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Deterministic Brownian Motions

大阪市大・理 釜江哲朗 (Teturo Kamae)

1 Introduction

Let Ω be a complete separable metrizable space. Let G be a nontrivial, closed, multiplicative subgroup of $\mathbf{R}_+$, the set of positive real numbers. That is, either $G = \mathbf{R}_+$ or there exists $\lambda > 1$ such that $G = \{\lambda^n; n \in \mathbf{Z}\}$. Assume that $(\mathbf{R}, G)$ **acts** on Ω, that is,

(1) For any $\omega \in \Omega$, $t \in \mathbf{R}$ and $\lambda \in G$, $\omega + t$ and $\lambda\omega$ are defined and belong to Ω so that the mappings $(\omega, t) \mapsto \omega + t$ and $(\omega, \lambda) \mapsto \lambda\omega$ are continuous.

(2) $\cdot + 0 = 1\cdot = \mathrm{id}_\Omega$, and

(3) for any $\omega \in \Omega$, $s, t \in \mathbf{R}$ and $\lambda \in G$, it holds that

$$(\omega + t) + s = \omega + (t + s), \quad \lambda(\eta\omega) = (\lambda\eta)\omega, \quad \lambda(\omega + t) = \lambda\omega + \lambda t.$$

Let $(\mathbf{R}, G)$ act on Ω. A continuous function $F : \Omega \times \mathbf{R} \to \mathbf{R}$ is called a **cocycle** on Ω if

$$F(\omega, t + s) = F(\omega, t) + F(\omega + t, s)$$

holds for any $\omega \in \Omega$ and $s, t \in \mathbf{R}$. A cocycle F on Ω is called to be α-G-**homogeneous** if

$$F(\lambda\omega, \lambda t) = \lambda^\alpha F(\omega, t)$$

for any $\omega \in \Omega$, $\lambda \in G$ and $t \in \mathbf{R}$, where α is a given real number with $0 < \alpha < 1$. It is simply called to be α-**homogeneous** if $G = \mathbf{R}_+$. We remark that the notion of homogeneuos cocycle is equivalent to the notion of cocycle with the scaling property in [5].

Example 1 *Let* $\Omega = \mathbf{R}$ *and* $(\mathbf{R}, \mathbf{R}_+)$ *act on* $\mathbf{R}$ *in the usual sense. Then, a cocycle* F *on* Ω *is a* **coboundary**, *that is, there exists a continuous function* $\varphi : \Omega \to \mathbf{R}$ *such that*

$$F(\omega, t) = \varphi(\omega + t) - \varphi(\omega)$$

for any $\omega \in \Omega$ *and* $t \in \mathbf{R}$. *Moreover, if* F *is* α*-homogeneous, then the above* φ *satisfies that*

$$\varphi(\omega) = \begin{cases} A|\omega|^\alpha + C & (\omega \geq 0) \\ B|\omega|^\alpha + C & (\omega < 0). \end{cases}$$

Example 2 *Let* Ω *be the space of all continuous function* $\omega : \mathbf{R} \to \mathbf{R}$ *with* $\omega(0) = 0$ *with the compact open topology. For any* $\omega \in \Omega$, $t \in \mathbf{R}$ *and* $\lambda \in \mathbf{R}_+$, *we define* $\omega + t \in \Omega$ *and* $\lambda\omega \in \Omega$ *by*

$$(\omega + t)(s) = \omega(t + s) - \omega(t) \quad \text{and} \quad (\lambda\omega)(s) = \lambda^\alpha \omega(\lambda^{-1} s)$$

for any $s \in \mathbf{R}$. *Then,* $(\mathbf{R}, \mathbf{R}_+)$ *acts on* Ω. *Let*

$$F(\omega, t) = \omega(t)$$

for any $\omega \in \Omega$ *and* $t \in \mathbf{R}$. *Then,* F *is a* α*-homogeneous cocycle. Let* μ *be an* $(\mathbf{R}, \mathbf{R}_+)$**-invariant** *probability Borel measure on* Ω, *that is,*

$$d\mu(\omega + t) = d\mu(\omega) \quad \text{and} \quad d\mu(\lambda\omega) = d\mu(\omega)$$

for any $t \in \mathbf{R}$ *and* $\lambda \in \mathbf{R}_+$. *Then,* $F(\omega, t)$ *is considered as a stochastic process on the probability space* (Ω, μ) *with the time parameter* $t \in \mathbf{R}$. *This process has stationary increments and is* α*-selfsimilar. The Wienner process is one of them for* $\alpha = 1/2$.

We are interessted in Ω on which $(\mathbf{R}, G)$ acts and which is **R-minimal**. That is,

(4) Ω is compact, and it holds that

$$\overline{\{\omega + t; t \in \mathbf{R}\}} = \Omega$$

for any $\omega \in \Omega$

We call Ω to be **R-strictly ergodic** if in addition,

(5) there exists a unique **R-invariant** probability Borel measure μ on Ω, that is,

$$d\mu(\omega + t) = d\mu(\omega)$$

for any $t \in \mathbf{R}$.

In this case, μ is also G-**invariant**, that is,

(6)

$$d\mu(\lambda\omega) = d\mu(\omega)$$

for any $\lambda \in G$.

We remark that a cocycle on **R**-minimal Ω is a minimal cocycle in the sense of [5] and vice versa.

Theorem 1 ([5]) *Let $(\mathbf{R}, G)$ act on Ω. Assume that Ω is $\mathbf{R}$-minimal. Then, for a nonzero α-G-homogeneous cocycle F, we have the following results.*

(i) There exists a constant C such that

$$|F(\omega, t) - F(\omega, s)| \leq C|t - s|^{\alpha}$$

for any $\omega \in \Omega$ and $s, t \in \mathbf{R}$. That is, the functions $F(\omega, t)$ on t for $\omega \in \Omega$ are uniformly α-H$\acute{}$older continuous.

(ii) For any $\omega \in \Omega$ and $t \in \mathbf{R}$,

$$\limsup_{s \downarrow 0} \frac{1}{s^{\alpha}} |F(\omega, t+s) - F(\omega, t)| > 0$$

holds. That is, for any $\omega \in \Omega$ the function $F(\omega, \cdot)$ is nowhere locally β-Hölder continuous for any $\beta > \alpha$. In special, $F(\omega, \cdot)$ is nowhere differentiable.

There are two important aspects of 'fractal' functions; almost periodicity and self-similarity. Our notion of homogeneous cocycles on minimal Ω is a formulation of 'fractal' functions from these points of view. We are also interested in self-similar processes with strictly ergodic, stationary increments which come from homogeneous cocycles on strictly ergodic Ω. Rudin-Shapiro process defined in [2] is one of them for $\alpha = \frac{1}{2}$ and $G = \{2^n; n \in \mathbf{Z}\}$ if it is restricted on an ergodic component.

We will construct such Ω and homogeneous cocycles on it. All results in this article will be published in [6].

2 Colored tiling

Let $\mathcal{R}$ be the set of nonempty rectangles $(a, b] \times [c, d)$ in $\mathbf{R}^2$ such that

(7)

$$e^{-b} = d - c.$$

Let Σ be a finite set with at least 2 elements, which will be called the set of **colors**.

A mapping $\omega : dom(\omega) \to \Sigma$ is called a **colored tiling** if $dom(\omega) \subset \mathcal{R}$ and $\bigcup_{S \in dom(\omega)} S$ gives a partition of $\mathbf{R}^2$. For $S \in dom(\omega)$, we call

$\omega(S)$ the **color** on the **tile** S. In addition, if $S = (a, b] \times [c, d)$, then the point $(b, c) \in \mathbf{R}^2$ is called the **corner** of S. For $x \in \mathbf{R}^2$, we define $\tilde{\omega}(x) := \omega(S)$ for the tile S with $x \in S \in dom(\omega)$. Let $\Omega(\Sigma)$ be the set of all colored tilings with the colors Σ. It is considered as a topological space in the sense that $\omega_n \in \Omega(\Sigma)$ converges to $\omega \in \Omega(\Sigma)$ as $n \to \infty$ if for every bounded region of $\mathbf{R}^2$, the **picture** drawn by ω_n converges to that of ω on it. This implies that for any bounded set K in $\mathbf{R}^2$, $\lim_{n\to\infty} \rho_K(\omega|\omega_n) = 0$, where

(8)

$$\rho_K(\omega|\omega_n) := \sup_{x\in K} \inf_{\substack{y\in\mathbf{R}^2 \\ \tilde{\omega}(x)=\tilde{\omega}_n(y)}} \parallel x - y \parallel = 0.$$

For $\omega \in \Omega(\Sigma)$, $t \in \mathbf{R}$ and $\lambda \in \mathbf{R}_+$, we define $\omega + t \in \Omega(\Sigma)$ and $\lambda\omega \in \Omega(\Sigma)$ as follows:

For $S := (a, b] \times [c, d)$ and $S' := (a, b] \times [c - t, d - t)$, $S' \in dom(\omega + t)$ if and only if $S \in dom(\omega)$, and in this case $(\omega + t)(S') = (\omega)(S)$. Also, for $S := (a, b] \times [c, d)$ and $S' := (a - \log\lambda, b - \log\lambda] \times [\lambda c, \lambda d)$, $S' \in dom(\lambda\omega)$ if and only if $S \in dom(\omega)$, and in this case $(\lambda\omega)(S') = \omega(S)$.

Then, it is easy to see that $(\mathbf{R}, \mathbf{R}_+)$ acts on $\Omega(\Sigma)$. We are interested in compact metrizable subsets of $\Omega(\Sigma)$ which are invariant under the action of $(\mathbf{R}, G)$. for some G.

Example 3 *Let* $\Sigma = \{0, 1\}$ *and*

$$\begin{aligned} B_2 := \{ \quad & \omega \in \Omega(\Sigma);\ \textit{for any } S := (a, b] \times [c, d) \in dom(\omega) \\ & \textit{it holds that } b = a + \log 2 \in (\log 2)\mathbf{Z} \textit{ and} \\ & S_i := (b, b + log2] \times [c + \tfrac{i}{2}(d - c), c + \tfrac{i+1}{2}(d - c)) \\ & \in dom(\omega) \textit{ with } \omega(S_i) = i \textit{ for } i = 0, 1\}. \end{aligned}$$

Then, $(\mathbf{R}, \{2^n; n \in \mathbf{Z}\})$ *acts an* B_2*. We can consider* B_2 *as the set of 2-sided, 2-adic expansions in the sense that* $\omega \in B_2$ *is identified with*

$$\begin{aligned} & \textstyle\sum_{i \in \mathbf{Z}} \tilde{\omega}(i \log 2, 0) 2^{-i} \\ = & \textstyle\sum_{i \le 0} \tilde{\omega}(i \log 2, 0) 2^{-i} \oplus \sum_{i > 0} \tilde{\omega}(i \log 2, 0) 2^{-i} \end{aligned}$$

where the convergence is in $\mathbf{Z}_2 \oplus [0,1]$ *with the identification of* $x \oplus 1$ *with* $(x+1) \oplus 0$ *for any* $x \in \mathbf{Z}_2$*.*

A **substitution** φ on a set Σ is a mapping $\Sigma \to \Sigma^+$, where $\Sigma^+ = \bigcup_{n=1}^{\infty} \Sigma^n$. For $\xi \in \Sigma^+$, we denote $L(\xi) := n$ if $\xi \in \Sigma^n$ and $\xi = \xi_0 \xi_1 \cdots \xi_{n-1}$. We can extend φ to be a homomorphism $\Sigma^+ \to \Sigma^+$ as follows:

$$\varphi(\xi) := \varphi(\xi_0)\varphi(\xi_1) \cdots \varphi(\xi_{n-1})$$

for $\xi \in \Sigma^n$. We can define $\varphi^2, \varphi^3, \cdots$ as the compositions of $\varphi : \Sigma^+ \to \Sigma^+$.

A **weighted substitution** (φ, η) on Σ is a mapping $\Sigma \to \Sigma^+ \times (0,1)^+$ such that $L(\varphi(\sigma)) = L(\eta(\sigma))$ and $\sum_{i < L(\eta(\sigma))} \eta(\sigma)_i = 1$ for any $\sigma \in \Sigma$. Note that φ is a substitution on Σ. We call η the **weight** on φ. We define $\eta^n : \Sigma \to (0,1)^+ \quad (n = 2, 3, ...)$ inductively by

$$\eta^n(\sigma)_k = \eta(\sigma)_i \eta^{n-1}(\varphi(\sigma)_i)_j$$

for any $\sigma \in \Sigma$ and i, j, k with

$$0 \le i < L(\varphi(\sigma)), 0 \le j < L(\varphi^{n-1}(\varphi(\sigma)_i)), k = \sum_{h<i} L(\varphi^{n-1}(\varphi(\sigma)_h)) + j$$

In this sense, (φ^n, η^n) is also a weighted substitution for $n = 2, 3, \cdots$.

A substitution φ on Σ is called to be **mixing** if there exists a positive integer n such that for any $\sigma, \sigma' \in \Sigma$ there exists i with $0 \le i < L(\varphi^n(\sigma))$ and $\varphi^n(\sigma)_i = \sigma'$.

For a weighted substitution (φ, η) on Σ, we always assume that

(9) the substitution φ is mixing.

We define the **base set** $B(\varphi, \eta)$ as the closed, multiplicative subgroup of $\mathbf{R}_+$ generated by the set

$$\{ \ \eta^n(\sigma)_i; \quad \sigma \in \Sigma, \ n = 0, 1, \cdots, \text{ and } 0 \leq i < L(\varphi^n(\sigma)) \text{ such that } \varphi^n(\sigma)_i = \sigma\}.$$

It is called to be **continuous** if $B(\varphi, \eta) = \mathbf{R}_+$, otherwise, **discrete**.

Let (φ, η) be a weighted substitution on a finite set Σ with $\sharp\Sigma \geq 2$ with $G := B(\varphi, \eta)$. Then, there exists a function $g : \Sigma \to \mathbf{R}_+$ such that

(10)
$$g(\varphi(\sigma)_i)G = g(\sigma)\eta(\sigma)_i G$$

for any $\sigma \in \Sigma$ and $0 \leq i < L(\varphi(\sigma))$. Note that if $G = \mathbf{R}_+$, then we can take $g \equiv 1$. In the discrete case, we can define g by $g(\sigma) := \eta^n(\sigma_0)_i$ for some n and i such that $\varphi^n(\sigma_0)_i = \sigma$, where σ_0 is a fixed element in Σ. For another g' satisfying (10), there exists a constant $C > 0$ such that $g'(\sigma)G = Cg(\sigma)G$ for any $\sigma \in \Sigma$.

Let $\Omega(\varphi, \eta, g)'$ be the set of all elements ω in $\Omega(\Sigma)$ such that

(i) if $(a, b] \times [c, d) \in dom(\omega)$, then $e^{-b} = d - c \in g(\omega((a, b] \times [c, d)))G$, and

(ii) if $(a, b] \times [c, d) \in dom(\omega)$ and $\omega((a, b] \times [c, d)) = \sigma$, then for $i = 0, 1, \cdots, L(\varphi(\sigma)) - 1$, $S_i \in dom(\omega)$ and $\omega(S_i) = \varphi(\sigma)_i$, where

$$S_i := (b, b - \log \eta(\sigma)_i] \times [c + (d - c)\sum_{j=0}^{i-1} \eta(\sigma)_j, c + (d - c)\sum_{j=0}^{i} \eta(\sigma)_j).$$

We call the tile S_i as above a **child** of the tile S, and S the **mother** of S_i. Let $\Omega(\varphi, \eta, g)''$ be the set of all $\omega \in \Omega(\varphi, \eta, g)'$ such that for any N, there exists $(a, b] \times [c, d) \in dom(\omega)$ with $(c, d] \supset [-N, N]$. Finally, we define $\Omega(\varphi, \eta, g)$ to be the closure of $\Omega(\varphi, \eta, g)''$. Then, $(\mathbf{R}, G)$ acts on $\Omega(\varphi, \eta, g)$. We denote $\Omega(\varphi, \eta, 1)$ simply by $\Omega(\varphi, \eta)$ in the continuous case.

Theorem 2 *For any weighted substitution* (φ, η) *satisfying (9) and* g *with (10),* $\Omega(\varphi, \eta, g)$ *is* $\mathbf{R}$*-strictly ergodic. Moreover, the toplogical entropy of the* $\mathbf{R}$*-action on* $\Omega(\varphi, \eta, g)$ *is 0.*

We prove only that there exists a unique $\mathbf{R}$-invariant probability Borel measure on $\Omega = \Omega(\varphi, \eta, g)$. Since Ω is a nonempty compact metrizable space and the $\mathbf{R}$-action is continuous, there exists an $\mathbf{R}$-invariant probability Borel measure μ on it. We prove that μ is the unique measure as this.

Let $\sigma, \sigma' \in \Sigma$. We define a random variable $X_{\sigma\sigma'}(y)$ on the probability space $y \in [0, 1)$ with the Lebesgue measure:

$$X_{\sigma\sigma'}(y) = -\log \eta^n(\sigma)_i,$$

where n is the minimum positive integer, if it exists, such that there exists i with $0 \leq i < L(\varphi^n(\sigma))$ satisfying that $\varphi^n(\sigma)_i = \sigma'$ and

$$\sum_{0 \leq j < i} \eta^n(\sigma)_j \leq y < \sum_{0 \leq j \leq i} \eta^n(\sigma)_j.$$

Then, $X_{\sigma\sigma'}$ exists with probability 1. Let $F_{\sigma\sigma'}$ be the distribution of the random variable $X_{\sigma\sigma'}$.

Let $S := (a, b] \times [c, d)$ be a tile in $\omega \in \Omega(\varphi, \eta, g)$ with $\omega(S) = \sigma'$. For $u > b$, let E be the number of the tiles in ω with color σ having the corner belonging to $[u, u + du) \times [c, d)$, where du stands for an arbitrary small positive number and we neglect all the terms with $o(du)$. Then we have

(11)

$$\frac{Ee^{-u}}{d-c} = \sum_{n=0}^{\infty} \int_{u-b\leq x<u-b+du} F_{\sigma'\sigma} * F_{\sigma\sigma}{}^{n*}(dx),$$

where $''*''$ implies the convolution of the distributions. It is well known by the renewal theory [1] that the above value converges to

$$(\int xF_{\sigma\sigma}(dx))^{-1}du$$

as $u \to \infty$ if $G = \mathbf{R}_+$ and to

$$(\int xF_{\sigma\sigma}(dx))^{-1}\log\lambda$$

as $u \to \infty$ satisfying that $e^{-u} \in g(\sigma)G$ if $G = \{\lambda^n; n \in \mathbf{Z}\}$ with $\lambda > 1$.

For $\sigma \in \Sigma$ and a Borel subset U of $\mathbf{R}^2$, let $\Pi(\sigma, U)$ be the subset of $\omega \in \Omega(\varphi, \eta, g)$ consisting of ω which has a tile S such that $\omega(S) = \sigma$ and S has the corner belonging to U. Let $dudv := [u, u + du) \times [v, v + dv)$ and $\sigma \in \Sigma$ satisfy that $e^{-u} \in g(\sigma)G$. Since μ is $\mathbf{R}$-invariant, $\mu(\Pi(\sigma, dudv)) = \mu(\Pi(\sigma, dudv + (0, y)))$ for any $y \in \mathbf{R}$. By integrating this equality with dy from 0 to N and applying Fubini's theorem we have

(12)

$$\mu(\Pi(\sigma, dudv)) = \frac{dv}{N}\int E(\omega)d\mu(\omega),$$

where we denote by $E(\omega)$ the number of the tiles in ω with color σ having the corner belonging to $[u, u+du) \times [0, N)$.

For any $\epsilon > 0$, take $L > 0$ such that the the value in (11) for any $\sigma' \in \Sigma$ with $u - b \geq L$ is close to A within ϵ, where

(13)

$$A = \begin{cases} (\int xF_{\sigma\sigma}(dx))^{-1}du & \text{if } G = \mathbf{R}_+ \\ (\int xF_{\sigma\sigma}(dx))^{-1}\log\lambda & \text{if } G = \{\lambda^n; n \in \mathbf{Z}\} \quad (\lambda > 1). \end{cases}$$

For any $\omega \in \Omega(\varphi, \eta, g)$ and $y \in \mathbf{R}$, let $S(y)$ be the tile in ω such that $S(y)$ intersects with $\mathbf{R} \times \{y\}$ and is contained in $(-\infty, u - L] \times \mathbf{R}$ but none of its children satisfies these conditions. Then, the vertical size of $S(y)$ is at most e^{L-u+u_0}, where

$$u_0 := \max_{\substack{\sigma\in\Sigma \\ 0\leq i<L(\varphi(\sigma))}} -\log\eta(\sigma)_i.$$

Let $S_1, \cdots, S_k$ be the set of all distinct $S(y)$'s for $y \in [0, N)$ such that the orthogonal projection to the vertical axis of $S(y)$ is contained in $[0, N)$. Then, the projections of S_i's are disjoint and we take N large enough so that their union covers large enough part of the inteval $[0, N)$. Let $\tilde{S}_i$ be the projection of S_i and $E_i(\omega)$ be the number of the tiles in ω with color σ having the corner belonging to $[u, u+du) \times \tilde{S}_i$. Then, by the assumption on L, (11) and (13), we have $|E_i(\omega)e^{-u} - |\tilde{S}_i|A| < |\tilde{S}_i|\epsilon$, where $|\tilde{S}_i|$ is the size of $\tilde{S}_i$. By adding the inequalities, we have $|E(\omega)e^{-u} - NA| < 2N\epsilon$. Thus, by integrating it with $d\mu(\omega)$, we have

(14)

$$|\int E(\omega)d\mu(\omega)e^{-u} - NA| < 2N\epsilon.$$

Conbining (12) and (14), we have

$$|\mu(\Pi(\sigma, dudv))e^{-u} - Adv| < 2\epsilon dv.$$

Since $\epsilon > 0$ was arbitrary, we have

(15)

$$\mu(\Pi(\sigma, dudv) = \begin{cases} (\int xF_{\sigma\sigma}(dx))^{-1}e^{u}dudv & \text{if } G = \mathbf{R}_+ \\ 1_{e^{-u}\in g(\sigma)G}(\int xF_{\sigma\sigma}(dx))^{-1}e^{u}\log\lambda dv \\ \quad \text{if } G = \{\lambda^n; n \in \mathbf{Z}\} \quad (\lambda > 1). \end{cases}$$

Thus, μ is determined and is unique, which completes the proof.

Example 4 (Fibonacci expansion) *Let $\Sigma = \{0, 1\}$. Let (φ, η) be the weighted substitution on Σ such that*

$$\begin{array}{l} 0 \to (0, \lambda^{-1})(1, \lambda^{-2}) \\ 1 \to (0, \lambda^{-1})(1, \lambda^{-2}), \end{array}$$

where $\lambda = \frac{1+\sqrt{5}}{2}$ and we arranged $(\varphi(\sigma)_i, \eta(\sigma)_i))$ in the order of i after "$\sigma \to$". Then, $B(\varphi, \eta) = \{\lambda^n; n \in \mathbf{Z}\}$. For $g \equiv 1$, (10) is satisfied. Let $\Omega := \Omega(\varphi, \eta, 1)$. Then, by Theorem 2, Ω is $\mathbf{R}$-strictly ergodic. Let μ be the unique $\mathbf{R}$-invariant probability Borel measure on Ω. By (15), μ satisfies that

$$\begin{array}{l} \mu(\Pi(0, dudv)) = A^{-1}e^{u}\log\lambda dv \\ \mu(\Pi(1, dudv)) = B^{-1}e^{u}\log\lambda dv \end{array}$$

for any $u, v \in \mathbf{R}$ with $e^{-u} \in G$, where

$$\begin{aligned} A &= \lambda^{-1}\log\lambda + \lambda^{-3}3\log\lambda + \cdots \\ &= (2\lambda - 1)log\lambda, \end{aligned}$$

$$\begin{aligned} B &= \lambda^{-2}2\log\lambda + \lambda^{-4}4\log\lambda + \cdots \\ &= (\lambda + 2)log\lambda. \end{aligned}$$

Thus, we have

$$\mu(\Pi(0, dudv)) = \tfrac{2\lambda-1}{5} e^u dv$$
$$\mu(\Pi(1, dudv)) = \tfrac{-\lambda+3}{5} e^u dv$$

for any $u, v \in \mathbf{R}$ *with* $e^{-u} \in G$.

Example 5 *Let* $\beta = \frac{1}{4} + \frac{\sqrt{3}}{8}$. *Let* (φ, η) *be the weighted substitution on* $\{0, 1\}$ *such that*

$$\begin{array}{l} 0 \to (0, \beta)(1, \frac{1}{2} - \beta)(1, \frac{1}{2} - \beta)(0, \beta) \\ 1 \to (1, \beta)(0, \frac{1}{2} - \beta)(0, \frac{1}{2} - \beta)(1, \beta) \end{array}.$$

Note that $\frac{\log(\frac{1}{2}-\beta)}{\log \beta}$ *is irrational and* $B(\varphi, \eta) = \mathbf{R}_+$. *Let* $\Omega = \Omega(\varphi, \eta)$. *Then, by Theorem 2,* Ω *is* $\mathbf{R}$*-strictly ergodic. Let* μ *be the unique* $\mathbf{R}$*-invariant probability Borel measure on* Ω. *Then,* μ *is also* $\mathbf{R}_+$*-invariant. By (15),* μ *satisfies that*

$$\mu(\Pi(0, dudv)) = \mu(\Pi(1, dudv)) = A^{-1} e^u dudv$$

for any $u, v \in \mathbf{R}$ *with*

$$\begin{aligned} A &= 2\beta(-log\beta) + \textstyle\sum_{n=0}^{\infty} (2\beta)^n (1 - 2\beta)^2 (-n \log \beta - 2 \log(\frac{1}{2} - \beta)) \\ &= -4\beta log\beta - 2(1 - 2\beta) log(\tfrac{1}{2} - \beta)). \end{aligned}$$

This example will be discussed later.

3 Homogeneous cocycle

Let (φ, η) be a weighted substitution on a finite set Σ with $\sharp\Sigma \geq 2$ satisfying (9). Let $G = B(\varphi, \eta)$ and g satisfy (10). For $0 < \alpha < 1$, let $M_\alpha = M_\alpha(\varphi, \eta)$ be the matrix $(m_{\sigma\sigma'}{}^{(\alpha)})_{\sigma,\sigma' \in \Sigma}$ such that

(16)

$$m_{\sigma\sigma'}{}^{(\alpha)} = \sum_{\substack{0 \leq i < L(\varphi(\sigma)) \\ \varphi(\sigma)_i = \sigma'}} \eta(\sigma)_i{}^\alpha$$

We assume that

(17) 1 is an eigen value of M_α with a nonzero eigen column vector $\xi = (\xi_\sigma)_{\sigma \in \Sigma}$.

Define $\tilde{\xi} : \Omega(\varphi, \eta, g) \times \mathbf{R}^2 \to \Sigma$ and $\tilde{S} : \Omega(\varphi, \eta, g) \times \mathbf{R}^2 \to \mathbf{R}$ by

$$\begin{array}{l} \tilde{\xi}(\omega, x, y) = \xi_{\tilde{\omega}(x,y)} \text{ and} \\ \tilde{S}(\omega, x, y) = |\tilde{S}| \text{ if } (x, y) \in S \in dom(\omega), \end{array}$$

where $|\tilde{S}|$ is the vertical size of S. We finally define $F : \Omega(\varphi, \eta, g) \times \mathbf{R} \to \mathbf{R}$ by

(18)

$$\begin{array}{l} F(\omega, t) = \lim_{x \to \infty} F(x, \omega, t), \text{ where} \\ F(x, \omega, t) = \int_0^t \tilde{\xi}(\omega, x, y) \tilde{S}(\omega, x, y)^{\alpha - 1} dy \end{array}$$

Theorem 3 *F is a nonzero α-G-homogeneous cocycle on $\Omega(\varphi, \eta, g)$.*

(We omit the proof.)

Corollary 1 *If G in Theorem 3 is continuous, then F defines a self-similar process with strictly ergodic, stationary increments having 0 entropy.*

Example 6 *Let us take $\Omega = \Omega(\varphi, \eta)$ in Example 5. Then, for the matrix $M_{\frac{1}{2}}$ in (16), we have*

$$M_{\frac{1}{2}} = \begin{pmatrix} \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \\ \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \end{pmatrix}.$$

Then $\xi = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigen vector of $M_{\frac{1}{2}}$ corresponding to the eigen value 1. Let F be the cocycle on Ω defined in (18) for this ξ. Then, F is a self-similar process with stationary increments of order $\frac{1}{2}$ which has 0 entropy.

4 Remarks

To represent a nonlinear f-expansion, we need a space of colored tilings with curved tiles S of the shape

$$S = \{(x, y); a(y) < x \leq b(y) \text{ and } c \leq y < d\},$$

where $c < d$ are real numbers and a, b are smooth functions on $[c, d)$ such that $a(y) < b(y)$ for any $y \in [c, d)$ and $\int_c^d e^{b(y)} dy = 1$. It is discussed in [4] in a somewhat different form.

The cocycle in Example 6 has the least possible complexity among the nonzero, α-homogeneous, minimal cocycles [5].

The transformation group $\{\lambda\cdot; \lambda \in G\}$ on the probability space (Ω, μ) with the unique $\mathbf{R}$-invariant probability measure μ can be proved to be ergodic. Therefore, by Theorem 1 and the ergodic theorem, for any α-G-homogeneous cocycle F on Ω,

$$C = \lim_{\epsilon \downarrow 0} \frac{1}{-\log \epsilon} \int_{\epsilon}^{1} \frac{|F(\omega, t+s) - F(\omega, t)|^{1/\alpha}}{s} \frac{ds}{s}$$

with probability 1, where

$$C = \int |F(\omega, 1)|^{1/\alpha} d\mu(\omega).$$

Using this, we can prove Itô's formula for the case $\alpha = 1/2$:

$$\begin{aligned} & f(F(\omega, B)) - f(F(\omega, A)) \\ = & \int_A^B f'(F(\omega, s)) dW(\omega, s) + \tfrac{C}{2} \int_A^B f''(F(\omega, s)) ds \end{aligned}$$

with probability 1, where the "martingale part" $W(\omega, s)$ is defined in a weak sense [3]. Therefore, 1/2-homogeneous cocycles on a $\Omega(\varphi, \eta)$ may well be called deterministic Brownian motions.

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