}$* Sf*-splits $\mathbf{K}.t$ iff there exists a 2-tree $\mathcal{T}$ with $\mathcal{G} = \mathcal{T}^{\mathbf{t}}$. (2) A finite and rooted frame $\mathcal{G}$ splits $\mathbf{S5} \otimes \mathbf{S5}$ iff $\mathcal{G} = \boxed{\mathsf{x}}^{\,\mathbf{S5}}$. (2') A finite and rooted frame $\mathcal{G}$* Sf*-splits $\mathbf{S5} \otimes \mathbf{S5}$ iff there exists a 2-tree $\mathcal{T}$ with $\mathcal{G} = \mathcal{T}^{\mathbf{S5}}$.*

5 Iterated splittings by $\mathcal{T}$-closed sets

Now that we know which frames Sf-split $\mathbf{K}_n$ we have to find a path to go on. We shall explain the idea how to do this by a simple example. Put $\mathbf{K5} = \mathbf{K} + \Diamond p \to \Box\Diamond p$. $\mathbf{K5}$ is an r-persistent logic such that $\mathsf{Fr}\mathbf{K5}$ is axiomatized by $(\forall x, y, z)(xRy \wedge xRz \Rightarrow yRz)$ (cf. [26]). How to prove that $\mathbf{K5}$ is an iterated Sf-splitting? Following correspondence theory it is readily checked that $\mathsf{Fr}\mathbf{K5} = \mathsf{Fr}_{\mathbf{F}}$, where $\mathbf{F}$ consists of the following frames.

$\mathcal{F}_1 =$ , $\mathcal{F}_2 =$ , $\mathcal{F}_3 =$ , $\mathcal{F}_4 =$ , $\mathcal{F}_5 =$ ,

$\mathcal{F}_6 =$  $\mathcal{F}_7 =$ .

In order to prove that $\mathbf{F}$ defines an iterated Sf-splitting which coincides with $\mathbf{K5}$ we have to decide with which frames to Sf-split first. It turns out that

$$\mathbf{K5} = \mathbf{K}/^{\mathsf{Sf}}\mathcal{F}_1/^{\mathsf{Sf}}\{\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_6\}/^{\mathsf{Sf}}\mathcal{F}_4/^{\mathsf{Sf}}\mathcal{F}_5/^{\mathsf{Sf}}\mathcal{F}_7,$$

where the right side of the equation is defined. In order to clarify this order we define

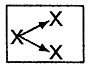
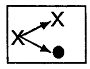
Arrow subframes.

Suppose that $\mathcal{F} = \langle f, \vec{R}\rangle$ and $\mathcal{G} = \langle g, \vec{S}\rangle$ are finite n-frames and $x \in g$. Then $\mathcal{F}$ is an *x-arrow subframe* of $\mathcal{G}$, in symbols $\mathcal{F} \leq_x \mathcal{G}$, if $f = g$, $R_i \subseteq S_i$, for $1 \leq i \leq n$, and x is a root of $\mathcal{F}$. $\mathcal{F}$ is a *strict x-arrow subframe* of $\mathcal{G}$, in symbols $\mathcal{F} <_x \mathcal{G}$, if $\mathcal{F} \leq_x \mathcal{G}$ and $\mathcal{F} \neq \mathcal{G}$.

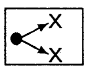
In general, for $\mathcal{F} <_x \mathcal{G}$, we should first Sf-split with $\mathcal{F}$ and then with $\mathcal{G}$. (Above, $\mathcal{F}_1$ is indeed the only frame in $\mathbf{F}$ which splits $\mathcal{S}\mathbf{K}$.) This motivates the order of the frames $\mathcal{F}_1 \ldots \mathcal{F}_5$. However, in order to proceed in this way, we need some knowledge about the intermediate steps. Roughly, we need to know that a $\mathcal{F}$ with $\mathcal{F} <_x \mathcal{G}$ does not occur in a frame for Λ if we want to show that $\mathcal{G}$ Sf-splits Λ. More precisely, we would like to be sure that

- $\Lambda_1 = \mathbf{K}/^{\mathsf{Sf}}\mathcal{F}_1$ is complete and that $\mathsf{Fr}\Lambda_1 = \mathsf{Fr}_{\{\mathcal{F}_1\}}$,
- $\Lambda_2 = \Lambda_1/^{\mathsf{Sf}}\{\mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_6\}$ is complete and $\mathsf{Fr}\Lambda_2 = \mathsf{Fr}_{\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_6\}}$,
- $\Lambda_3 = \Lambda_2/^{\mathsf{Sf}}\mathcal{F}_4$ is complete and $\mathsf{Fr}\Lambda_3 = \mathsf{Fr}_{\{\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4, \mathcal{F}_6\}}$.

and so on, for all the other intermediate steps. Often this will be achieved by proving that those intermediate logics are r-persistent. The frames $\mathcal{F}_6$ and $\mathcal{F}_7$ fit into this scheme since

$\mathcal{F}_{11} =$  and $\mathcal{F}_{12} =$ 

do not occur as subframes of frames in $\mathsf{Fr}\Lambda_1 = \mathsf{Fr}_{\{\mathcal{F}_1\}}$. The argument is similar for $\mathcal{F}_7$. Let us note already that the heuristic ideas above are not valid in general. For instance, it will be shown (cf. Theorems 5.2 (1.) and 6.4) that $\mathcal{F}_{11}$ Sf-splits $\mathbf{K}$ but that $\mathcal{F}_{12}$ does not Sf-split $\mathbf{K}/^{\mathsf{Sf}}\mathcal{F}_{11}$. On the other hand



Sf-splits $\mathbf{K}/^{\mathsf{Sf}}\mathcal{F}_{11}$.

Following the idea of forming x-arrow-subframes we shall first investigate Sf-splittings by sequences of frames from

$\mathcal{T}$-closed sets.

Suppose that $\mathcal{T} = \langle t, \vec{<} \rangle$ is an n-tree with root 0. Then we put

$$\langle \mathcal{T} \rangle = \{\mathcal{F} \in \mathsf{Fr} | \mathcal{T} \leq_0 \mathcal{F}\}.$$

Then $(\langle \mathcal{T} \rangle, \leq_0)$ is a finite partially ordered set with smallest element $\mathcal{T}$ and greatest element $\langle t, \langle t \times t : 1 \leq i \leq n \rangle \rangle$. For $\mathcal{F} \in \langle \mathcal{T} \rangle$ we denote by $[\mathcal{T}, \mathcal{F}]$ the interval $\{\mathcal{G} : \mathcal{T} \leq_0 \mathcal{G} \leq_0 \mathcal{F}\}$. Also put $[\mathcal{T}, \mathcal{F}[= [\mathcal{T}, \mathcal{F}] - \{\mathcal{F}\}$. A set $\mathbf{F} \subseteq \langle \mathcal{T} \rangle$ is *$\mathcal{T}$-closed* if $\mathcal{F} \in \mathbf{F}$ implies $[\mathcal{T}, \mathcal{F}] \subseteq \mathbf{F}$.

n-trees.

Above, and in what follows, a n n-frame $\mathcal{G} = \langle g, \vec{S} \rangle$ is an *n-tree* if it is cycle free and rooted and $S_i \cap S_j = \emptyset$, for all $i \neq j$, and each $x \in g$ has not more than one predecessor with respect to the relation $S = \bigcup\{S_i | 1 \leq i \leq n\}$. 1-trees are called trees.

Since we want to define iterated Sf-splittings by $\mathcal{T}$-closed sets $\mathbf{F}$ it is important to know whether $\mathsf{Fr}_{\mathbf{F}}$ is a class definable by modal formulas. This can be checked by using Proposition 2.4. However, it is instructive to have an axiomatization. For a finite n-tree $\mathcal{T} = \langle t, \vec{<} \rangle$ and $\mathcal{T} \leq_0 \mathcal{G} = \langle g, \vec{S} \rangle$ define

$$\begin{aligned} \nabla_{[\mathcal{T},\mathcal{G}]} \;=\; & \bigwedge \langle p_x \rightarrow \neg p_y | x \neq y \rangle \\ & \wedge \bigwedge \langle p_y \rightarrow \Diamond_i p_x | y <_i x \rangle \\ & \wedge \bigwedge \langle p_y \rightarrow \neg \Diamond_i p_x | \neg(x S_i y) \rangle \end{aligned}$$

Proposition 5.1 *$\Lambda = \mathbf{K}_n + \Box^{(dp(\mathcal{T}))} \nabla_{[\mathcal{T},\mathcal{G}]} \rightarrow \neg p_0$ is the r-persistent subframe logic with* $\mathsf{Fr}\Lambda = \mathsf{Fr}_{[\mathcal{T},\mathcal{G}]}$.

Obviously we get an axiomatization for each logic of the form $\mathsf{Th}(\mathsf{Fr}_{\mathbf{F}})$, where $\mathbf{F}$ is $\mathcal{T}$-closed for some $\mathcal{T}$. The formulas $\nabla_{[\mathcal{T},\mathcal{G}]}$ look quite similar to the formulas axiomatizing iterated Sf-splittings, but they are not equivalent. This will follow from the fact that quite often logics of type $\mathsf{Th}(\mathsf{Fr}_{\mathbf{F}})$ are not iterated Sf-splittings. The following Lemma is proved in a similar way. We are ready to prove the first general splitting result. In one of the cases of this result the following frames will play a major role. Put, for $m \in \omega$,

$$\mathsf{line}_m = \langle \{0, \ldots, m\}, < \rangle,$$

$$\mathsf{disc}_m = \langle \{0, \ldots, m\}, S \rangle, \text{ where } iSj \text{ iff } j = i + 1,$$

$$\mathsf{HELP}_m = [\mathsf{disc}_m, \mathsf{line}_m].$$

It follows immediately from Theorem 4.1 that HELP_m defines an iterated Sf-splitting such that $\mathbf{H}_m := \mathbf{K}/^{\mathsf{Sf}}\mathsf{HELP}_m$ is r-persistent and $\mathsf{Fr}\mathbf{H}_m = \mathsf{Fr}_{\mathsf{HELP}_m}$. Roughly, in presence of $\mathbf{H}_m$ a lot of frames will Sf-split which do not Sf-split without $\mathbf{H}_m$.

Transitive closure.

For a frame $\mathcal{G} = \langle g, \vec{S} \rangle$ define the *transitive closure* $\vec{S}^*$ of $\vec{S}$ by putting

$$x\vec{S}^*y \text{ iff there is a path of length} > 0 \text{ from } x \text{ to } y \text{ in } \mathcal{G}.$$

Two classes of frames.

For an n-tree $\mathcal{T} = \langle t, \vec{<} \rangle$ define the set $\langle \mathcal{T} \rangle_{\mathsf{R}}$ by

$$\langle t, \vec{R} \rangle \in \langle \mathcal{T} \rangle_{\mathsf{R}} \Leftrightarrow \mathcal{T} \leq_0 \langle t, \vec{R} \rangle \text{ and } R_i \subseteq \vec{<}^* \cup \{(x, 0) : x \in t\}, 1 \leq i \leq n.$$

For a n-tree $\mathcal{T} = \langle t, \vec{<} \rangle$ define the set $\langle \mathcal{T} \rangle_{\mathsf{H}}$ by

$$\langle t, \vec{R} \rangle \in \langle \mathcal{T} \rangle_{\mathsf{H}} \Leftrightarrow \mathcal{T} \leq_0 \langle t, \vec{R} \rangle \text{ and } R_i \subseteq \vec{<}^* \cup \{(x,0) : x \in t\} \cup \{(x,x) : x \in t\}, 1 \leq i \leq n.$$

Theorem 5.2 *Suppose that $\mathcal{G} \in \langle \mathcal{T} \rangle$ and that Λ is a r-persistent subframe logic with $\mathsf{Fr}\Lambda \subseteq \mathsf{Fr}_{[\mathcal{T},\mathcal{G}[}$. Suppose that one of the following cases holds.*

1. *$\mathcal{G} \in \langle \mathcal{T} \rangle_{\mathsf{R}}$.*
2. *$\mathcal{G} \in \langle \mathcal{T} \rangle_{\mathsf{H}}$ and $\Lambda \supseteq \otimes^n \mathbf{T}$.*
3. *$\mathcal{G}, \mathcal{T}$, and Λ are monomodal, $\mathcal{G} \in \langle \mathcal{T} \rangle_{\mathsf{H}}$ and $\Lambda \supseteq \mathbf{K}/^{\mathsf{Sf}}\mathsf{HELP}_m$, for an $m \in \omega$.*
4. *Λ has conjugates, i.e. $\Lambda \supseteq \mathbf{K}_n + \mathbf{cn}_\pi$ (see the Introduction).*

Then $\mathcal{G}$ Sf-splits Λ and $\Lambda/^{\mathsf{Sf}}\mathcal{G}$ is r-persistent with $\mathsf{Fr}(\Lambda/^{\mathsf{Sf}}\mathcal{G}) = \mathsf{Fr}\Lambda \cap \mathsf{Fr}_{\{\mathcal{G}\}}$.

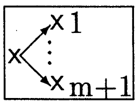
We proceed with some applications of this Theorem. Often it will be more convenient to allow Sf-splittings of logics Λ by frames $\mathcal{G}$ with $\mathcal{G} \not\models \Lambda$. In this case we simply put $\Lambda/^{\mathsf{Sf}}\mathcal{G} = \Lambda$. This mainly applies to iterated Sf-splittings.

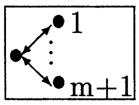
Define an n-tree $\mathcal{T}$ by putting $\mathcal{T} = \langle \{0\}, \langle \emptyset | 1 \leq i \leq n \rangle \rangle$. By Theorem 5.2 (1.) $\langle \mathcal{T} \rangle_{\mathsf{R}}$ defines an iterated Sf-splitting of $\mathbf{K}_n$ and

$$\otimes^n \mathbf{T} = \mathbf{K}_n/^{\mathsf{Sf}}\langle \mathcal{T} \rangle_{\mathsf{R}}.$$

Corollary 5.3 *For all π with $\pi \circ \pi = id$ the logics $\otimes^n \mathbf{T} + \mathbf{cn}_\pi$ are iterated Sf-splittings of $\mathbf{K}_n$ by finite frames. They have the fsp, the ssp and the Sf-axiomatization problem is decidable. Examples are $\mathbf{T}.t$ and $\otimes^n \mathbf{T}.\mathbf{B}_1$.*

For $m > 0$ put

$\mathsf{wd}_m =$ 

and $\mathsf{rwd}_m =$ 

. (7.1)

It is readily checked that $\mathsf{Fr}_{\mathsf{WD}_m} = \mathsf{Fr}(\mathbf{K} + \mathbf{I}_m)$, where $\mathsf{WD}_m = [\mathsf{wd}_m, \mathsf{rwd}_m]$. We also have $\mathsf{WD}_m \subseteq \langle \mathsf{wd}_m \rangle_{\mathsf{H}}$. So we can apply Theorem 5.2 (3.) to WD_m and have that $\Lambda + \mathbf{I}_m$ is an iterated Sf-splitting by finite and rooted frames whenever $\Lambda \supseteq \mathbf{H}_m$, for some $m \in \omega$. This applies, for instance, to $\mathbf{T} + \mathbf{I}_m$.

Recall from Proposition 3.2 that the logics $\mathbf{K}_n.\mathbf{Tr}_m$ play an important role in lattices of subframe since they are those which allow splittings by arbitrary frames in rFr. So, it would be nice if we could get $\mathbf{K}_n.\mathbf{Tr}_m$ as an iterated Sf-splitting. In the next section we shall see that this not the case. However, for conjugated logics, more can be said.

Corollary 5.4 *$\mathbf{K}.t + \mathbf{tr}_n$ is an iterated Sf-splitting of $\mathbf{K}.t$ by finite frames and has the ssp in $\mathcal{S}\mathbf{K}.t$. $\mathbf{T}.t + \mathbf{tr}_n$ is an iterated Sf-splitting of $\mathbf{K}_2$ by finite frames and has ssp in $\mathcal{S}\mathbf{K}_2$. (Here, $\mathbf{tr}_n = (\Box_{\mathcal{L}_2}^{(n)} \to \Box_{\mathcal{L}_2}^{(n+1)} p)^{\mathsf{Sf}}$).*

We see that Part 4 of Theorem 5.2 is indeed the strongest result so far. We also get the following result for minimal tense extensions $\Lambda.t$ of monomodal logics Λ. For a monomodal frame $\langle h, \lhd \rangle$ define

$$\langle h, \lhd \rangle^t = \langle g, \lhd, \lhd^{-1} \rangle.$$

and for a set of monomodal Kripke frames $\mathbf{F}$ put $\mathbf{F}^t = \{h^t : h \in \mathbf{F}\}$. A straightforward proof shows

$$\mathsf{Fr}(\Lambda.t) = (\mathsf{Fr}\Lambda)^t \text{ and } (\mathsf{Fr}(\Theta.t) = \mathsf{Fr}(\Lambda.t) \Leftrightarrow \mathsf{Fr}\Lambda = \mathsf{Fr}\Theta), \tag{7.3}$$

for all monomodal logics Λ and Θ. The following Corollary follows immediately with Theorem 5.2 (4.).

Corollary 5.5 *Let $\mathcal{T}$ be a tree and $\mathbf{F} \subseteq \langle \mathcal{T} \rangle$ be $\mathcal{T}$-closed. Then $\mathbf{F}^t$ defines an iterated* Sf*-splitting of $\mathbf{K}.t$ such that $\mathbf{K}.t/^{\mathsf{Sf}}\mathbf{F}^t$ is r-persistent and*

$$\mathsf{Fr}(\mathbf{K}.t/^{\mathsf{Sf}}\mathbf{F}^t) = (\mathsf{Fr}_{\mathbf{F}})^t.$$

Put, for $m > 0$,

$$\mathsf{tr}_m = \langle \{0, \ldots, m+1\}, \{(i,j) | j \leq i+1\} \rangle. \tag{7.2}$$

It is readily checked that $\mathsf{Fr}_{\mathsf{TR}_m} = \mathsf{Fr}(\mathbf{K}.\mathbf{Tr}_m)$, for $\mathsf{TR}_m = [\mathsf{disc}_{m+1}, \mathsf{tr}_m]$. Thus, the minimal tense extensions $(\mathbf{K}.\mathbf{Tr}_n).t$ as well as $(\mathbf{K} + \mathbf{I}_n).t$ are iterated Sf-splittings of $\mathbf{K}.t$ by finite frames. We even get $\mathbf{K4}.t$. Put

$$\mathcal{F}_3 = \boxed{\times \longleftrightarrow \times}, \; \mathcal{F}_4 = \boxed{\times \longleftrightarrow \bullet}\,.$$

Now it is not difficult to show that

$$\mathbf{K4}.t = (\mathbf{K}.\mathbf{Tr}_1.t)/^{\mathsf{Sf}}\mathcal{F}_3^t/^{\mathsf{Sf}}\mathcal{F}_4^t.$$

By the previous Corollary $(\mathbf{K}.\mathbf{Tr}_1).t = \mathbf{K}.t/^{\mathsf{Sf}}\mathsf{TR}_1^t$. So we conclude

Corollary 5.6 *(1) The logic $\mathbf{K4}.t$ is an iterated* Sf*-splitting of $\mathbf{K}.t$ by finite frames. It has the ssp and the fsp and the* Sf*-axiomatization problem is decidable in $\mathcal{S}\mathbf{K}.t$. (2) The logic $\mathbf{S4}.t$ is an iterated* Sf*-splitting of $\mathbf{K}_2$ by finite frames. It has the ssp, the fsp and the* Sf*-axiomatization problem is decidable in $\mathcal{S}\mathbf{K}_2$.*

6 Negative Results

In this section we deliver some general counterexamples. Denote by $\mathbf{G.3}$ the logic $\mathbf{G} + \mathbf{I}_1$. $\mathbf{G.3}$ has the finite model property and its frames are precisely the strict orderings without infinite ascending chains. Hence

$$\mathbf{G.3} = \mathsf{Th}\{\mathsf{line}_n : n \in \omega\}.$$

Theorem 6.1 *Suppose that Λ is a r-persistent subframe logic with $\mathsf{line}_m \models \Lambda$, for all $m \in \omega$ (or, equivalently, $\Lambda \subseteq \mathbf{G.3}$). Suppose that there exists $\mathcal{G} \in \mathsf{rFr}\mathbf{T}$ which is r-cycle-free and $\mathcal{G} \not\models \Lambda$. Then Λ is not an iterated* Sf*-splitting of $\mathbf{K}$ by frames in* rFr *and Λ does not have the ssp.*

The Theorem above applies, for instance, for $\mathbf{K}.\mathbf{I}_n$ and $\mathbf{K}.\mathbf{Tr}_n$ but certainly also to uncountably many other subframe logics.

Theorem 6.2 *Suppose that $\mathbf{K}.\mathbf{Tr}_n \subseteq \Lambda \subseteq \mathbf{G.3}$ for an $n > 0$. Then Λ is not a join-*Sf*-splitting of $\mathbf{K}.\mathbf{Tr}_{n+1}$ by finite and rooted frames and does not have the ssp in $\mathcal{S}\mathbf{K}.\mathbf{Tr}_{n+1}$.*

Recall that all frames in $\mathsf{rFr}(\mathbf{K.Tr}_{n+1})$ Sf-split $\mathbf{K.Tr}_{n+1}$. Thus, Theorem 6.2 states that as long as we do not Sf-split with finite $\mathbf{G.3}$-frames we shall not get a logic containing $\mathbf{K.Tr}_n$ as a join-Sf-splitting of $\mathbf{K.Tr}_{n+1}$; hence not as an iterated Sf-splitting of $\mathbf{K}$ by frames in rFr. Thus, the following result is just a reformulation of Theorem 6.2.

Theorem 6.3 *Suppose that* $\mathbf{F} \subseteq \mathsf{rFr}(\mathbf{K.Tr}_{n+1})$ *such that* $\mathbf{F} \cap \{\mathsf{line}_m : m \in \omega\} = \emptyset$ *and* $\mathsf{Fr}(\mathbf{K.Tr}_{n+1})_{\mathbf{F}} \subseteq \mathsf{Fr}\mathbf{Tr}_n$. *Then*

$$\mathbf{K.Tr}_{n+1}/^{\mathsf{Sf}}\mathbf{F}$$

is incomplete.

This result shows that splittings also form a powerful tool for establishing numerous simple examples of finitely axiomatizable incomplete subframe logics. So far we did not disprove certain quite plausible conjectures as concerns extensions of Theorem 5.2 (1.). Consider a tree $\mathcal{T} = \langle t, < \rangle$ and a point $x \in t$ without successors (i.e. with $\{y : x < y\} = \emptyset$). Denote by $\mathcal{T}_{r(x)}$ the frame $\langle t, < \cup \{(x,x)\}\rangle$. The following result states that with each tree $\mathcal{T}$ with $|t| > 2$ there is associated a strictly descending chain of (incomplete) subframe logics Θ_n, $n \in \omega$, with $\mathsf{Fr}\Theta_n = \mathsf{Fr}_{\{\mathcal{T}, \mathcal{T}_{r(x)}\}}$.

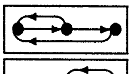
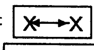
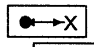
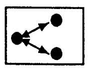
Theorem 6.4 *Suppose that* $\mathcal{T} = \langle t, < \rangle$ *is a tree with* $|t| > 2$ *and* x *has no successors. Then there is a sequence* $\langle k(n) : n \in \omega \rangle$ *such that, for*

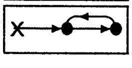
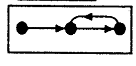
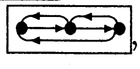
$$\Theta_n = (\mathbf{K}/^{\mathsf{Sf}}\mathcal{T}) + (\Box^{(k(n))}\nabla_{\mathcal{T}_{r(x)}} \to \neg p_0)^{\mathsf{Sf}}, \ n \in \omega,$$

1. *For all* $n \in \omega$, $\mathsf{Fr}\Theta_n = \mathsf{Fr}_{\{\mathcal{T}, \mathcal{T}_{r(x)}\}}$ *and* Θ_n *is incomplete.*
2. *For all* $n \in \omega$, $\Theta_n \supset \Theta_{n+1}$.
3. $\mathcal{T}_{r(x)}$ *does not* Sf*-split* $\mathbf{K}/^{\mathsf{Sf}}\mathcal{T}$.
4. $\mathsf{ThFr}_{\{\mathcal{T}, \mathcal{T}_{r(x)}\}}$ *does not have the ssp and is not an iterated* Sf*-splitting by finite and rooted frames.*

7 Subframe Logics above K4

The main result of this section is a classification of subframe logics containing $\mathbf{K4}$. In order to prove it we have to Sf-split with a number of frames. But some work was already done in Theorem 5.2. Put

$\mathcal{G}_1 =$  $\mathcal{G}_2 =$  $\mathcal{G}_3 =$  $\mathcal{G}_4 =$ 

$\mathcal{G}_5 =$  $\mathcal{G}_6 =$  $\mathcal{G}_7 =$ ,

and put

$$\mathsf{TR} = [\mathsf{disc}_2, \mathcal{G}_1] \cup \{\mathcal{G}_2 \ldots \mathcal{G}_7\}.$$

Based on the following two Lemmas one can show the main Theorem.

Lemma 7.1 $\mathsf{Fr}\mathbf{K4} = \mathsf{Fr}_{\mathsf{TR}}$.

Lemma 7.2 *For all $n \in \omega$,*

$$\Lambda_n := \mathbf{K}/^{\mathsf{Sf}}\mathsf{line}_n/^{\mathsf{Sf}}[\mathsf{disc}_2, \mathcal{G}_1]/^{\mathsf{Sf}}\mathcal{G}_2/^{\mathsf{Sf}} \ldots /^{\mathsf{Sf}}\mathcal{G}_7$$

is well-defined and

$$\Lambda_n = \mathbf{K} + \{(\Box^{(3)}\nabla_{\mathcal{G}} \to p_0)^{\mathsf{Sf}} : \mathcal{G} \in \mathsf{TR} \cup \{\mathsf{line}_n\}\} = \mathbf{K4}/^{\mathsf{Sf}}\mathsf{line}_n.$$

Theorem 7.3 *Suppose that $\mathbf{F}$ is a set of finite, transitive and rooted frames and that $n \geq 2$. Then the following conditions are equivalent.*
(1) $\mathbf{K4}/^{\mathsf{Sf}}\mathbf{F}$ *has ssp in* $\mathcal{S}\mathbf{K}$.
(2) $\mathbf{K.Tr}_n/^{\mathsf{Sf}}(\mathsf{TR} \cup \mathbf{F}) = \mathbf{K4}/^{\mathsf{Sf}}\mathbf{F}$.
(3) $\mathbf{K.Tr}_n/^{\mathsf{Sf}}(\mathsf{TR} \cup \mathbf{F})$ *has fmp.*
(4) $\mathbf{K.Tr}_n/^{\mathsf{Sf}}(\mathsf{TR} \cup \mathbf{F})$ *is complete.*
(5) $(\exists m \in \omega)(\mathsf{line}_m \in \mathbf{F})$.
(6) $\mathbf{K4}/^{\mathsf{Sf}}\mathbf{F}$ *is an iterated* Sf*-splitting of* $\mathbf{K}$ *by finite and rooted frames.*

Corollary 7.4 *For $\Lambda \in \mathcal{S}\mathbf{K4}$ the following conditions are equivalent:*
(1) $\Lambda \not\subseteq \mathbf{G.3}$.
(2) Λ *is an iterated* Sf*-splitting of* $\mathbf{K}$ *by finite and rooted frames.*
(3) Λ *is a join-*Sf*-splitting of* $\mathbf{K.Tr}_2$ *by finite and rooted frames.*
(4) Λ *has ssp in* $\mathcal{S}\mathbf{K}$.
(5) Λ *has fsp in* $\mathcal{S}\mathbf{K}$.
(6) Λ *has ssp in* $\mathbf{K.Tr}_2$.
(7) Λ *has fsp in* $\mathbf{K.Tr}_2$.

Thus, only a minor weakening of transitivity to $\mathbf{tr}_2$ destroys all the nice properties of subframe logics containing $\mathbf{K4}$. Now, for a logic Θ without the ssp in $\mathcal{S}\Lambda$ one would like to know the cardinality of the set of logics $\Theta_1 \in \mathcal{S}\Lambda$ with $\mathsf{Fr}\Theta = \mathsf{Fr}\Theta_1$. Put

$$\delta_{\mathcal{S}\Lambda}(\Theta) = |\{\Theta_1 \in \mathcal{S}\Lambda : \mathsf{Fr}\Theta = \mathsf{Fr}\Theta_1\}|.$$

The following partial answer for logics containing $\mathbf{K4}$ is delivered in [33].

Theorem 7.5 *If $\Lambda \in \mathcal{S}\mathbf{K4}$ and $\Lambda \subseteq \mathbf{G}$, then $\delta_{\mathcal{S}\mathbf{K.Tr}_2}(\Lambda) = 2^{\aleph_0}$.*

It is an open (and from a structural point of view interesting) problem whether also $\delta_{\mathcal{S}\mathbf{K}}(\mathbf{G.3}) = 2^{\aleph_0}$. We note that is is easy to show with the frames from the last section that $\delta_{\mathcal{S}\mathbf{K}}(\mathbf{G.3}) \geq \aleph_0$.

8 The upper part of $\mathcal{S}K_n$

Investigating the upper part of a lattice is a classical problem in modal logic (cf. [5], [12], [24]). So it is interesting whether the upper part of the lattice of subframe logics behaves better than the upper part of the lattice of all normal modal logics. Recall from the Introduction that $\mathbf{K.Alt}_n$ is the (mono)-modal theory of all frames $\langle h, \lhd \rangle$ satisfying $|\{y : x \lhd y\}| \leq n$, for all $x \in h$. First we need the foolowing result on the finite model property.

Theorem 8.1 *For all $n, m > 0$ all subframe logics containing $\otimes^n\mathbf{K.Alt}_m$ have the fmp.*

Based on this result one can show

Theorem 8.2 *For all $n, l > 0$ all subframe logics containing $\otimes^n\mathbf{K.Alt}_l$ have the ssp and the fsp.*

We note that, in a certain sense, this result is optimal since there exists a finitely axiomatizable undecidable monomodal subframe logic such that in all rooted frames only one point is allowed to have more than 4 successors. (This is not shown in [33] or [37] but will be shown elsewhere).

Recall that a logic Λ is called tabular iff $\Lambda = \mathsf{Th}\mathcal{G}$, for a finite frame $\mathcal{G}$. Since each tabular logic contains some $\otimes^n\mathbf{K.Alt}_m$, we conclude

Corollary 8.3 *All tabular subframe logics have ssp and fsp.*

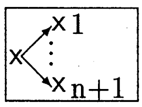
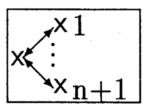
Call a logic Λ Sf-*pretabular* iff it is a maximal non-tabular subframe logic in $\mathcal{S}\mathbf{K}$. By Zorn's Lemma, all non-tabular subframe logics are contained in a Sf-pretabular subframe logic. Examples are $\mathbf{K.Alt}_1$, $\mathbf{G.3}$, $\mathbf{S5}$ and $\mathbf{Grz.3}$. (Recall that $\mathbf{Grz.3}$ is the reflexive counterpart of $\mathbf{G.3}$, i.e.

$$\mathbf{Grz.3} = \mathsf{Th}\{\mathsf{rline}_m : m \in \omega\},$$

where $\mathsf{rline}_m = \langle\{0, \ldots, m\}, \leq\rangle$, for $m \in \omega$, (cf. [15])). Now we can formulate all desirable properties of the upper part of $\mathcal{S}\mathbf{K}_n$. The *codimension* of a logic Λ in $\mathcal{S}\mathbf{K}_n$ is the length of the longest $\subset$-chain in $\mathcal{S}\mathbf{K}_n$ from Λ to $\mathcal{L}_n$.

Corollary 8.4 *(1) All* Sf*-pretabular logics have infinite codimension in $\mathcal{S}\mathbf{K}_n$. (2) A subframe logic is tabular if and only if it has finite codimension in $\mathcal{S}\mathbf{K}_n$. (3) The* Sf*-axiomatization problem in $\mathcal{S}\mathbf{K}_n$ is decidable, for all tabular subframe logics. (4) All* Sf*-pretabular logics have the fmp.*

We shall now restrict attention to monomodal logics. One of the classical problems of modal logic is the description of the pretabular logics in certain lattices, if possible. However, even for logics containing $\mathbf{K4}$ such a description is not available by Blok's result that there are $2^{\aleph_0}$ pretabular logics containing $\mathbf{K4}$ (cf. [5]). (On the other hand, there are precisely 5 pretabular logics containing $\mathbf{S4}$ (cf. [12])). The situation is different for monomodal lattices of subframe logics. Here we shall describe those Sf-pretabular logic which do not contain $\mathbf{K.Alt}_n$, for all $n > 0$. (A description of all monomodal Sf-pretabular logics seems possible. However, it is readily checked that there exist $\geq \aleph_0$ such logics containing $\mathbf{K.Alt}_3$). Consider the following sets of frames

$\mathbf{F}^{00} = \{$  $: n \in \omega\}$ and $\mathbf{G}^{00} = \{$  $: n \in \omega\}$.

From $\mathbf{F}^{00}$ and $\mathbf{G}^{00}$ we obtain sets $\mathbf{F}^0$ and $\mathbf{G}^0$ by adding $\{(0,0)\}$ to the relations, and we obtain $\mathbf{F}^1$ and $\mathbf{G}^1$ by adding $\{(x,x) : 1 \leq x \leq n+1\}$ to the relations. Finally we obtain sets $\mathbf{F}^{01}$ and $\mathbf{G}^{01}$ by adding $\{(x,x) : 0 \leq x \leq n+1\}$ to the relations. Recall the definition of $\mathbf{Grz.3}$ from the Introduction (1.5).

Theorem 8.5 *A monomodal subframe logic* Λ *with* $\Lambda \not\supseteq \mathbf{K.Alt}_n$*, for all* $n > 0$*, is an* Sf*-pretabular logic iff* Λ *is one of the logics* **G.3**, **Grz.3**, **S5**, *or one of the following logics*

$$\mathsf{Th}(\mathsf{Sf}\mathbf{F}^{00}),\ \mathsf{Th}(\mathsf{Sf}\mathbf{F}^{0}),\ \mathsf{Th}(\mathsf{Sf}\mathbf{F}^{1}),\ \mathsf{Th}(\mathsf{Sf}\mathbf{F}^{01}),$$

$$\mathsf{Th}(\mathsf{Sf}\mathbf{G}^{00}),\ \mathsf{Th}(\mathsf{Sf}\mathbf{G}^{0}),\ \mathsf{Th}(\mathsf{Sf}\mathbf{G}^{1}),\ \mathsf{Th}(\mathsf{Sf}\mathbf{G}^{01}).$$

The following result shows once more the fundamental role of the logic **G.3** for the structure of the lattice of monomodal subframe logics.

Theorem 8.6 *All monomodal* Sf*-pretabular logics not equal to* **G.3** *have the ssp in* $\mathcal{S}\mathbf{K}$.

9 Tense Extensions

In this final section we compare monomodal subframe logics Λ with their minimal tense extension $\Lambda.t$. The object is to get insight into the different lattice theoretic behavior of subframe logics with and without conjugates. Denote by $\mathcal{L}$ the monomodal language with $\Box$. We shall assume that monomodal logics Λ are always formulated in $\mathcal{L}$, and we assume that $\Lambda.t$ is formulated in the bimodal language with modal operators $\Box$ and $\Box^-$. Bimodal logics containing $\mathbf{K}.t$ are called *tense logics.* We denote, for a tense logic Θ, by Θ_+ the monomodal logic $\Theta \cap \mathcal{L}$. The first important question is whether $(\Lambda.t)_+ = \Lambda$, i.e., whether $\Lambda.t$ is a conservative extension of Λ. That this is so, has been assumed in some articles, e.g. in [22]. Here we shall show that this is not the case. Recall from (7.3) the definition $\langle h, \lhd\rangle^t = \langle h, \lhd, \lhd^{-1}\rangle$, for each (mono-)modal Kripke frame $\langle h, \lhd\rangle$, and that

$$\mathsf{Fr}(\Lambda.t) = (\mathsf{Fr}\Lambda)^t \text{ and } (\mathsf{Fr}\Theta.t = \mathsf{Fr}\Lambda.t \Leftrightarrow \mathsf{Fr}\Lambda = \mathsf{Fr}\Theta),$$

for all monomodal logics Λ and Θ. The following Proposition follows immediately.

Proposition 9.1 *Let* $\Lambda \in \mathcal{S}\mathbf{K}$. *(1) If* Λ *is complete, then* $(\Lambda.t)_+ = \Lambda$. *Especially,* $(\Lambda.t)_+ = \Lambda$*, for all subframe logics containing* **K4**. *(2) If* Λ *has ssp in* $\mathcal{S}\mathbf{K}$,

$$\{\Theta \in \mathcal{S}\mathbf{K} : \Theta.t = \Lambda.t\} = \{\Lambda\}.$$

(3) If $\Lambda.t$ *has ssp in* $\mathbf{K}.t$*, then*

$$\{\Theta \in \mathcal{S}\mathbf{K} : \Theta.t = \Lambda.t\} = \{\Theta \in \mathcal{S}\mathbf{K} : \mathsf{Fr}\Theta = \mathsf{Fr}\Lambda\}.$$

Thus, by (3), combining Corollaries 5.5 and 5.6 with the counterexamples of section 8, we obtain numerous subframe logics Λ with $|\{\Theta \in \mathcal{S}\mathbf{K} : \Theta.t = \Lambda.t\}| > 1$. This holds, for instance, for $\Lambda \in \{\mathbf{K.I}_n, \mathbf{K.Tr}_n, \mathbf{K4}\}$. Even more can be said, also as a straightforward consequence of Corollary 5.6 and Corollary 7.4, for logics containing **K4**.

Theorem 9.2 *For all subframe logics* Λ *containing* **K4**,

$$\{\Theta \in \mathcal{S}\mathbf{K} | \Theta.t = \Lambda.t\} = \{\Theta \in \mathcal{S}\mathbf{K} : \mathsf{Fr}\Lambda = \mathsf{Fr}\Theta\}.$$

Hence, $|\{\Theta \in \mathcal{S}\mathbf{K} : \Theta.t = \Lambda.t\}| > 1$ *iff* $\Lambda \not\subseteq$ **G.3**.

After this easy application of the results we already had we now come to the main result of this section. We first note the following result on intrinsic properties of the subframe logics under consideration, which already indicate some interesting connections between usually independent properties (cf.[34] and [35]).

Theorem 9.3 *If Λ is a subframe logic containing* $\mathbf{K4}$ *then* $\Lambda.t$ *has the fmp iff* Λ *is elementary.* $\Lambda.t$ *is complete, for all* $\Lambda \in \mathcal{S}\mathbf{K4}$ *and* $\Lambda.t$ *is decidable, for all finitely axiomatizable* $\Lambda \in \mathcal{S}\mathbf{K4}$.

Note that this result does not extend to all logics above $\mathbf{K4}$. [36] presents an example where Λ has the fmp but $\Lambda.t$ is incomplete. Now, what about the lattice theoretic behavior of subframe logics of type $\Lambda.t$?

Theorem 9.4 *Let Λ be a monomodal subframe logic containing* $\mathbf{K4}$. *Then the following conditions are equivalent:*

(1a) $\Lambda.t$ *is an iterated* Sf*-splitting of* $\mathbf{K}.t$ *by frames in* rFr.
(1b) $\Lambda.t$ *is an iterated* Sf*-splitting of* $\mathbf{K4}.t$ *by frames in* rFr.
(2a) $\Lambda.t$ *has fsp in* $\mathcal{S}\mathbf{K}.t$.
(2b) $\Lambda.t$ *has fsp in* $\mathcal{S}\mathbf{K4}.t$.
(3a) $\Lambda.t$ *has ssp in* $\mathcal{S}\mathbf{K}.t$.
(3b) $\Lambda.t$ *has ssp in* $\mathcal{S}\mathbf{K4}.t$.
(4a) Λ *is elementary.*
(4b) $\Lambda.t$ *is elementary.*

$\{\phi^{\mathsf{Sf}} : \mathbf{K4} + \phi^{\mathsf{Sf}}$ *is elementary* $\}$ *is recursive. For* $\Lambda \supseteq \mathbf{S4}$ *we can replace* $\mathbf{K4}$ *by* $\mathbf{S4}$ *and* $\mathbf{K}.t$ *by* $\mathbf{K}_2$, *in the equivalences above.*

So we find, for minimal tense extensions, a remarkable connection between the lattice theoretic properties of a logic and intrinsic properties like elementarity and fmp. Compared with the results on $\mathcal{S}\mathbf{K4}$ we see that conjugates change the behavior in an unexpected way. While $\mathbf{Grz}$ behaves nice in $\mathcal{S}\mathbf{K}$ the logic $\mathbf{Grz}.t$ does not in $\mathcal{S}\mathbf{K}.t$ (since it is not elementary). On the other hand $\mathbf{K4}$ is problematic in $\mathcal{S}\mathbf{K}$ but $\mathbf{K4}.t$ is fine in $\mathcal{S}\mathbf{K}.t$.

References

[1] F. Bellisima, On the Lattice of Extensions of the Modal Logic $K.Alt_n$, Arch. Math. Logic 27 (1988) 107 – 114.

[2] J. van Benthem, Modal and Classical Logic (Bibliopolis, 1983).

[3] W. Blok, Varieties of interior algberas, Ph.D. Thesis, University of Amsterdam, 1976.

[4] W. Blok, On the degree of incompleteness in modal logic and the covering relation in the lattice of modal logics, Report 78-07, University of Amsterdam 1978.

[5] W. Blok, Pretabular Varieties of Modal Algebras, Studia Logica 39 (1980) 101 – 124.

[6] W. Blok, The Lattice of Modal Logics. An algebraic Investigation, Journal of Symbolic Logic 45 (1980) 221 – 236.

[7] W. Blok & D. Pigozzi, On the structure of varieties with equationally definable principle congruences I, Algebra Universalis 15 (1982) 195 - 227.

[8] G. Boolos, The Logic of Provability, (Cambridge University Press, 1993)

[9] A. Bull, That all normal extensions of $S4.3$ have the finite model property, Zeitschrift für mathematische Logik und Grundlagen der Mathematik 12 (1966) 341 – 343.

[10] J. Burgess, Basic Tense Logic, In Handbook of Philosophical Logic, Vol. 2, edited by D. Gabbay and F. Guenthner, Dordrecht, 1984.

[11] A. Chagrov & M. Zakharyaschev, Modal companions of intermediate propositional logics, Studia Logica 51 (1992) 49 – 82.

[12] L. Esakia & V. Meskhi, Five critical systems, Theoria 43 (1977) 52 – 60.

[13] K. Kine, Some connections between elementary and modal logic, in S. Kanger, ed., Proceedings of the Third Scandinavian Logic Symposium (North-Holland, Amsterdam, 1975).

[14] K. Fine, Logics containing K4, Part I, Journal of Symbolic Logic 39 (1974) 229 – 237.

[15] K. Fine, Logics containing K4, Part II, Journal of Symbolic Logic 50 (1985) 619 – 651.

[16] K. Fine & G. Schurz, Transfer Theorems for stratified multimodal logics, In Proceedings of the Arthur Prior Memorial Conference, Christchurch, 1991

[17] D. Gabbay & I. Hodkinson & M. Reynolds, Temporal Logic, (Oxford, 1994)

[18] R. Goldblatt, Metamathematics of modal logic, Reports on Mathematical Logic 6,7 (1976) 41 – 77, 21 – 52.

[19] R. Goldblatt, Varieties of Complex Algebras, Annals of Pure and Applied Logic 38 (1989) 173 – 241.

[20] Henkin & Monk & Tarski, Cylindric Algebras, Part I (North Holland, Amsterdam, 1971).

[21] M. Kracht, An Almost General Splitting Theorem for Modal Logic, Studia Logica 49 (1990) 455 – 470.

[22] M. Kracht, Even more about the lattice of tense logics, Arch Math. Logic 31 (1992) 243 – 257.

[23] M. Kracht & F. Wolter, Properties of independently axiomatizable bimodal logics, Journal of Symbolic Logic 56 (1991) 469 – 485.

[24] L. Maksimova, Pretabular extensions of Lewis $S4$, Algebra i Logica 14 (1975) 28 – 55.

[25] R. McKenzie, Equational bases and non-modular Lattice Varieties, Trans. Am. Math. Soc. 174 (1972) 1 – 43.

[26] M. Nagle & S. Thomason, The extensions of the Modal Logic $K5$, Journal of Symbolic Logic 50 (1985) 102 – 109.

[27] W. Rautenberg, Der Verband der normalen verzweigten Modallogiken, Math. Z. 156 (1977) 123 – 140.

[28] W. Rautenberg, Klassische und nichtklassische Aussagenlogik (Wiesbaden, 1979).

[29] G. Sambin & V. Vaccaro, Topology and duality in modal logic, Annals of Pure and Applied Logic 37 (1988) 249 – 296.

[30] K. Segerberg, An essay in classical modal logic, (Mimeograph, Uppsala, 1971)

[31] K. Segerberg, Modal Logics with Functional Alternative Relations, NDJFL 27 (1986) 504 – 522.

[32] S. K. Thomason, Reduction of second order logic to modal logic, Zeitschrift für mathematische Logik und Grundlagen der Mathematik 21 (1975) 107 – 114.

[33] F. Wolter, Lattices of Modal Logics, Ph. D. Thesis, FU Berlin 1993.

[34] F. Wolter, The finite model property in Tense logic, to appear in Journal of Symbolic Logic, 1995.

[35] F. Wolter, Completeness and Decidability of tense logics closely related to logics above K4, to appear in Journal of Symbolic Logic, 1996

[36] F. Wolter, A counterexample in tense logic, to appear in Notre Dame Journal of Formal Logic, 1996

[37] F. Wolter, The Structure of Lattices of Subframe Logics, to appear in Annals of Pure and Applied Logic, 1996

[38] F. Wolter, Properties of Tense logics, manuscript