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Permutation Groups of Finite Morley Rank

Ali Nesin

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1 Introduction

The concept of Morley rank arose in 1965 when Morley treated a problem of Łoś concerning first order theories which have only one model of a given uncountable cardinality, up to isomorphism, a property known as $\aleph_1$-categoricity. The prototypical example of this is the theory of algebraically closed fields of specified characteristic, and Morley showed quite generally that the models of any such theory admits a notion of dimension, to which he assigned the not entirely felicitous name "rank", which Baldwin later showed is finite (Morley defined an ordinal rank which in a more general context can in fact be infinite) [**B**].

A first order theory is said to have finite Morley rank if it has a saturated structure whose definable subsets are endowed with a positive integer in such a way that these integers behave like the "dimension" of the sets they are attached to. For example every variety over an algebraically closed field is a structure of finite Morley rank and in this context the Morley rank coincides with the Zariski dimension.

The main problem in the subject is the Cherlin–Zil'ber Conjecture: *A simple group of finite Morley rank is an algebraic group over an algebraically closed field.*

In this paper, I will first define the concept of Morley rank and then review some results and open problems on permutation groups of finite Morley rank. All the results and definitions can be found in [**BN**].

2 Morley Rank

Let T be a complete theory and let $\mathcal{M}$ be a saturated model of T. For a definable set X in $\mathcal{M}$ (may be with parameters from M) we attempt to define the Morley rank of X, an ordinal denoted by $\mathrm{rk}(X)$, as follows: $\mathrm{rk}(X) \geq 0$ if $X \neq \emptyset$. $\mathrm{rk}(X) \geq n+1$ if X has infinitely many disjoint definable subsets Y_i such that $\mathrm{rk}(Y_i) \geq n$. The theory T is ω-stable if and only if every definable subset of M has an ordinal Morley rank. (Having a Morley rank is independent of the choice of the saturated model $\mathcal{M}$ of T). The theory T has **finite Morley rank** if every definable subset of M has finite Morley rank. A structure $\mathcal{M}$ is said to have finite Morley rank if $Th(\mathcal{M})$ has finite Morley rank.

In an ω-stable group G, the definable subgroups satisfy the descending chain condition, so that the intersection of all the definable subgroups of finite index of G is a definable subgroup of finite index of G. This subgroup is called the **connected component** of G and is denoted by G°. We say that G is connected if $G = G^{\circ}$.

3 Permutation Groups

A group G is said to **act** on a set X if there is a group homomorphism ϕ from G into $\mathrm{Sym}(X)$. For $g \in G$ and $x \in X$, we write gx for the image of x under $\phi(g)$. With this notation, we have the following two equalities: $1x = x$ and $g(hx) = (gh)x$ for all $g, h \in G$ and $x \in X$. The pair (G, X) – without mentioning the homomorphism ϕ, but strictly speaking we should – is called a **permutation group**. For $x_1, \ldots, x_n \in X$, we let

$$G_{x_1,\ldots,x_n} = \{g \in G : gx_i = x_i \text{ for } i = 1, \ldots, n\}.$$

This is clearly a subgroup of G.

If G is any group and H is any subgroup of G, denote by G/H the left coset space of H in G. Then the group G acts on the set G/H by $g(xH) = gxH$. This permutation group has the property that for all $x, y \in G/H$, there is a $g \in G$ such that $gx = y$. A permutation group (G, X) that satisfies this property is called a **transitive** permutation group. Any transitive permutation group can be taken to be of the form $(G, G/G_x)$ for any fixed $x \in G$.

More generally, the permutation group (G, X) is said to be n-**transitive** if for any distinct $x_1, \ldots, x_n \in X$ and any distinct $y_1, \ldots, y_n \in X$, there is a $g \in G$ such that $gx_i = x_i$ for $i = 1, \ldots, n$. If further the element $g \in G$ is unique, then (G, X) is said to be **sharply n-transitive**. One uses the term **regular** rather than "sharply 1-transitive".

A 2-transitive group of finite Morley rank where G_x is definable (this means exactly that the action of G on X may be taken to be interpretable in G) has involutions [**BN**].

4 Frobenius Groups

A **Frobenius group** is a group B with a proper subgroup T, called **Frobenius complement**, such that for $b \in B$, $T^b \cap T \neq 1$ implies $b \in T$. Equivalently a Frobenius group is a group G acting transitively on a set X of cardinality > 1 in such a way that $G_x \neq 1$ and $G_{x,y} = 1$ for all distinct $x, y \in G$. Note that a sharply 2-transitive group is a Frobenius group.

The phrase "$T < B$ is a Frobenius group" will mean that B is a Frobenius group and T is a Frobenius complement of B. Note that if $T < B$ is a Frobenius group, then so is $T^b < B$ for any $b \in B$. Whenever $B = U \rtimes T$ for some normal subgroup U, B is said to be a **split** Frobenius group. The subgroup U is called the **Frobenius kernel** of B.

Finite Frobenius groups are of great importance in the classification of finite simple groups and for this reason they have been subject to intensive study. Let $T \leq B$ stand for a finite Frobenius group for a while. It is well-known that B splits, say as $U \rtimes T$. The proof of the splitting is quite easy when T has an involution [**P**, Proposition 8.3]. But the general proof, due to Frobenius, is much more involved and uses character theory (see e.g. [**B**, page 172], [**P**, Theorem 17.1] or [**S**, Chapter 6, Theorem 2.2]). It was well-known that if U is solvable then it is nilpotent (see [**H**]). For a long time the structure of U was unknown and was conjectured to be nilpotent by Frobenius. In 1959, Thompson proved Frobenius' Conjecture [**T**]. The structure of T is investigated by Zassenhaus (see [**P**, §18]).

We conjecture the following:

Conjecture 1 *A Frobenius group $T < B$ of finite Morley rank splits as $U \rtimes T$ for some nilpotent group U.*

It is known that if $T < B$ is a Frobenius group of finite Morley rank, then the following hold:

1) The Frobenius complement T is necessarily definable in the pure group structure of B [**N3**].

2) If $B = U \rtimes T$ is split, then U is also definable in the pure group structure of B. Furthermore if U is solvable then U is nilpotent [**N3**].

3) If B is solvable, then $B = U \rtimes T$ is split. If, further, B is connected, then $B = B' \rtimes T$ [**N3**].

4) The subgroup T has finitely many involutions and if T is connected then it has at most one involution [**DN1**].

5) If T is finite, then B splits [**EN**].

6) A minimal counterexample to the splitting of Frobenius groups of finite Morley rank is a counterexample to the Cherlin-Zil'ber conjecture [**EN**].

7) If $B = U \rtimes T$ with nonnilpotent U and if B has smallest Morley rank with this property, then U is simple and a counterexample to the Cherlin-Zil'ber Conjecture [**EN**].

5 Sharply 2-Transitive Groups

Let K be a field and consider the affine group

$$G = \left\{ \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} : t \in K^*, u \in K \right\} \simeq K^+ \rtimes K^*.$$

The group G acts on the set

$$X = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in K \right\}$$

sharply 2-transitively. We will call a permutation group of this kind a **standard** sharply 2-transitive group.

We make the following conjecture about sharply 2-transitive group of finite Morley rank:

Conjecture 2 *An infinite sharply 2-transitive group G of finite Morley rank is a standard sharply 2-transitive group.*

A sharply 2-transitive group where a one point stabilizer has a normal complement is called *split.* It is well-known such a complement is necessarily abelian. It is not known whether or not there are nonsplit sharply 2-transitive groups. The following problem is one of the main obstacles to Conjecture 2:

Conjecture 3 *A sharply 2-transitive group G of finite Morley rank splits, i.e. $G = N \rtimes H$ for some normal subgroup N where H is a one-point stabilizer.*

The reader should note that Conjecture 3 is a special case of Conjecture 1. Even the split sharply 2-transitive groups of finite Morley rank do not allow an easy treatment and it is worth while to restate Conjecture 2 for this particular case:

Conjecture 4 *An infinite split sharply 2-transitive group of finite Morley rank is isomorphic to a standard sharply 2-transitive.*

Till the end of this section we assume that G is an infinite sharply 2-transitive group of finite Morley rank. We let H to be a one-point stabilizer. Since $H < G$ is a Frobenius group, H is definable. and if G splits as $U \rtimes H$, then we know that U is also definable. Therefore, replacing X by G/H, we may assume that the action of G on X is interpretable. The subgroup H is known to have at most one involution and if it does have one, then for two distinct involutions i, j of G, the order of the element ij is independent of the choice of i and j and either is a prime p or is infinite. When H has no involution, one says that the characteristic of G is 2. If H has an involution, then characteristic of G is defined to be the order of ij for two distinct involutions i, j of G.

We know the following about G:

1. G and H are connected [**N1**].

2. If H is abelian, then G is standard [**K**]. (This is a general result that does not need any model theoretic assumption on G).

3. If G is split and H has an infinite normal solvable subgroup, then G is standard [**BN**].

4. Assume $G = U \rtimes H$ is split and $C_{\mathrm{End}(U)}(H)$ is infinite. Then G is standard [**CGNV**].

5. If $G = U \rtimes H$ is split and has characteristic ∞, then G is standard [**CGNV**].

6. If $G = U \rtimes H$ is split and $Z(H)$ is infinite then G is standard [**CGNV**].

7. If G is split, then the connected solvable subgroups of H are abelian [**BN**].

8. Assume the characteristic of G is not 2. Let $i \in H$ be the unique involution and let w be another involution. Assume that for some nilpotent subgroup H_1, we have $N_H C_G(wi) \leq H_1 \lhd H$. Then G is standard [**N2, DN1**].

9. In particular, if $\mathrm{char}(G) \neq 2$ and H is nilpotent, then G is standard [**N2**, **DN1**].

6 Zassenhaus Groups

A doubly transitive permutation group is called a **Zassenhaus group** if the stabilizer of two distinct points is nontrivial and if the stabilizer of any three distinct points is trivial. In particular, a sharply 3-transitive group is a Zassenhaus group. Let G be a Zassenhaus group acting on a set X of cardinality ≥ 3. Set $B = G_x$, the stabilizer of the point $x \in X$, and $T = G_{x,y}$ the pointwise stabilizer of the two distinct points $x, y \in X$. Since B acts transitively on the set $X \setminus \{x\}$, and since only the identity element of B fixes two distinct points, $T < B$ is a Frobenius group. When B splits as $U \rtimes T$, we say that G is a **split** Zassenhaus group.

Conjecture 5 *An infinite Zassenhaus group of finite Morley rank is isomorphic to* $\mathrm{PSL}_2(K)$ *for some algebraically closed field* K*. Furthermore the action of* G *is isomorphic to the action of* $\mathrm{PSL}_2(K)$ *acting naturally on the projective line.*

We know the following about infinite Zassenhaus groups of finite Morley rank.

1. G is not solvable; B is definable; if$G$ is split then G° is also a split Zassenhaus group and in this case U is connected [**DN2**].

2. Conjecture 5 holds if G is sharply 3-transitive group [**N1**].

3. If G is split and if T contains an involution then Conjecture 5 holds [**DN2**].

4. If G is split and if U contains a central involution then Conjecture 5 holds [**DBN**].

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