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Gevrey singularities for nonlinear wave equations

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1. INTRODUCTION

We consider the following semilinear wave equations,

$$\Box u = f(u) \quad \text{in } \Omega \subset R_t \times R_x^2, \tag{1}$$

where u is a real valued function, $\Box = \partial^2/\partial t^2 - \triangle$ with $\triangle = \sum_{j=1}^{2} \partial^2/\partial x_j^2$, Ω is a bounded domain which contains the origin and $f(u)$ is a polynomial of u with $f(0) = 0$.

We study the interaction of Gevrey singularities for this equation. We assume that solutions that we study here are all in $H^s(\Omega)$ with $s > 3/2$ where $H^s(\Omega)$ is a Sobolev space of order s in Ω. In 1982, J. Rauch and M. Reed [4] have made an example in which three singularities produce new singularities. In 1984, J. M. Bony [2] and R. Melrose and N. Ritter [3] have had a general result for C^∞singularity independently. We put $\Sigma_j = \{(t,x) \in R^3; t = \omega_j \cdot x\}$ $(j = 1,2,3)$ with $\omega_j \in S^1$. Their result for the equation (1) is as follows.

Theorem 1.1 (J. M. Bony [2], R. Melrose and N. Ritter [3]). *If u is conormal with respect to $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ in $\Omega_- = \Omega \cup \{t < 0\}$, then the solution u is C^∞ in $K \backslash (\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \{t^2 = |x|^2\})$ where K is a domain of determine with respect to Ω_-.*

In this talk, we shall make the Gevrey version of the above result.

Definition 1.1 (Gevrey conormal distribution). For $s > 3/2, \sigma \leq 1$, we call that $u \in H^s(\Sigma, G^{(\sigma)}; \Omega)$, if and only if for any compact set $K \subset \Omega$ and for any vector fields $V_1, \dots, V_l$ with analytic coefficients and any integer l which are tangent to Σ, there exist constants $C, A > 0$ such that

$$\|V_1^{\alpha_1} \cdots V_l^{\alpha_l} u\|_{H^s(K)} \leq C A^{|\alpha|} (|\alpha|!)^\sigma \tag{2}$$

for any integers $\alpha_1, \dots, \alpha_l$.

Theorem 1.2. *Suppose that u is in $H^s(\Omega)$ for some $s > 5/2$, u satisfies the equation (1) and $u \in H^s(\Sigma_1, G^{(\sigma)}; \Omega_-)$. Then we have*

$$u \in H^s(\Sigma_1, G^{(\sigma)}; K), \tag{3}$$

where $\Omega_- = \Omega \cap \{(t,x); t < 0\}$, *$K$ is the domain of dertermine with respect to* Ω_-.

Theorem 1.3. *Suppose that u is in $H^s(\Omega)$ for some $s > 5/2$, u satisfies the equation* (1) *and $u \in H^s(\Sigma_1 \cup \Sigma_2, G^{(\sigma)}; \Omega_-)$. Then we have*

$$u \in H^s(\Sigma_1 \cup \Sigma_2, G^{(\sigma)}; K), \tag{4}$$

where $\Omega_- = \Omega \cap \{(t,x); t < 0\}$, *$K$ is the domain of dertermine with respect to* Ω_-.

Theorem 1.4 (Main result). *Suppose that $u \in H^s(\Omega)$ $(s > 5/2)$, u satisfies the equation* (1) *and*

$$u \in H^s(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3, G^{(\sigma)}; \Omega_-). \tag{5}$$

Then u is a Gevrey class function of order σ in $K \backslash \Sigma_1 \cup S_2 \cup \Sigma_3 \cup \Gamma_+$, where $\Gamma_+ = \{t^2 = |x|^2, t > 0\}$, $\Omega_- = \Omega \cap \{t < 0\}$ and K is a domain of determine with respect to Ω_-.

Corollary 1.1. *Suppose that $u \in H^s(\Omega)$ $(s > 5/2)$, u satisfies the equation* (1) *and*

$$u \in H^s(\Sigma_1 \cup \Sigma_2 \cup \Sigma_3, G^{(1)}; \Omega_-). \tag{6}$$

Then u is real analytic in $K \backslash \Sigma_1 \cup S_2 \cup \Sigma_3 \cup \Gamma_+$, where $\Gamma_+ = \{t^2 = |x|^2, t > 0\}$, $\Omega_- = \Omega \cap \{t < 0\}$ and K is a domain of determine with respect to Ω_-.

2. Preliminaries

Let K be a relatively compact set in $R^3 = R_t \times R_x^2$ such that each subset $K \cap \{(t,x); s \le t \le T\}$ is a domain of determine with respect to $K \cap \{(t,x); t \le s\}$ for $S \le s \le T$. For $m > 5/2$ and $f \in H^m(K)$, we put

$$E_m(t)[f] = \|f(t)\|_{H^{m-1/2}(K(t))} + \|\partial_t f(t)\|_{H^{m-3/2}(K(t))} \tag{7}$$

with $K(s) = K \cap \{(t,x); t = s\}$.

Proposition 2.1 (Energy estimate). *For $f \in H^m(K)$, we have*

$$E_m(t_2)[f] \le E_m(t_1)[f] + C(T) \int_{t_1}^{t_2} \|\Box f\|_{H^{m-3/2}(K(s))} ds \tag{8}$$

for $S \le t_1 < t_2 \le T$.

Proposition 2.2. *For $u, v \in H^m(K)$, we have*

$$E_m(t)[uv] \le C(n) E_m[u] E_m[v]. \tag{9}$$

Let Q be a analytic vector field on K. We define a quantity $\|f(t)\|_{G^s_A(Q;E_m)}$ by

$$(10) \qquad \|f(t)\|_{G^s_A(Q;E_m)} = \sum_{l=0}^{\infty} \frac{A^l}{l!^s} E_m(t)[P^l]$$

and we put $\|f(t)\|_{X(Q)} = \|f\|_{G^s_A(Q;E_m)}$ and $|\!|\!|f|\!|\!|_{Y_A([t_1,t_2];Q)} |\!|\!|f|\!|\!|_{Y([t_1,t_2];Q)} = \sup_{t_1\le t\le t_2} \|f(t)\|_{X(Q)}$ for abbreviation.

Proposition 2.3. *For $c \in R$,*

$$(11) \qquad \|f\|_{G^s_A(Q+c;E_m)} \le e^{|c|A}\|f\|_{X(Q)}.$$

Proposition 2.4.

$$\|uv\|_{X(Q)} \le C(n)\|u\|_{X(Q)}\|v\|_{X(Q)}.$$

For Σ_1 and Σ_2, we put $\tilde\omega_j = (1, -\omega_j)$, $\tilde\omega_j^* = (1, \omega)$, $\nabla = (\partial_t, \partial_{x_1}, \partial_{x_2})$ and put

$$(12) \qquad X_1 = (\tilde\omega_1 \times \tilde\omega_2) \cdot \nabla$$

$$(13) \qquad X_2 = (t - \omega_2 \cdot x)\tilde\omega_1^* \cdot \nabla$$

$$(14) \qquad X_3 = \tilde\omega_2^* \cdot \nabla$$

$$(15) \qquad X_4 = (t - \omega_1 \cdot x)\tilde\omega_2^* \cdot \nabla$$

Proposition 2.5. *We have*

$$(16) \qquad [X_j, X_k] = 0 \quad \text{for} \quad 1 \le j, k \le 4,$$

$$(17) \qquad [\Box, X_1] = [\Box, X_3] = 0,$$

$$(18) \qquad [\Box, X_2] = [\Box, X_3] = C_1\Box + C_2X_1^2,$$

for some C_1 and C_2.

Proposition 2.6. (1) *X_1, X_2 and X_3 are all tangent to Σ_1.*
(2) *X_1, X_2 and X_4 are all tangent to $\Sigma_1 \cup \Sigma_2$.*

Proposition 2.7. (1) *X_1, X_2 and X_3 are linearly independent in Σ_1^c.*
(2) *X_1, X_2 and X_4 are linearly independent in $(\Sigma_1 \cup \Sigma_2)^c$.*

3. Lemmas

In this section, we prepare several lemmas which are used to prove the theorems. We put $P = t\partial_t + x \cdot \partial_x$. Let K' be a relatively compact open set in K satisfying the same condition K of the section 2. We consider the following linearized equation,

$$\begin{cases} \Box v = F(w), \\ v = u(-\epsilon, x) \quad \partial_t v = \partial_t u \quad \text{for} \quad t = -\epsilon, \end{cases}$$

where we take ϵ is so small that $K'(-\epsilon)$ determines $K' \cap \{-\epsilon < t < T\}$. Let S denote the mapping that corresponds w to v. We put $u_0 = S[0]$ and $u_n = Su_{n-1}$. Since u_0

is a solution to the homogenous linear wave equation, there exists a constant A such that $\|u_0\|_{Y_A([-\epsilon,T];P)} < \infty$. We put $B_0 = \max(\|u_0\|_{Y([-\epsilon,T];P)}, 2\|u(t_1)\|_{X(P)}$.

Lemma 3.1. *If u satisfies the assumption of Theorem 1.2 or 1.3 or 1.4, we have*

$$\|u\|_{Y([t_1,t_2];P)} \leq \|u(t_1)\|_{X(P)}, \tag{19}$$

for $-\epsilon \leq t_1 < t_2 \leq T$ with $t_2 - t_1 < 1/(2C(T)F(C(n))G(B_0))$.

Proof. Using Propositions 2.1, 2.3 and 2.4, we have the lemma. □

Lemma 3.2 (the Energy estimate). *If u satisfies the assumption of Thoeorem 1.2 or 1.3 or 1.4, we have*

$$\sup_{0\leq t\leq T} \|u(t)\|_{X(P)} \leq \|\phi\|_{G^s_A(x\cdot\nabla;E_m)}. \tag{20}$$

Proof. Using the lemma 3.1 several times, we have the lemma. □

4. Proof of Theorem 1.1 and 1.2

First we prove Theorem 1.1. From Propositions 2.6 and 2.7 it suffices to show that for every compact set $K' \subset K$ there exist constants C_1 and A_1 such that

$$\|X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} u\|_{H^m(K')} \leq C_1 A_1^{|\alpha|} |\alpha|^\sigma, \tag{21}$$

for all non negative integers α_1, α_2 and α_3 with $|\alpha| = \alpha_1 + \alpha_2 \alpha_3$. We can prove the above by the same argument as in the proof of Lemma 3.2.

5. Regularity in the interior of the cone

Let σ be a real number greater than or equal to 1. We put $P = t\partial_t + x \cdot \partial_x$. The following lemma is a key kemma to prove Theorem 1.4.

Lemma 5.1 (Key lemma). *Suppose that*

$$\|P^l u\|_{H^s(K)} \leq C_1 A_1^l (l!)^\sigma \quad \text{for } \forall l \in N \cup \{0\} \tag{22}$$

and u satisfies the equation (1). Then u is a Gevrey class function of order σ in Γ_+, where $\Gamma_+ = \{(t,x) \in R^3; t^2 > |x|^2, t > 0\}$.

Proof. For simplicity, we prove only the case $f(u) = u^m$. Let $B \subset \Gamma_+$ be a relatively compact ball. It suffices to show that u is a Gevrey class function of order σ in each $B \subset \Gamma_+$. We put $M = \Box^2 + P^4$.

Let $\chi(x)$ be a C^∞ function in B such that $0 < \chi$ in B and $\chi(x) = \mathrm{dist}(x, \partial B)$ near ∂B. We put $\psi(x) = \chi(x)^N$ and we take N sufficiently large that $\|\partial^\beta(\psi u)\|_B \leq c\|M\psi u\|_B$ for $|\beta| \leq 4$.

We show that

$$\|\psi^{|\alpha|} \partial^\alpha P^l u\| \leq C_2 A_2^{|\alpha|+l} ((|\alpha| + l)!)^\sigma \tag{23}$$

for some $C_2 > 0$ and $A_2 > 0$ for all $\alpha \geq 0$ and all $l \geq 0$. We show (23) by induction with respect to α.

When $|\alpha| = 0$, (23) is nothing but the assumption (22). We assume that (23) is valid until $|\alpha| = m$.

First we prove the case $0 \leq m \leq 3$. For $|\alpha| = m+1$, we have

$$\|\psi^{|\alpha|}\partial^\alpha P^l u\|_B \leq \|\partial^\alpha \psi^{|\alpha|} P^l u\|_B + \|[\psi^{|\alpha|}, \partial^\alpha] P^l u\|_B. \tag{24}$$

The second term of the right hand side is estimated by

$$\sum_{\alpha' < \alpha} C_3 \|\psi^{|\alpha'|}\partial^{\alpha'} P^l u\| \leq C_4 C_2 A_2^{m+l}((m+l)!)^\sigma \tag{25}$$

$$\leq \frac{1}{2} C_2 A_2^{m+l+1}((m+l)!)^\sigma \tag{26}$$

if we take $A_2 \geq 2C_4$. Since $|\alpha| = m+1 \leq 4$, the first term is estimated by

$$\|M\psi^{|\alpha|}P^l u\|_B \leq \|\psi^{|\alpha|} M P^l u\|_B + \|[M, \psi^{|\alpha|}] P^l u\|_B. \tag{27}$$

The second term of the right hand side of the above inequality can be estimated by

$$C_5 \sum_{|\alpha'| \leq 3} \|\psi^{|\alpha'|}\partial^{\alpha'} P^l u\|_B \leq \frac{1}{4} C_2 A_2^{m+l+1}((m+l+1)!)^\sigma \tag{28}$$

if we take A_2 sufficiently large. The first term is estimated by

$$\|\psi^{|\alpha|}\Box^2 P^l u\|_B + \|\psi^{|\alpha|} P^{l+4} u\|_B. \tag{29}$$

The second term of the right hand side of the above can estimated by

$$\begin{aligned} C_6 \|P^{l+4} u\|_B &\leq C_6 C_2 A_2^{l+4}((l+4)!)^\sigma \\ &\leq \frac{1}{8} C_2 A_2^{l+m+1}((l+m+1)!)^\sigma, \end{aligned}$$

if we take C_2 and A_2 sufficiently large. The first term of the right hand side of the above is estimated by

$$\begin{aligned} \|\psi^{|\alpha|}(P+4)^l \Box^2 u\|_B &= \|\psi^{|\alpha|}(P+4)^l \Box(u^m)\|_B \\ &\leq m\|\psi^{|\alpha|}(P+4)^l u^{2m-1}\| + \binom{m}{2} \sum_{j=0}^{2} \|\psi^{|\alpha|}(P+4)^l u^{m-2}(\partial_j u)^2\|_B. \end{aligned}$$

The second term of the right hand side of the above is esitmated by

$$\begin{aligned} \frac{m(m-1)}{2} \sum_{j=0}^{2} \sum_{\alpha_1 + \cdots + \alpha_m = \alpha} \frac{\alpha!}{\alpha_1! \cdots \alpha_m!} \sum_{l_1 + \cdots + l_m = l} \frac{l!}{l_1! \cdots l_m!} \|\psi^{|\alpha_1|}(P+4)^{l_1} u\|_B \times \\ \|\psi^{|\alpha_2|} P^{l_2} u\|_B \cdots \|\psi^{|\alpha_{m-2}|} P^{l_{m-2}} u\|_B \|\psi^{|\alpha_{m-1}|} P^{l_{m-1}} \partial_j u\|_B \|\psi^{|\alpha_m|} P^{l_m} \partial_j u\|_B. \end{aligned} \tag{30}$$

This can be estimated by $\frac{1}{16}C_2A_2^{l+m+1}((l+m+1)!)^\sigma$ if we take C_2 and A_2 sufficiently large. The first term can be also estimated by $\frac{1}{16}C_2A_2^{l+m+1}((L+m+1)!)^\sigma$.

Next we prove the case $m \geq 4$. For $|\alpha| = m-3$ and $|\beta| = 4$, we have

$$\|\psi^{m+1}\partial^{\alpha+\beta}P^l u\|_B \leq \|\partial^\beta \psi^{m+1}\partial^\alpha P^l u\|_B + \|[\psi^{m+1}, \partial^\beta]\partial^\alpha P^l u\|_B. \tag{31}$$

The second term of the right hand side of the above can be estimated by

$$C_7 \sum_{\beta'<\beta} \|\psi^{m-3+|\beta'|}\partial^{\beta'} P^l u\|_B \leq C_7 C_2 A_2^{m+l}((m+l)!)^\sigma \tag{32}$$

$$\leq \frac{1}{2}C_2 A_2^{m+l+1}((m+l+1)!)^\sigma \tag{33}$$

if we take $A_2 \geq 2C_7$. Since $|\beta| = 4$, the first term is extimated by

$$\|M\psi^{m+1}\partial^\alpha P^l u\|_B \leq \|\psi^{m+1}M\partial^\alpha P^l u\|_B + \|[M, \psi^{m+1}]\partial^\alpha P^l u\|_B. \tag{34}$$

Using the same argument as in the case $m \leq 3$, we can estimate the right hand side of the above by $\frac{1}{2}C_2A_2^{m+l+1}((m+l+1)!)^\sigma$. But we note that we do not change C_2 at each step of induction in the case $m \geq 4$ not as in the case $m \leq 3$.

□

6. Proof of Main result

We devide $K \backslash \Sigma_1 \cup S_2 \cup \Sigma_3 \cup \Gamma_+$ into 4 parts, $\bigcup_{i=1}^4 \mathcal{O}_i$ with

$$\mathcal{O}_1 = \{(t,x) \in R^3; t - \omega_1 \cdot x > 0, t - \omega_2 \cdot x < 0, t - \omega_3 \cdot x < 0\} \cup \cdots \tag{35}$$

$$\mathcal{O}_2 = \{(t,x) \in R^3; t - \omega_1 \cdot x > 0, t - \omega_2 \cdot x > 0, t - \omega_3 \cdot x < 0\} \cup \cdots \tag{36}$$

$$\mathcal{O}_3 = \{(t,x) \in R^3; t - \omega_1 \cdot x > 0, t - \omega_2 \cdot x > 0, t - \omega_3 \cdot x > 0, t^2 - |x|^2 < 0\} \tag{37}$$

$$\mathcal{O}_4 = \{(t,x) \in R^3; t - \omega_1 \cdot x > 0, t - \omega_2 \cdot x > 0, t - \omega_3 \cdot x > 0, t^2 - |x|^2 > 0\}. \tag{38}$$

For x in $\mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}_3$, the backward light cone Γ_x^- from x does not contain the origin. So we can prove that u is in $G^{(\sigma)}$ in this area by the same argument as in the proof of Theorems 1.2 and 1.3.

To prove that u is in $G^{(\sigma)}$ in $\mathcal{O}_4$, we use the operator $P = t\partial_t + x \cdot \partial_x$. Using this operator, M. Beals[1] has given another proof of the theorem 1.1 of Bony and Melrose–Ritter. Note that for all relatively compact open set $L \subset \Omega_-$, there exist constants $C, A_1 > 0$ such that

$$\|P^k u\|_{H^s(L)} \leq CA_1^k(k!)^\sigma \quad \text{for} \forall k, \tag{39}$$

from the assumptions of Theorem 1.2. Since $[\Box, P] = 2\Box$,

$$\Box(Pu) = P\Box u + [\Box, P]u = (P+2)f(u). \tag{40}$$

So we have

$$\Box(P^k u) = (P+2)^k f(u). \tag{41}$$

Using the energy inequality 3.2, we have for all relatively compact open set $L \subset K$, there exist constants $C, A > 0$ such that

$$\|P^k u\|_{H^s(L)} \leq CA^k (k!)^\sigma \quad \text{for } \forall k. \tag{42}$$

From Lemma 5.1, we have that u is in $G^{(\sigma)}$ in $\mathcal{O}_4$.

References

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