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On the equivalence of the condition (S) of Kawai and the property of regular growth

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§1. Introduction

For a hyperfunction $\mu(x)$ defined on $\mathbf{R}^n$ with compact support, we consider the convolution operator $\mu*$. we denote by $\widehat{\mu}(\zeta)$ the Fourier transform of $\mu(x)$. Professor T. Kawai [K] introduced the following condition (S) to $\widehat{\mu}(\zeta)$:

$$(\mathrm{S})\quad \begin{cases} \text{For every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that} \\ \text{for any } \eta \in \mathbf{R}^n \text{ with } |\eta| > N \\ \text{we can find } \zeta \in \mathbf{C}^n, \text{ which satisfies} \\ |\eta - \zeta| < \varepsilon|\eta| \\ |\widehat{\mu}(\zeta)| \geq e^{-\varepsilon|\eta|}. \end{cases}$$

and proved the existence of solutions of the convolution equation $\mu * f = g$ in the category of hyperfunctions.

On the other hand, J. F. Korobeĭnik, O. V. Epifanov, and V. V. Morzhakov have shown that the question of solvability of convolution equations in convex domains of $\mathbf{C}^n$ are closely related to the notion of entire function of completely regular growth

Let Ω be a convex domain in $\mathbf{C}^n$ and K a compact convex set in $\mathbf{C}^n$. Let $\mathcal{O}(\Omega)$ be the space of holomorphic functions on Ω equipped with the topology of uniform convergence on compact subsets of Ω and let $\mathcal{O}(K)$ be the space of germs of the holomorphic functions on K provided with the usual topology of the inductive limit, $\mathcal{O}'(\Omega)$ and $\mathcal{O}'(K)$ denote dual spaces to $\mathcal{O}(\Omega)$ and $\mathcal{O}(K)$, respectively. For an analytic functional $T \in \mathcal{O}'(K)$, we denote by $\widehat{T}(\zeta)$ its Fourier-Borel transform, and we consider the convolution operator:

$$T* \,:\, \mathcal{O}(\Omega + K) \longrightarrow \mathcal{O}(\Omega)\,.$$

Using the condition that $\widehat{T}(\zeta)$ is an entire function of exponential type of completely regular growth in $\mathbf{C}^n$, Morzhakov [M] gave some results on the surjectivity of $T*$.

Moreover, R. Ishimura - Y. Okada [I-Y.O] considered the convolution operator $\mu*$, operating on holomorphic functions in tube domains of the form $\mathbf{R}^n + \sqrt{-1}\omega$ with $\sqrt{-1}\omega$ an open set in $\sqrt{-1}\mathbf{R}^n$, and under the condition (S), they proved the existence of holomorphic solutions in any open tube domain. Conversely, by the method of Morzhakov, they showed that the existence of solutions in some special tube domain implies the condition (S). That means the condition (S) is sufficient and almost necessary for the existence of solutions.

Comparing these results, we have the following natural question.

Problem 1.1
Are there some relation between the condition (S) and the property of completely regular growth ?

We get the positive answer to this problem.

§2. Regular growth

In this section, we shall recall principal notions of regular growth of entire functions and of subharmonic functions. We refer to P. Lelong - L. Gruman [L-G] for terminologies.

Let γ and Γ be open connected cones with vertex at the origin in $\mathbf{R}^m$ and in $\mathbf{C}^n$

respectively, and let $\rho(r)$ be a proximate order. For $L > 0$, we put

$$\gamma_L := \gamma \cap \{\xi \in \mathbf{R}^m | \, |\xi| > L\} \quad \text{and} \quad \Gamma_L := \Gamma \cap \{\zeta \in \mathbf{C}^n | \, |\zeta| > L\} .$$

We shall denote by $SH^{\rho(\|\xi\|)}(\gamma_L)$ the family of functions u subharmonic in γ_L such that there exist constants A and $B > 0$ (depending on u) with $u(\xi) \leq A + B\|\xi\|^{\rho(\|\xi\|)}$. For such a function, we put

$$\widehat{h}_u(\xi) := \limsup_{r\to\infty} \frac{u(r\xi)}{r^{\rho(r)}} \quad \text{and} \quad \widehat{h}^*_u(\xi) := \limsup_{\xi'\to\xi\in\gamma_L} \widehat{h}_u(\xi')$$

and we call them the indicator of u and the regularized indicator of u respectively. For a holomolphic function f with $|f(z)| \leq Ae^{B|z|^{\rho(|z|)}}$ in Γ_L , which we denote by $f \in Exp^{\rho(|z|)}(\Gamma_L)$, identifying $\mathbf{C}^n$ with $\mathbf{R}^{2n}$, we consider the subharmonic function $\log|f(z)|$ in place of $u(\xi)$. Then for $\zeta \in \Gamma_L$, we put

$$h_f(\zeta) := \widehat{h}_{\log|f(\zeta)|}(\zeta) \quad \text{and} \quad h^*_f(\zeta) := \widehat{h}^*_{\log|f(\zeta)|}(\zeta)$$

and we call $h_f(\zeta)$ the radial indicator of f and $h^*_f(\zeta)$ the regularized radial indicator of f. For $\xi \in \gamma_L$, $r > 0$ and $\delta > 0$, we put

$$I^r_u(\xi,\delta) := \frac{1}{\omega_m \delta^m} \int_{|\eta-\xi|<\delta} \frac{u(r\eta)}{r^{\rho(r)}} d\eta$$

(where ω_m is the volume of the unit ball in $\mathbf{R}^m$)

We shall use the following definition of Lelong - Gruman [L-G].

Definition 2.1 (Lelong - Gruman)
A function $u \in SH^{\rho(\|\xi\|)}(\gamma_L)$ will be said to be of regular growth in $\xi_0 \in \gamma_L$, if

$$\liminf_{\delta\to 0} \liminf_{r\to\infty} I^r_u(\xi_0,\delta) = \widehat{h}^*_u(\xi_0) \, , \quad \widehat{h}^*_u(\xi_0) \neq -\infty \tag{2.1}$$

and also $f \in Exp^{\rho(|z|)}(\Gamma_L)$ is called a function of regular growth in $\zeta_0 \in \Gamma_L$, if for $u = \log|f|$, we have (2.1).

Remark 2.2
In the case of one complex variable, the property of regular growth coincides with the classical notion of property of completely regular growth. (see B. Ya. Levin [L])

§3. Generalization of the condition (S)

We will generalize the condition (S) to subharmonic functions in γ_L with the proximate order $\rho(r)$.

Definition 3.1
Let $u(\xi) \in SH^{\rho(\|\xi\|)}(\gamma_L)$, $\xi_0 \in \gamma_L$ and we assume $\widehat{h}_u^*(\xi_0) \neq -\infty$. We define that $u(\xi)$ satisfies the condition (S) in ξ_0 , if $u(\xi)$ satisfies the following condition $(S)_{\xi_0}$,

$$(\mathrm{S})_{\xi_0} \quad \begin{cases} \text{For every } \varepsilon > 0, \text{ there exists } N > 0 \text{ such that} \\ \text{for any } r \in \mathbf{R} \text{ with } r > N \\ \text{we can find } \xi \in \mathbf{R}^m, \text{ which satisfies} \\ |\xi - \xi_0| < \varepsilon, \\ \dfrac{u(r\xi)}{r^{\rho(r)}} \geq \widehat{h}_u^*(\xi_0) - \varepsilon. \end{cases}$$

and also $f \in Exp^{\rho(|z|)}(\Gamma_L)$ satisfies the condition (S) in $\zeta_0 \in \Gamma_L$, if for $u = \log|f|$, we have the condition $(S)_{\zeta_0}$.

Remark 3.2
For a hyperfunction $\mu(x)$ with compact support, its Fourier transform $\widehat{\mu}(\zeta)$ is a function of infra-exponential type in real directions, namely $h_{\widehat{\mu}}^*(\xi) = 0$ on $\xi \in \mathbf{R}^n$. Therefore, our definition is a natural generalization of the condition (S) of Professor T. Kawai.

§4. Result

We have the following theorem answering to Problem 1.1.

Theorem 4.1
Let $u(\xi) \in SH^{\rho(\|\xi\|)}(\gamma_L)$, $\xi_0 \in \gamma_L$ and suppose $\widehat{h}_u^*(\xi_0) \neq -\infty$. Then u is of regular growth in ξ_0 if and only if u satisfies the condition $(S)_{\xi_0}$.

For a proof, see [I-J.O]

Let constant $\rho > 0$ fixed. Following Morzhakov [M], for $f \in Exp^{\rho}(\Gamma_L)$, we denote by $Fr[f]$ the set of plurisubharmonic functions in Γ_L which are limits in $L^1_{loc}(\Gamma_L)$ of

the sequences of the form $\dfrac{\log|f(t_j z)|}{t_j^\rho}$, with $t_j > 0$.
The lower indicator of f is given by

$$\underline{h}_f(\zeta) := \inf_{g\in Fr[f]} g(\zeta)$$

The following was introduced by S. Ju. Favorov:
For $f \in Exp^{\rho(|z|)}(\Gamma_L)$, $\zeta_0 \in \Gamma_L$, we assume $h_f^*(\zeta_0) \neq -\infty$. f is said to be of completely regular growth in ζ_0, if $h_f^*(\zeta_0) = \underline{h}_f(\zeta_0)$

Therefore, from Lemma 1 of Morzhakov [M] and Thorem 4.1, we have

Corollary 4.2
Let $f \in Exp^\rho(\Gamma_L)$, $\zeta_0 \in \Gamma_L$, suppose $h_f^*(\zeta_0) \neq -\infty$. Then the following conditions are equivalent.
1) f is of regular growth in ζ_0.
2) f satisfies the condition $(S)_{\zeta_0}$.
3) f is of completely regular growth in ζ_0
4) for any $g \in Exp^\rho(\Gamma_L)$,

$$h_{fg}^*(\zeta_0) = h_f^*(\zeta_0) + h_g^*(\zeta_0)$$

("addition theorem for indicators")

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