数理解析研究所講究録
935巻 1996年 9-20

Transformation $\tilde{G}$ for analytic functionals and its applications

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1. Introduction

In [1] and [8] Avanissian and Supper studied Abel interpolation problems of entire functions of exponential type by using analytic functionals with compact carrier. To derive the results they used the sequence $\{D^m f(m)\}_{m \in \mathbb{N}^n}$. In this report we derive analogous results for non - entire functions of exponential type defined in the direct product of half planes. We will make use of the sequence $\{D^{-m} f(-m)\}_{m \in \mathbb{N}^n}$ instead of $\{D^m f(m)\}_{m \in \mathbb{N}^n}$. In Section 2 we describe notations which we needed. The following Section is devoted to results obtained by Avanissian and Supper. The definitions and properties of transform $\tilde{G}$ of analytic functionals with unbounded carrier are given in Section 4. In the last Section we will present our main results.

2. Notations

In what follows we will use following notations. Following [1] and [8], we put

$$\tilde{U} = \{t = r\exp(ix) : 0 \le r < (\pi - |x|)/|\sin(x)|,\ |x| \le \pi\}.$$

$D_r = \{ t \in \mathbb{C} : |t| < r \}$.

$\varphi(t) = t^{-1}\exp(-t)$. φ is bi-holomorphic map between $\tilde{U} - \{0\}$ and $\mathbb{C} - [-e,0]$. ([5])

$\Lambda = \{ t \in \mathbb{C} : |\varphi(t)| > e \} \cup \{0\}$.

$\tilde{U} \supset D_1 \supset \Lambda$. (For the figure of $\tilde{U}$ and Λ, see [2]).

$\psi = \varphi^{-1}$. $\psi(w) = \sum_{n=1}^{\infty} \frac{(-n)^{n-1}}{n!} w^{-n}$ $(|w| > e)$.

K_i denotes i-th projection of $K \subset \mathbb{C}^n$.

For $S \subset \mathbb{C}$, $S^* = S-\{0\}$.

$d(S)$ denotes the transfinite diameter of S.

For $m = (m_1,\ldots,m_n) \in \mathbb{N}^n$, we put $\|m\| = m_1+\ldots+m_n$,

$D^m = \partial_{x_1}^{m_1} \ldots \partial_{x_n}^{m_n}$, $D^{-m} = D_1^{-m_1} \ldots D_n^{-m_n}$,

where $D_i^{-m_i} f(x) = \frac{1}{(m_i-1)!} \int_0^{\infty} f(x-a)a^{m_i-1}\,da$.

$\langle t,z\rangle = t_1z_1 + \ldots + t_nz_n$ for $t = (t_1,\ldots t_n)$ and $z = (z_1,\ldots z_n) \in \mathbb{C}^n$.

3. Results of Avanissian and Supper

In this section we will recall some results obtained by Avanissian and Supper. For the details, we refer the reader to [1] and [8].

Let T be an analytic functional carried by a compact set K in $\mathbb{C}^n$. $\tilde{T}(z) = \langle T_t, \exp(\langle t,z\rangle)\rangle$ is Fourier - Borel transform of T. Now we assume that $K_i \subset U$ for $i = 1,\ldots,n$. Transform $\tilde{G}_K(T)(w)$ is defined as follows :

$\tilde{G}_K(T)(w) = \langle T_t, \Pi_{i=1}^{n}(1-w_i t_i \exp(t_i))^{-1}\rangle$.

These transformations have following properties.

<u>Proposition 1</u>.

(1) $\tilde{G}_K(T)(w)$ is holomorphic in $\Pi_{i=1}^{n}(\mathbb{C}-\varphi(K_i^*))$.

(2) $\tilde{G}_K(T)(w) = \Sigma_{m \in \{\mathbb{N} \cup 0\}^n} D^m \tilde{T}(m) w^m$.

(3) (Inversion formula)

$$\tilde{T}(z) = (2\pi i)^{-n} \int_\Gamma \tilde{G}_K(T)(w) \exp(\Sigma_{i=1}^{n} z_i \psi(w_i)) \frac{dw_1}{w_1} \cdots \frac{dw_n}{w_n} ,$$

where $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$ and Γ_i $(i=1,\ldots n)$ is a countour surrounding $[-e,0]$.

(4) $K = \Pi_{i=1}^{n} K_i$. Suppose that $K_i \subset \tilde{U}$ ($i = 1.,,,n$) and $K \ni 0$. Then $\tilde{G}$ is isomorphism between $\mathcal{O}'(K)$ and $\mathcal{O}(\Pi_{i=1}^{n}(\mathbb{C}-\varphi(K_i^*)))$.

<u>Example 1</u>. $\delta(t)$ (Dirac's delta function)

$$\tilde{G}_{\{0\}}(\delta)(w) = 1$$

<u>Example 2.</u> $\delta'(t)$

$$\tilde{G}_{\{0\}}(\delta')(w) = -w$$

<u>Example 3</u>. ([8])(hypergeometric function) It is well known that Hypergeometric function $F(\alpha,\beta,\gamma,w)$ is holomorphic in $\mathbb{C}-[1,\infty]$. By

(4) in Prop.1 there exists an analytic functional (hyperfunction) T $\in \mathcal{O}'(K)$ such that $\tilde{G}_K(T)(w) = F(\alpha,\beta,\gamma,w)$, where K is $[0,\psi(1)]$. ($\psi(1) = 0.567...$).

Example 4. ([8]) (confluent hypergeometric function)
Confluent hypergeometric function $\Phi(\alpha,\gamma,w)$ is an entire function of w. Hence there exists an analytic functional (hyperfunction) $T_{\alpha,\gamma}$ supported by the origin such that $\tilde{G}_{\{0\}}(T_{\alpha,\gamma}) = \Phi(\alpha,\gamma,w)$.

Example 5. (Hypergeometric function with two variables)
$F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y)$ is holomorphic in $(\mathbb{C}-[1,\infty)) \times (\mathbb{C}-[1,\infty))$. Hence there exists an analytic functional (hyperfunction) T supported by K such that

$$\tilde{G}_K(T)(x,y) = F_3(\alpha,\alpha',\beta,\beta',\gamma,x,y),$$

where $K = [0, \psi(1)] \times [0, \psi(1)]$.

Example 6. (Confluent hypergeometric function with two variables) $\Phi_3(\beta,\gamma,x,y)$ and $\Phi_2(\beta,\beta',\gamma,x,y)$ are entire functions. So there exist analytic functionals (hyperfunctions) T_2 and T_3 supported by the origin such that

$$\tilde{G}_{\{0\}}(T_2)(x,y) = \Phi_2(\beta,\beta',\gamma,x,y)$$
$$\tilde{G}_{\{0\}}(T_3)(x,y) = \Phi_3(\beta,\gamma,x,y)$$

For the details of hypergeometric functions of two variables, we refer the reader to [4] and [6].

Theorem 1. ([1] and [8]) Let K be a compact set in $\mathbb{C}^n$. Suppose that entire function f(z) satisfies following conditions:

(1) For arbitrary $\varepsilon > 0$, there exists a constant $C_\varepsilon \geq 0$ such that $|f(z)| \leq C_\varepsilon \exp(H_K(z) + \varepsilon |z|)$ $\quad (z = (z_1, \ldots z_n) \in \mathbb{C}^n)$.

(2) For any $m = (m_1, \ldots m_n) \in \{\mathbb{N} \cup 0\}^n$,
$D^m f(m) = 0$.

If all K_i (i= 1,...n) are contained in U, then f(z) vanishes identically.

Remark 1. Assumption $K_i \subset U$ is crucial. Suppose that $a \in \partial U$. We put $f(a,z) = \exp(az) - \exp(\bar{a} z)$. f(a,z) satisfies (1) and (2) in theorem 1. But f(z) doesn't vanish identically. $\sin(\frac{\pi}{2} z)$ is a special case of this example. $(\sin(\frac{\pi}{2} z) = (2i) f(\frac{\pi}{2} i, z))$
Another example ze^{-z} is obtained by following manner:

$$ze^{-z} = \lim_{\substack{a \in \partial U \\ a \to -1}} (a - \bar{a})^{-1} f(a,z).$$

Theorem 2. (Abel interpolation formula. [1] and [8]) Suppose that K is a compact set in $\mathbb{C}^n$ and entire function f(z) satisfies condition (1) in theorem 1.
If $K_i \subset \Lambda$ for i=1,...n, then following expansion is valid:

$$f(z) = \sum_{m \in \mathbb{N}^n} \frac{D^m f(m)}{m!} z_1 \ldots z_n (z_1 - m_1)^{m_1 - 1} \ldots (z_n - m_n)^{m_n - 1}$$

To prove theorem 2. we need following lemma.

Lemma 1

$$\exp(zt) = \sum_{n=0}^{\infty} (te^{t})^{n} \frac{z(z-n)^{n-1}}{n!} \qquad (t \in \Lambda).$$

(Proof) By Stirling's formula, $z(z-n)^{n-1}/n!$ behaves like $O(e^{n})$ for sufficiently large n. Hence if t belongs to Λ then the series in the right hand side converges uniformly .

$$\frac{a^{n}}{n!} = (2\pi i)^{-1} \int z^{-n-1} \exp(az)dz,$$

$$\frac{z(z-n)^{n-1}}{n!} = \frac{(z-n)^{n}}{n!} + \frac{(z-n)^{n-1}}{(n-1)!} .$$

Applying these identities and residue theorem to right hand side in lemma, we obtain lemma.

(Proof of Theorem 2)

By Martineau - Ehrenpreis's theorem, there exists an analytic functional $T \in \mathcal{O}'(K)$ such that $f(z) = \tilde{T}(z)$. From the definition, $\tilde{T}(z) = \langle T_t, \exp(\langle t,z\rangle)\rangle$.

Inserting the identity in lemma, we obtain Abel interpolation series.

Example 7. (confluent hypergeometric function $\Phi(\alpha,\gamma,w)$)

As shown in example 4, there exists an analytic functional $T_{\alpha,\gamma}$ supported by the origin such that $\tilde{G}_{\{0\}}(T)(w) = \Phi(\alpha,\gamma,w)$. Since $\{0\}$ is included in Λ, $\tilde{T}_{\alpha,\gamma}$ can be expanded to Abel interpolation series.

<u>Example 8.</u> (Abel's identity) We apply Abel's interpolation formula to $(y + z)^n$. Then we have

$$(y+z)^n = z \sum_{k=0}^{n} \binom{n}{k} (y+k)^{n-k}(z-k)^{k-1}.$$

Putting $y = r/q$, $z = p/q$. We obtain

$$(r+p)^n = p \sum_{k=0}^{n} \binom{n}{k} (r+kq)^{n-k}(p-kq)^{k-1}.$$

This is so - called Abel's identity.([3]) If $q=0$, this is binomial expansion.

<u>Remark 2.</u> We can not omit condition $K_i \subset \Lambda$. ze^{-z} is Fourier - Borel transform of $\delta'(t+1)$. support of $\delta'(t+1)$ is $\{-1\}$. $\{-1\}$ is a boundary point of Λ. Hence ze^{-z} is not expressed by Abel interpolation series. $f(a,z)$ ($a \in \partial U$) also give such example.

<u>Remark 3.</u> In the case of $K \subset U$, Abel interpolation series is Mittag - Leffler summable in general. ([2]).

<u>Theorem 3.</u> Suppose that entire function $f(z)$ satisfies following assumptions :

(3) There exists a constant $C \geq 0$ such that

$|f(z)| \leq C \exp(\Sigma_{k=1}^{n} a_k |z_k|)$.

(4) $D^{i+j}f(i+j) = D^{i}f(i)D^{j}f(j)$ (for any $i,j \in \mathbb{N}^n$).

If $a_k < \psi(1)$ for all $k = 1,...n$, then $f(z)$ is constant.

4. Transform $\tilde{G}$ for analytic functionals with unbounded carrier.

In this section we will consider transform $\tilde{G}$ of analytic functionals with unbounded carrier. Let L be a closed convex set bounded in the imaginary direction. Holomorphic test function space $Q(L:k')$ is defined as follows :

$$Q(L:k') = \lim_{\substack{\varepsilon' \searrow 0 \\ \varepsilon \searrow 0}} \text{ind}\, Q_b(L_\varepsilon : k'+\varepsilon'),$$

$$Q_b(L_\varepsilon : k'+\varepsilon') = \{f \in \mathcal{O}(\mathring{L}_\varepsilon) \cap C(\overline{L}_\varepsilon);\ \sup_{t \in \overline{L}_\varepsilon} |f(t)||\exp((k'+\varepsilon)t| < +\infty \}.$$

$\mathcal{O}(\mathring{L}_\varepsilon)$ and $C(\overline{L}_\varepsilon)$ denote the space of holomorphic functions in $\mathring{L}_\varepsilon$ (interior of L_ε) and the space of continuous functions in $\overline{L}_\varepsilon$ (closurer of L_ε) respectively. $Q'(L:k')$ denotes the dual space of $Q(L:k')$. The element of $Q'(L:k')$ is called analytic functional with carrier L and of type k'. $\tilde{T}(z) = \langle T_t, \exp(\langle t,z \rangle)\rangle$ is Fourier - Borel transform of $T \in Q'(L:k')$. $\tilde{T}(z)$ is holomorphic in the direct product of half planes $\prod_{i=1}^{n}\{ \text{Re } z_i < -k'\}$ and of exponential type $H_L(z)$ (supporting function of L). Converse statement also valids. ([7])

Now we put following assumptions :

(i) $0 \le k' < 1$,

(ii) $L_i \subset U \cap \{ \operatorname{Re} t_i > a_i \}$ for some $a_i > 0$. ($i = 1,\dots n$).
Under these two conditions we can define transformation $\tilde{G}_L(T)$ for $T \in Q'(L:k')$ as follows :

$$\tilde{G}_L(T)(w) = \langle T_t, \Pi_{i=1}^n (1-w_i t_i \exp(t_i))^{-1} \rangle.$$

$\tilde{G}_L(T)(w)$ has following properties.

Proposition 2. ([9])

(5) $\tilde{G}_L(T)(w)$ is holomorphic in $\Pi_{i=1}^n (\mathbb{C} - \overline{\varphi(L_i)})$.

(6) $\tilde{G}_L(T)(w) = (-1)^n \Sigma_{m \in \mathbb{N}^n} D^{-m} \tilde{T}(-m) w^{-m}$.

(7) (inversion formula)

$$\tilde{T}(z) = (2\pi i)^{-n} \int_\Gamma \tilde{G}_L(T)(w) \exp(\Sigma_{i=1}^n z_i \psi(w_i)) \frac{dw_1}{w_1} \dots \frac{dw_n}{w_n}$$

$\Gamma = \Gamma_1 \times \dots \times \Gamma_n$ and Γ_i ($i=1,\dots n$) is a boundary of sector with vertex at zero surrounding $\overline{\varphi(L_i)}$.

5. Main results.

In this section we show our main results.

Theorem 4. Suppose that $0 \le k' < 1$ and $f(z)$ satisfies following conditions:

(8) $f(z)$ is holomorphic in $\Pi_{i=1}^{n}\{ z_i \in \mathbb{C} ;\ \ \mathrm{Re}\, z_i < -k' \}$

(9) for all $\varepsilon > 0$ and $\varepsilon' > 0$, there exists a constant $C_{\varepsilon,\varepsilon'} \geq 0$ such that

$|f(z)| \leq C_{\varepsilon,\varepsilon'} \exp(H_L(z) + \varepsilon | z |)$, $(\mathrm{Re}\, z_i \leq -k'-\varepsilon',\ i=1,\ldots n)$.

(10) $D^{-m} f(-m) = 0$, $(m = (m_1,\ldots m_n) \in \mathbb{N}^n)$.

If L satisfies (ii) in sec.4, then $f(z)$ vanishes identically.

(Proof) By the assumptions (8) and (9), there exists $T \in Q'(L:k')$ such that $f(z) = \tilde{T}(z)$.([7]) From assumption (10) and expansion (6) in Prop.2. $\tilde{G}_L(T)(w)$ vanishes identically. Hence by inversion formula (7) in Prop.2, $f(z)$ vanishes identically.

<u>Remark 4.</u> We can not omit condition (ii) in theorem 4. Suppose that $a \in \partial U$ and $\mathrm{Re}\, a > 0$. Then $f(a,z)$ satisfies all assumptions in theorem 4. But $f(a,z)$ does not vanishes identically.

<u>Corollary</u>. We assume (8), (9) in theorem 4 and (i),(ii) in sec.4. Suppose that $f(z)$ satisfies following conditions :

(11) $D^{-i-j}f(-i-j) = D^{-i}f(-i)\, D^{-j}f(-j)$ (for all $i,\ j \in \mathbb{N}^n$),

(12) $D^{-i}f(-i) \in \mathbb{Z}$, (for all $i \in \mathbb{N}^n$).

If $a_i > \psi(1)$ (= 0.567...), then $f(z)$ vanishes identically.

Remark 5 Condition $a_i > \psi(1)$ is crucial. Put $f(z) = \exp(\psi(1)z)$. Then $f(z)$ satisfies (11) and (12). But this function doesn't vanish identically.

Now we assume that F is a algebraic number field with $[F, \mathbb{Q}] = d$. We put $\delta = d$ if $F \subset \mathbb{R}$ and $\delta = d/2$ if $F \not\subset \mathbb{R}$. O_F denotes the set of algebraic integers in F. For algebraic integer a. we put $|a| = \max\{ |a_i| \; ; \; a_i$' are cojugates of a over $\mathbb{Q}$ }.

Theorem 5. We put same assumptions (8) and (9) in theorem 4 and (i) and (ii) in sec.4. Suppose that f(z) satisfies following conditions :

(13) $D^{-m}f(-m) \in O_F$, (for all $m \in \mathbb{N}^n$).

(14) $\limsup_{||m|| \to \infty} (||m||)^{-1} \log |\overline{D^{-m}f(-m)}| \le c$, (for some $c > 0$)

If $\log(d(\overline{\varphi(L_i)})) < -(\delta-1)c$ valid for i=1,...n, then f(z) is exponential polynomial.

Corollary Let $L = \Pi_{i=1}^{n}[a_i, \infty)$. Suppose that f(z) satisfies (8) and (9) in theorem 4 and $D^{-m}f(-m) \in \mathbb{Z}$ for all $m \in \mathbb{N}^n$. If $a_i > \psi(4)$, then f(z) is exponential polynomial.

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