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Hyperbolicity of Localizations

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1. Introduction

Let $P(x,D)$ be a differential operator of order m in an open set $\Omega\subset\mathbf{R}^{n+1}$ with coordinates $x=(x_0,x')=(x_0,x_1,\ldots,x_n)$, hence a sum of differential polynomials $P_j(x,D)$ of order j $(j\le m)$ with symbols $P_j(x,\xi)$. In [7] Ivrii-Petkov has proved a necessary condition for the Cauchy problem to $P(x,D)$ is correctly posed which asserts that $P_{m-j}(z)$ must vanish of order $r-2j$ at z if $P_m(z)$ vanishes of order r at z with $z=(x,\xi)\in T^*\Omega\setminus 0$. This enables us to define the localization $P_{z_0}(z)$ at a multiple characteristic z_0 (of $P_m(z)$), which is a polynomial on $T_{z_0}(T^*\Omega)$, following Helffer [4].

In this note we show that $P_{z_0}(z)$ is hyperbolic, that is verifies Gårding's condition if the Cauchy problem to $P(x,D)$ is correctly posed. The proof is based on the arguments of Svensson [9] and Nishitani [8].

Since $P_{z_0}(z)$ is hyperbolic, following Atiyah-Bott-Gårding [1], one can define the localizations $P_{(z_0,z_1,\ldots,z_s)}(z)$ successively as the localization of $P_{(z_0,z_1,\ldots,z_{s-1})}(z)$ at z_s which are hyperbolic polynomials on $T_{z_0}(T^*\Omega)\cong\cdots\cong T_{z_s}(T^*\Omega)$ (see also Hörmander [5, II]). It may occur the case that the lineality $\Lambda_{(z_0,z_1,\ldots,z_s)}(P_m)$ of $P_{m(z_0,z_1,\ldots,z_s)}(z)$ (see (2.8) below) is an involutive subspace with respect to the canonical symplectic structure on $T_{z_0}(T^*\Omega)$. In this case we prove that for the Cauchy problem to be correctly posed it is necessary that

$$P_{(z_0,z_1,\ldots,z_s)}(z)=P_{m(z_0,z_1,\ldots,z_s)}(z),$$

that is, no lower order terms of $P_{(z_0,\ldots,z_s)}(z)$ occur. This argument was also used in Bernardi-Bove-Nishitani [2] with $s=1$.

2. Localization is hyperbolic

We denote by $L^{m,r}_{z_0}$ the set of pseudodifferential operators P near z_0 with symbol $P(x,\xi)$ verifying

$$P(x,\xi)\sim\sum_{j=0}^{\infty}P_{m-j}(x,\xi)$$

in every homogeneous symplectic coordinates around z_0 where $P_{m-j}(x,\xi)$ are positively homogeneous of degree $m-j$ in ξ and vanish of order at least $r-2j$ and $P_m(x,\xi)$ vanishes exactly to the order r at z_0. Note that we may replace in the definition "every" by "some".

Lemma 2.1 (Helffer [4]). *Let $P\in L^{m,r}_{z_0}$. Then*

$$Q(x,\xi)=\exp\{\frac{i}{2}\sum_{j=0}^{n}\frac{\partial^2}{\partial x_j\partial\xi_j}\}P(x,\xi)\tag{2.1}$$

is invariantly defined in $L^{m,r}_{z_0}/L^{m,r+1}_{z_0}$: Let χ be a homogeneous symplectic coordinates around z_0 and let F be a Fourier integral operator associated with χ and $\hat{P}=FPF^{-1}$.

Then we have

$$\hat{Q}(\chi(x,\xi)) = Q(x,\xi)$$

in $L^{m,r}_{z_0}/L^{m,r+1}_{z_0}$ *where* $\hat{Q}$ *is associated with* $\hat{P}$ *by (2.1).*

Definition 2.1. We define the localization $P_{z_0}(x,\xi)$ of $P \in L^{m,r}_{z_0}$ at $z_0 = (x_0,\xi_0)$ as the lowest order term of the Taylor expansion of

$$\mu^{2m} Q(x_0 + \mu x, \mu^{-2}\xi_0 + \mu^{-1}\xi)$$

as $\mu \to 0$ which is invariantly defined as a polynomial on $T_{z_0}(T^*\Omega)$: If y are local coordinates around the origin and $\hat{P}(y,\eta)$ is the full symbol of P for the coordinates $(y, \eta dy)$ then we have

$$\hat{P}_{w_0}(y'(x_0), {}^t y'(x_0)^{-1}\xi + {}^t(y'\xi_0)'(x_0)x) = P_{z_0}(x,\xi), \; w_0 = (y(x_0), {}^t y'(x_0)^{-1}\xi_0).$$

Writing $Q(x,\xi)$ as the sum of homogeneous parts $Q_{m-j}(x,\xi)$, it is clear that

$$\begin{aligned} &P_{z_0}(x,\xi) = \sum_{r-2j\geq 0} Q_{m-j,z_0}(x,\xi), \\ &Q_{m-j,z_0}(z) = P_{m-j,z_0}(z) + \sum_{i<j,|\alpha|=j-i} c_\alpha P^{(\alpha)}_{m-i,z_0(\alpha)}(z) \end{aligned} \tag{2.2}$$

with some constants c_α where $Q_{m-j,z_0}(x,\xi)$ and $P_{m-j,z_0}(x,\xi)$ are defined by

$$P_{m-j,z_0}(z) = \lim_{\mu\to 0} \mu^{-(r-2j)} P_{m-j}(z_0 + \mu z).$$

Let $P(x,D) = \sum_{j=0}^{m} P_j(x,D)$ be a differential operator of order m on Ω containing the origin where $P_j(x,D)$ is the homogeneous part of degree j with symbol $P_j(x,\xi)$. Assume that the plane $x_0 = 0$ is non characteristic and we are concerned with the Cauchy problem with respect to x_0 =const.. Let $z_0 \in T^*\Omega \setminus 0$ be a characteristic of P_m of order r;

$$d^j P_m(z_0) = 0 \quad \text{for} \quad j < r, \quad d^r P_m(z_0) \neq 0.$$

By the necessary condition of Ivrii-Petkov [7] stated in Introduction we conclude that $P \in L^{m,r}_{z_0}$ provided that the Cauchy problem for P is correctly posed. Then we have from Lemma 2.1 that

Proposition 2.2 (cf. Ivrii and Petkov [7]). *Assume that the Cauchy problem for* $P(x,D)$ *is correctly posed near the origin and let* $z_0 \in T^*\Omega \setminus 0$ *be a multiple characteristic of* P_m. *Then the localization* $P_{z_0}(z)$ *is an invariantly defined polynomial on* $T_{z_0}(T^*\Omega)$.

Let us denote by $\tilde{P}_{z_0}(x,\xi)$ the lowest order term of the Taylor expansion of

$$\mu^{2m} P(x_0 + \mu x, \mu^{-2}\xi_0 + \mu^{-1}\xi)$$

as $\mu \to 0$. Note that $\tilde{P}_{z_0}(x,\xi)$ is not coordinates free but we have

Lemma 2.3. *The following two conditions are equivalent.*
(i) $\tilde{P}_{z_0}(z)$ *is hyperbolic with respect to* $\theta = (0, e_0)$,
(ii) $P_{z_0}(z)$ *is hyperbolic with respect to* θ.

Proof. Recall that $\tilde{P}_{z_0}(z) = \sum_{r-2j\geq 0} P_{m-j,z_0}(z)$. Since $\tilde{P}_{z_0}(z)$ is hyperbolic if and only if $P_{m-j,z_0}(z)$ are weaker than $P_{m,z_0}(z) = Q_{m,z_0}(z)$ (see Hörmander [5, II], Svensson [9]) the proof is immediate by (2.2).

Now our aim is to prove

Theorem 2.4. *Assume that the Cauchy problem for* $P(x, D)$ *is correctly posed near the origin and let* $z_0 \in T^*\Omega\backslash 0$ *be a multiple characteristic of* P_m. *Then the localization* $P_{z_0}(z)$ *is a hyperbolic polynomial with respect to* $\theta = (0, e_0)$.

Let z_0 be a characteristic of order r_0 of $P_m(z)$ so that $P_{z_0}(z)$ is a polynomial of degree r_0. We denote by $P_{(z_0,z_1)}(z)$ the localization of $P_{z_0}(z)$ at z_1, that is the first coefficient of $\mu^{r_0}P_{z_0}(\mu^{-1}z_1 + z)$ that does not vanish identically in z:

$$\mu^{r_0}P_{z_0}(\mu^{-1}z_1 + z) = \mu^{r_1}(P_{(z_0,z_1)}(z) + O(\mu)), \ \mu \to 0$$

(see Hörmander [5, II] and Atiyah-Bott-Gårding [1]). We call r_1 the order of z_1. From Lemma 3.4.2 in Atiyah-Bott-Gårding [1] it follows that $P_{(z_0,z_1)}(z)$ is again hyperbolic with respect to θ. Furthermore z_1 is a characteristic of P_{m,z_0} of order r_1 and $P_{m(z_0,z_1)}(z)$ is the principal part of $P_{(z_0,z_1)}(z)$. On the other hand Corollary 12.4.9 in Hörmander [5, II] shows that

$$d^\nu Q_{m-j,z_0}(z_1) = 0, \ \nu < r_1 - 2j$$

where $d^\nu Q(z)$ denotes the ν-th differential of Q with respect to z. Since $Q_{m-j,z_0}(z)$ are homogeneous of degree $r_0 - 2j$ it is clear that

$$P_{(z_0,z_1)}(z) = \sum_{r_1-2j\geq 0} Q_{m-2j(z_0,z_1)}(z)$$

where

$$Q_{m-j(z_0,z_1)}(z) = \lim_{\mu\to 0} \mu^{-(r_1-2j)}Q_{m-j,z_0}(z_1 + \mu z)$$

which is homogeneous of degree $r_1 - 2j$ in z. Repeating the same arguments we get

Lemma 2.5. *Let* $P_{(z_0,\ldots,z_k)}(z)$ *be the localization of* $P_{(z_0,\ldots,z_{k-1})}(z)$ *at* z_k *of which order is* r_k (≥ 2);

$$P_{(z_0,\ldots,z_k)}(z) = \left(P_{(z_0,\ldots,z_{k-1})}\right)_{z_k}(z).$$

Then we have for every j *with* $r_k - 2j > 0$

$$d^\nu Q_{m-j(z_0,\ldots,z_{k-1})}(z_k) = 0, \ \nu < r_k - 2j$$

and hence

$$Q_{m-j(z_0,\ldots,z_k)}(z) = \lim_{\mu\to 0} \mu^{-(r_k-2j)}Q_{m-j(z_0,\ldots,z_{k-1})}(z_k + \mu z)$$

exists. Moreover $P_{(z_0,\ldots,z_k)}(z)$ *is equal to*

$$\sum_{r_k-2j\geq 0} Q_{m-j(z_0,\ldots,z_k)}(z)$$

and hyperbolic with respect to θ.

Corollary 2.6. *Let z_k be a characteristic of $P_{m(z_0,\dots,z_{k-1})}(z)$ of order r_k (≥ 2). Then we have*

$$d^\nu P_{m-j(z_0,\dots,z_{k-1})}(z_k) = 0, \quad \nu < r_k - 2j \tag{2.3}$$

and then

$$P_{m-j(z_0,\dots,z_k)}(z) = \lim_{\mu\to 0} \mu^{-(r_k-2j)} P_{m-j(z_0,\dots,z_{k-1})}(z_k + \mu z) \tag{2.4}$$

exists.

Proof. Assume that (2.3) and

$$\begin{aligned} Q_{m-j(z_0,\dots,z_{k-1})}(z) &= P_{m-j(z_0,\dots,z_{k-1})}(z) \\ &+ \sum_{i<j,|\alpha|=j-i} c_\alpha P^{(\alpha)}_{m-i(z_0,\dots,z_{k-1})(\alpha)}(z) \end{aligned} \tag{2.5}$$

hold with $k = p$ where c_α are constants. Then it is easy to see that (2.5) with $k = p+1$ holds. Thus (2.3) with $k = p+1$ follows from Lemma 2.5. By induction on k we get the desired conclusion.

Here we give another formula which defines $P_{(z_0,\dots,z_s)}(z)$ directly. Let $0 < \mu_0 < \mu_1 < \cdots < \mu_s$ be a sequence of parameters with

$$\mu_j = O(\mu_{j+1}^{m+1}) \quad \text{as} \quad \mu_{j+1} \to 0. \tag{2.6}$$

Then we have

$$\begin{aligned} (\mu_0 \cdots \mu_s)^{2m} Q(&x_0 + \mu_0 x_1 + \cdots + \mu_0 \cdots \mu_{s-1} x_s + \mu_0 \cdots \mu_s x, \\ &(\mu_0 \cdots \mu_s)^{-2}(\xi_0 + \mu_0 \xi_1 + \cdots + \mu_0 \cdots \mu_{s-1}\xi_s + \mu_0 \cdots \mu_s \xi) \\ &= \mu_0^{r_0} \cdots \mu_s^{r_s} \left(P_{(z_0,\dots,z_s)}(z) + O(\mu_s)\right) \end{aligned} \tag{2.7}$$

where $z_j = (x_j, \xi_j)$ and r_j is the order of z_j.

Let $\Lambda_{(z_0,\dots,z_s)}(P_m)$ be the lineality of $P_{m(z_0,\dots,z_s)}$ which is a linear subspace defined by

$$\{z | P_{m(z_0,\dots,z_s)}(w + tz) = P_{m(z_0,\dots,z_s)}(w), \forall t \in \mathbf{R}, \forall w \in T_{z_0}(T^*\Omega)\} \tag{2.8}$$

and let $\sigma = \sum_{j=0}^n d\xi_j \wedge dx_j$ be the canonical symplectic two form on $T^*\Omega$. For $S \subset T_{z_0}(T^*\Omega)$ we denote by S^σ the annihilator of S with respect to σ:

$$S^\sigma = \{z \in T_{z_0}(T^*\Omega) | \sigma(z, w) = 0, \forall w \in S\}.$$

Theorem 2.7. *Assume that the Cauchy problem for $P(x, D)$ is correctly posed near the origin and*

$$\Lambda_{(z_0,\dots,z_s)}(P_m)^\sigma \subset \Lambda_{(z_0,\dots,z_s)}(P_m).$$

Then we have

$$P_{(z_0,\dots,z_s)}(z) = P_{m(z_0,\dots,z_s)}(z),$$

that is, no lower order terms occur in $P_{(z_0,\dots,z_s)}(z)$.

Example 2.1. Let

$$P(x,\xi) = (\xi_0^2 - x_1^2\xi_n^2 - \xi_1^2)(\xi_0^2 - x_1^2\xi_n^2 - 2\xi_1^2) + p_2(\xi_0, x_1\xi_n, \xi_1)\xi_n$$

where p_2 is a homogeneous polynomial of degree 2. With $z_0 = (0, e_n)$ it is clear that

$$P_{4,z_0} = (\xi_0^2 - x_1^2 - \xi_1^2)(\xi_0^2 - x_1^2 - 2\xi_1^2),\ Q_{3,z_0} = 6ix_1\xi_1 + p_2(\xi_0, x_1, \xi_1).$$

Let z_1 be $\xi_0 = x_1 = a$, $a \in \mathbf{R}$, $\xi_1 = 0$ so that

$$P_{4(z_0,z_1)} = 4a^2(\xi_0 - x_1)^2,\ Q_{3(z_0,z_1)} = p_2(a, a, 0).$$

Since $\Lambda_{(z_0,z_1)}(P_4)^\sigma \subset \Lambda_{(z_0,z_1)}(P_4)$ it follows from Theorem 2.7 that $p_2(a, a, 0) = 0$. Similarly choosing z_1 to be $\xi_0 = a$, $x_1 = -a$, $\xi_1 = 0$ we get $p_2(a, -a, 0) = 0$. Thus

$$p_2(\xi_0, x_1, \xi_1) = c(\xi_0^2 - x_1^2) + \xi_1 p_1(\xi_0, x_1, \xi_1)$$

where p_1 is linear. Finally one can write

$$P(x,\xi) = (\xi_0^2 - x_1^2\xi_n^2 - \xi_1^2 + c\xi_n)(\xi_0^2 - x_1^2\xi_n^2 - 2\xi_1^2) + \xi_1 L(\xi_0, x_1\xi_n, \xi_1)\xi_n$$

with a linear function L.

Example 2.2. Let

$$P(x,\xi) = (\xi_0 - x_0\xi_n)^2(\xi_0 + x_0\xi_n) + \alpha(\xi_0 - x_0\xi_n)\xi_n + \beta(\xi_0 + x_0\xi_n)\xi_n$$

where $\alpha, \beta \in \mathbf{C}$. With $z_0 = (0, e_n)$ we have

$$P_{3,z_0} = (\xi_0 - x_0)^2(\xi_0 + x_0),\ Q_{2,z_0} = \alpha(\xi_0 - x_0) + (\beta - i)(\xi_0 + x_0).$$

Taking z_1 to be $\xi_0 = 1$, $x_0 = 1$ it follows that

$$P_{3(z_0,z_1)} = 2(\xi_0 - x_0)^2,\ Q_{2(z_0,z_1)} = 2(\beta - i).$$

Since $\Lambda_{(z_0,z_1)}(P_3)^\sigma \subset \Lambda_{(z_0,z_1)}(P_3)$ we have $\beta = i$ by Theorem 2.7. Set

$$p_1(x,\xi) = \xi_0 - x_0\xi_n,\ p_2(x,\xi) = (\xi_0 - x_0\xi_n)(\xi_0 + x_0\xi_n) + (\alpha + i)\xi_n$$

then $\beta = i$ implies that

$$P(x, D) = p_1^w(x, D)p_2^w(x, D)$$

where $p_j^w(x, D)$ are Weyl realizations of $p_j(x,\xi)$, see Hörmander [5, III].

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