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A commuting difference system arising from the elliptic R-matrix *

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Abstract

For Belavin's elliptic R-matrix, we constract an "L-operator" as a set of difference operators acting on the functions on a type A weight space. According to Baxter's argument for commuting transfer matrices, the trace of the L-operater gives a commutative difference system. We show that for the above mentioned L-operator this approach gives the elliptic Macdonald-type operators, actually equivalent to Ruijsenaars' operators. We briefly mentioned about some interesting invariant subspaces.

1 Introduction

In [M1], [M2], I. G. Macdonald defined a commuting system of difference operators for each root system and thereby define a new family of orthogonal polynomials containing two rational parameters (q, t) (in case all the roots have the equal length). Up to now, at least two ways of understanding for these systems are known. One is the work by Etingof and Kirillov [EK1], who obtained these operators as the image of central elements of the quantum enveloping algebra. The other is the work by Cherednik [C92], who uses double affine Hecke algebras, their representation via difference operators (Dunkl operators), and the center of the algebra.

Here we wish to suggest yet another approach for the system.

Needless to say, the Yang-Baxter equation is one of the important background of the above two works. Originally, in Baxter's study of solvable lattice statistical models, the Yang-Baxter equation arose as the condition to provide sufficiently many commuting operators.This is done by taking the traces of the so called L-operators, the operators which satisfy the "$RLL = LLR$ relation" (1), which is nothing but a variant of the Yang-Baxter equation.

So the following question is quite natural to ask: "what kind of operators arise if we start with the L-operator realised as difference operators for appropriate functions?" This is our approach and we will show this idea actually works quite well at least for one interesting case.

The case we consider in this paper is for the elliptic R-matrix of Belavin [Be]. In the trigonometric limit, up to a certain simple "gauge transformation" [Re] , this R-matrix

*この報告の詳細については、論文 [H3] および報告集 [城崎] を御覧ください。

degenerates to the image of the universal R-matrix for the quantum affine enveloping algebra $U_q(A^{(1)}_{n-1})$ ([J],[Dr]) in the vector representation.

2 The difference operators

For $n > 1$ let $V = \oplus_{k\in \mathbf{Z}/n\mathbf{Z}} \mathbf{C}e^k$ $(e^k = e^{k+n})$ and let $g, h \in \mathrm{GL}(V)$to be $ge^k := e^k \exp\frac{2\pi i k}{n}$, $he^k := e^{k+1}$. We have $gh = hg\exp\frac{2\pi i}{n}$. Let $\hbar, \tau \in \mathbf{C}, \hbar \neq 0, \mathrm{Im}\tau > 0$.

Belavin's R-matrix $R(u) = R_\hbar(u)$ is characterized as the unique solution of the following five conditions.

- $R_\hbar(u)$ is a holomorphic $\mathrm{End}(\mathbf{C}^n \otimes \mathbf{C}^n)$ -valued function in u,
- $R_\hbar(u) = (x\otimes x)R_\hbar(u)(x\otimes x)^{-1}$ for $x = g, h$,
- $R_\hbar(u+1) = (g\otimes 1)^{-1}R_\hbar(u)(g\otimes 1) \times (-1)$,
- $R_\hbar(u+\tau) = (h\otimes 1)R_\hbar(u)(h\otimes 1)^{-1} \times (-\exp 2\pi i(u + \frac{\hbar}{n} + \frac{\tau}{2}))^{-1}$,
- $R_\hbar(0) = P : x\otimes y \mapsto y\otimes x$.

It holds that 1)there is a unique solution to the above conditions and 2) the solution satisfies the Yang-Baxter equation.

By an L-operator we mean the matrix $L(u) = [L(u)^i_j]_{i,j=1,\cdots n}$ of operators (noncommutative letters) that satisfy

$$\check{R}(u-v)L(u)\otimes L(v) = L(v)\otimes L(u)\check{R}(u-v), \tag{1}$$

where $\check{R}(u) := PR(u)$.

For Belavin's R-matrix we shall construct such an L-operator in the following way. Let $\mathbf{h}^*$ be the weight space for $sl_n(\mathbf{C})$ and realize $\mathbf{h}^*$ in $\mathbf{C}^n = \oplus_{i=1,\cdots,n}\mathbf{C}\epsilon_i$, $<\epsilon_i, \epsilon_j> = \delta_{i,j}$, as the orthogonal complement to $\sum_{i=1,\cdots,n}\epsilon_i$. We denote the orthogonal projection of ϵ_i by $\bar{\epsilon}_i$. For each $\lambda, \mu \in \mathbf{h}^*$ and $j = 1, \cdots, n$ we put

$$\phi(u)^\mu_{\lambda j} := \begin{cases} \theta_j(\frac{u}{n} - <\lambda, \bar{\epsilon}_k>) & : \mu - \lambda = \hbar\bar{\epsilon}_k \quad \text{for some } k = 1, \cdots, n, \\ 0 & : \text{otherwise} \end{cases} \tag{2}$$

where

$$\theta_j(u) := \sum_{\mu\in\frac{n}{2}-j+\mathbf{Z}} \exp 2\pi i(\mu(u+\frac{1}{2}) + \frac{\mu^2}{2l}\tau).$$

Also we let $\bar{\phi}(u)^{\mu+\hbar\bar{\epsilon}_k, j}_\mu$ to be the entry in the inverse matrix to $[\phi(u)^{\mu+\hbar\bar{\epsilon}_k}_\mu{}_j]_{j,k=1,\cdots,n}$, namely

$$\sum_{j=1}^n \bar{\phi}(u)^{\mu+\hbar\bar{\epsilon}_k j}_\mu \phi(u)^{\mu+\hbar\bar{\epsilon}_{k'}}_\mu{}_j = \delta_{k,k'}, \qquad \sum_{k=1}^n \phi(u)^{\mu+\hbar\bar{\epsilon}_k}_\mu{}_j \bar{\phi}(u)^{\mu+\hbar\bar{\epsilon}_k j'}_\mu = \delta_{j,j'}. \tag{3}$$

Then generalizing a result in the celebrated paper [S] we have

Theorem 1 ([H1], [H2]) *For a function f on $\mathbf{h}^*$, put*

$$(L(c|u)^i_j f)(\mu) := \sum_{k=1,\cdots,n} \phi(u+c\hbar)^{\mu+\hbar\bar{\epsilon}_k}_\mu{}_j \bar{\phi}(u)^{\mu+\hbar\bar{\epsilon}_k, i}_\mu f(\mu+\hbar\bar{\epsilon}_k). \tag{4}$$

Then for any $c\in\mathbf{C}$, the collection of difference operaors $L(c|u) = [L(c|u)^i_j]_{i,j=1\cdots,n}$ satisfies the desired relation (1). i.e., $L(c|u)$ gives a 1-parameter (c) family of L-operators.

Recall $V = \oplus_{j=1,\cdots,n}\mathbf{C}e^j = \mathbf{C}^n$ and let $\mathcal{O}$ be the ring of meromorphic functions. Then the above L-operator is an endomorphism on the space $V \otimes \mathcal{O}(\mathbf{h}^*)$,

$$L(c|u) \in \mathrm{End}(V \otimes \mathcal{O}(\mathbf{h}^*)).$$

Here the first space $V = \mathbf{C}^n$ can be regarded as the space of defining comodule (vector "co"representation) for the bialgebra $A(R)$ defined by the relation (1). We can consider more complicated comodules for this bialgebra as well: actually for each young diagram Y we can construct a $A(R)$-comodule $V(Y)$ whose dimension is just the same as for the GL_n-module that corresponds to Y. This is an early result known as the fusion technique [KRS][C] and done by taking the appropriate sub/quotient of the tensor comodule of $V(\square) = V = \mathbf{C}^n$. It follows that we get a collection of difference operators

$$L^Y(c|u) \in \mathrm{End}(V(Y) \otimes \mathcal{O}(\mathbf{h}^*))$$

for each Y and they satisfy the relation

$$\check{R}^{Y,Y'}(u-v)L^Y(c|u) \otimes L^{Y'}(c|v) = L^{Y'}(c|v) \otimes L^Y(c|u)\check{R}^{Y,Y'}(u-v), \tag{5}$$

where $\check{R}^{Y,Y'}$ is the "fused R-matrices" which is nothing but the isomorphism between the $A(R)$-modules $V(Y) \otimes V(Y') \to V(Y') \otimes V(Y)$.

These structures are of course now well understood for the trigonometric R-matrix case, where we have the quantised enveloping algebra and its universal R-matrix as the origin of those fused R-matrix or the fused L-operators.

Now we are in the position to ask the concretized question in Section 1: "what kind of operators arise as the traces of these $L^Y(c|u)$'s?". We shall consider the case $Y = 1^k$, the vertical k boxes case. Then L^{1^k} is a matrix of size dim $\wedge^k \mathbf{C}^n$ whose matrix element is a difference operator. We denote the Jacobi theta function by

$$\theta(u) = \sqrt{-1}p^{1/8}(z^{1/2} - z^{-1/2}) \prod_{m=1}^{\infty}(1 - zp^m)(1 - z^{-1}p^m)(1 - p^m)$$

with $p = \exp 2\pi i\tau$ and $z = \exp 2\pi i u$.

Theorem 2 *1. Let $M^{(k)}(c|u) := Trace_{1^k}L^{1^k}(c|u)$ $(k = 1, \cdots, n)$. Then we have*

$$M^{(k)}(c|u) = \frac{\theta(u + \frac{kc\hbar}{n})}{\theta(u)} \sum_{I\subset\{1,\cdots,n\},|I|=k} \left(\prod_{s\notin I, t\in I} \frac{\theta(<\lambda, \bar{\epsilon}_s - \bar{\epsilon}_t> + \frac{c\hbar}{n})}{\theta(<\lambda, \bar{\epsilon}_s - \bar{\epsilon}_t>)} \right) T_I^{\hbar}, \tag{6}$$

where $T_I^{\hbar}$ stands for the $\hbar$-shift operator:

$$(T_i^{\hbar}f)(\lambda) := f(\lambda + \hbar\bar{\epsilon}_i), \quad T_I^{\hbar} := \prod_{i\in I} T_i^{\hbar}.$$

2.(cf.[D]) Define a function on $\mathbf{h}^*$ *by*

$$\Phi(\lambda) := \prod_{k\neq k'} d^+(z_k/z_{k'}), \quad d^+(z) := \prod_{k=0}^{\infty}\prod_{m=0}^{\infty} \frac{1 - zq^{m+1}p^k}{1 - zq^{m+g+1}p^k} \frac{1 - z^{-1}q^{m-g}p^{k+1}}{1 - z^{-1}q^m p^{k+1}}, \tag{7}$$

where $p = \exp 2\pi i\tau, q = \exp 2\pi i\hbar(|q| < 1)$ *and* $z_j := \exp 2\pi i < \lambda, \bar{\epsilon}_j >$. *Then the conjugation by the square root* $\Phi^{1/2}$ *yields* [1] *the Ruijenaars' [R] commuting operators:*

$$
\begin{aligned}
&(\frac{\theta(u+\frac{kc\hbar}{n})}{\theta(u)})^{-1} \cdot \Phi^{-1/2} M^{(k)}(c|u)\Phi^{1/2} \\
= &\sum_{I\subset\{1,\cdots,n\},|I|=k} \left(\prod_{s\notin I, t\in I} \sqrt{\frac{\theta(c\hbar + <\lambda, \epsilon_s - \epsilon_t>)}{\theta(<\lambda,\epsilon_s-\epsilon_t>)}}\right) T_I^{\hbar} \left(\prod_{s\notin I, t\in I} \sqrt{\frac{\theta(c\hbar + <\lambda, \epsilon_t - \epsilon_s>)}{\theta(<\lambda,\epsilon_t-\epsilon_s>)}}\right).
\end{aligned}
$$

In the operator $M^{(k)}(c|u)$ (6), the spectral parameter u appears only in the overall factor. It is easy to see that, in the trigonometric limit $p \to 0$ the commuting system $M^{(k)}(c|u)$ (6) falls into the Macdonald's one with the parameters $q = \exp 2\pi i\hbar$ and $t = q^{-c/n}$.

It is an important remark that these operators obviously commute, as we mentioned before. This is because the extended "RLL=LLR" relation (5) can be rewritten as

$$\check{R}^{Y,Y'}(u-v)L^Y(c|u) \otimes L^{Y'}(c|v)\check{R}^{Y,Y'}(u-v)^{-1} = L^{Y'}(c|v) \otimes L^Y(c|u),$$

and then taking the trace simply gives

$$M^Y(c|u)M^{Y'}(c|v) = M^{Y'}(c|v)M^Y(c|u)$$

where $M^Y(c|u) := trace_{V(Y)} L^Y(c|u)$. This simple argument and the resulting operators, "the commuting transfer matriecs", was effectively used in Baxter's analysis of the spin chain models [Bax71], [B]; see also [TF]. Thus we may state the ideology:

commuting transfer matrices = commuting difference system.

The proof for the above theorem is fully computational. We only mention here that the proof relies on the formula below, which is quite interesting itself.

Lemma 1 *Recall* $Y_{r<s} := 1$ *if* $r < s$ *holds,* $Y_{r<s} := 0$ *otherwise. The following formula holds:*

$$
\begin{aligned}
&\det\left[\prod_{r=1}^{d} \theta(\mu_r - \lambda_{s'} + \hbar Y_{r<s} + \delta_{r,s}(u-(s-1)\hbar))\right]_{s,s'=1,\cdots,d} \qquad (8)\\
= &\prod_{s=1}^{d-1}\theta(u-s\hbar) \prod_{1\le s<s'\le d} \theta(\lambda_{s'}-\lambda_s)\theta(\hbar+\mu_s-\mu_{s'}).
\end{aligned}
$$

Proof of this formula can be done by the induction on k. The $\hbar = 0$ case of (8) is easily transformed into the Cauchy type determinant formula

$$\det\left[\frac{\theta(\mu_s - \lambda_{s'} + u)}{\theta(\mu_s-\lambda_{s'})\theta(u)}\right]_{s,s'=1,\cdots,d} = \frac{\prod_{1\le s<s'\le d}\theta(\mu_s-\mu_{s'})\theta(\lambda_{s'}-\lambda_s)}{\prod_{s,s'=1,\cdots,d}\theta(\mu_s-\lambda_{s'})},$$

which is also known as the genus 1 case of Fay's trisecant formula [Fay]. But I do not know whether (8) is previously known or not.

It is also interesting to remark that the $\hbar = 0$ case of this formula was quite relevant in [R] although his approach for the commuting system is completely different with ours.

[1] as long as it makes sense

3 An invariant subspace spanned by symmetric theta functions

Let $Th_l^{S_n}$ denote the space of level l $A_{n-1}^{(1)}$- characters [Kac]. This space is of dimension $\frac{(l+n)!}{l!n!}$ and spanned by the theta series on $\mathbf{h}^*$ invariant under the action of the symmetric group S_n. Then we have

Theorem 3 ([H2]) *For nonnegative integer l, we have $L(l|u)_j^i Th_l^{S_n} \subset Th_l^{S_n}$, hence $M^{(k)}(l|u)Th_l^{S_n} \subset Th_l^{S_n}$.*

We can state more precisely the following: As an $A(R)$-module, $Th_l^{S_n}$ is isomorphic to the representation $V(\square \cdots \square)$, the module corresponding to the Young diagram of l horizontal boxes.

Thus there is an interesting "representation theoretic" invariant subspace for our operators. This space would be identified with the space of Weyl group invariant theta functions in [EK2], where they considered the affine analogue of Sutherland operator and its diagonalization.

As an elliptic analogue of Macdonald polynomial theory [M1], we may define a family of orthogonal polynomials as the simultaneous eigenfunction for our operators $M^{(k)}(c|u)$. This diagonalization problem is now under investigation and we hope to report the result in a near future.

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