

A HAMILTONIAN PATH INTEGRAL FOR A DEGENERATE PARABOLIC PSEUDO-DIFFERENTIAL OPERATOR

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ABSTRACT. In this paper, using a Hamiltonian path integral, we give an expression of the symbol of the fundamental solution for a degenerate parabolic pseudo-differential operator. This Hamiltonian path integral converges in the topology of the symbol class $S_{\lambda,\rho,\delta}^{2m}$ and in the weak topology of the symbol class $S_{\lambda,\rho,\delta}^0$.

0. Introduction

In this paper, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator in a different way from that in C.Tsutsumi [10]. In [10], she constructed the fundamental solution by Levi-Mizohata method. On the other hand, in this paper, we construct the fundamental solution by a Hamiltonian path integral. If we use a Hamiltonian path integral, we can actually write the symbol of the fundamental solution. Furthermore, this Hamiltonian path integral converges in the topology of the symbol class $S_{\lambda,\rho,\delta}^{2m}$ and in the weak topology of the symbol class $S_{\lambda,\rho,\delta}^0$.

In Section 1, we introduce some basic properties of pseudo-differential operators, which we use in Section 2. For the details, see Chapter 7 § 1 and § 2 in H.Kumano-go [6]. In Section 2, we construct the fundamental solution for a degenerate parabolic pseudo-differential operator by a Hamiltonian path integral. Theorem 2.1 is the main theorem in this paper.

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1. Pseudo-Differential Operators

For $x = (x_1, \dots, x_n) \in \mathbf{R}_x^n$, $\xi = (\xi_1, \dots, \xi_n) \in \mathbf{R}_\xi^n$ and multi-indices of non-negative integers $\alpha = (\alpha_1, \dots, \alpha_n)$, $\beta = (\beta_1, \dots, \beta_n)$, we employ the usual notation:

$$|\alpha| = \alpha_1 + \dots + \alpha_n, \quad |\beta| = \beta_1 + \dots + \beta_n,$$

$$\alpha! = \alpha_1! \dots \alpha_n!, \quad \beta! = \beta_1! \dots \beta_n!,$$

$$x \cdot \xi = x_1 \xi_1 + \dots + x_n \xi_n, \quad \langle x \rangle = (1 + |x|^2)^{1/2}, \quad \langle \xi \rangle = (1 + |\xi|^2)^{1/2},$$

$$\partial_{\xi_j} = \frac{\partial}{\partial \xi_j}, \quad D_{x_j} = -i \frac{\partial}{\partial x_j}, \quad \partial_\xi^\alpha = \partial_{\xi_1}^{\alpha_1} \dots \partial_{\xi_n}^{\alpha_n}, \quad D_x^\beta = D_{x_1}^{\beta_1} \dots D_{x_n}^{\beta_n}.$$

\mathcal{S} denotes the Schwartz space of rapidly decreasing C^∞ -functions on \mathbf{R}^n . For $u \in \mathcal{S}$, we define semi-norms $|u|_{l, \mathcal{S}}$ ($l = 0, 1, 2, \dots$) by

$$|u|_{l, \mathcal{S}} \equiv \max_{k+|\alpha| \leq l} \sup_x |\langle x \rangle^k \partial_x^\alpha u(x)| \quad (l = 0, 1, 2, \dots).$$

Then, \mathcal{S} is a Fréchet space with these semi-norms.

For simplicity, we set $\bar{d}\eta \equiv (2\pi)^{-n} d\eta$ and $\bar{d}\xi \equiv (2\pi)^{-n} d\xi$.

Oscillatory integral of a function $a(\eta, y)$, is defined by the equality

$$\text{O}_s \text{---} \iint e^{-iy \cdot \eta} a(\eta, y) dy \bar{d}\eta \equiv \lim_{\epsilon \rightarrow 0} \iint e^{-iy \cdot \eta} \chi(\epsilon \eta, \epsilon y) a(\eta, y) dy \bar{d}\eta,$$

where $\chi(\eta, y) \in \mathcal{S}$ in $\mathbf{R}_{\eta, y}^{2n}$ and $\chi(0, 0) = 1$. For the details, see Chapter 1 § 6 in H.Kumano-go [6].

Definition 1.1 (A weight function $\lambda(\xi)$).

We say that a real-valued C^∞ -function $\lambda(\xi)$ on \mathbf{R}_ξ^n is a weight function, if there exist constants $A_0, A_\alpha > 0$ such that

$$1 \leq \lambda(\xi) \leq A_0 \langle \xi \rangle, \quad (1.1)$$

$$|\partial_\xi^\alpha \lambda(\xi)| \leq A_\alpha \lambda(\xi)^{1-|\alpha|}. \quad (1.2)$$

Examples.

$$1^\circ \lambda(\xi) = \langle \xi \rangle.$$

$$2^\circ \lambda(\xi) = \left\{ 1 + \sum_{j=1}^n |\xi_j|^{2m_j} \right\}^{1/(2m)}, \quad (m_j \in \mathbb{N}, \quad m \equiv \max_{1 \leq j \leq n} \{m_j\}).$$

Definition 1.2 (Pseudo-differential operators $S_{\lambda, \rho, \delta}^m$).

We say that a C^∞ -function $p(x, \xi)$ on $\mathbf{R}_{x, \xi}^{2n}$ is a symbol of class $S_{\lambda, \rho, \delta}^m$ ($m \in \mathbf{R}$, $0 \leq \delta \leq \rho \leq 1$), if for any α, β , there exists a constant $C_{\alpha, \beta}$ such that

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha, \beta} \lambda(\xi)^{m + \delta|\beta| - \rho|\alpha|}, \quad (1.3)$$

where $p_{(\beta)}^{(\alpha)}(x, \xi) \equiv \partial_\xi^\alpha D_x^\beta p(x, \xi)$.

The pseudo-differential operator $p(X, D_x)$ with the symbol $p(x, \xi)$ is defined by

$$p(X, D_x)u(x) \equiv \iint e^{i(x-x') \cdot \xi} p(x, \xi) u(x') dx' d\xi \quad (u \in \mathcal{S}), \quad (1.4)$$

where $d\xi \equiv (2\pi)^{-n} d\xi$.

Remark.

1° For simplicity, we set $p_{(\beta)}^{(\alpha)}(x, \xi) \equiv \partial_\xi^\alpha D_x^\beta p(x, \xi)$, $p^{(\alpha)}(x, \xi) \equiv \partial_\xi^\alpha p(x, \xi)$ and

$$p_{(\beta)}(x, \xi) \equiv D_x^\beta p(x, \xi) \text{ for any } \alpha, \beta.$$

2° The symbol class $S_{\lambda, \rho, \delta}^m$ is a Fréchet space with the semi-norms

$$|p|_l^{(m)} \equiv \max_{|\alpha + \beta| \leq l} \sup_{(x, \xi)} \{ |p_{(\beta)}^{(\alpha)}(x, \xi)| \lambda(\xi)^{-(m + \delta|\beta| - \rho|\alpha|)} \} \quad (l = 0, 1, 2, \dots). \quad (1.5)$$

3° The continuity of $p(X, D_x) : \mathcal{S} \rightarrow \mathcal{S}$ is clear. Furthermore, we can extend

$p(X, D_x) : \mathcal{S} \rightarrow \mathcal{S}$ to $p(X, D_x) : \mathcal{S}' \rightarrow \mathcal{S}'$ by means of

$$(p(X, D_x)u, v) \equiv (u, p(X, D_x)^*v) \text{ for } u \in \mathcal{S}', v \in \mathcal{S}. \quad (1.6)$$

Theorem 1.3 (Multi-products).

Let M be a positive constant and let $\{m_j\}_{j=1}^{\infty}$ be a sequence of real numbers satisfying

$$\sum_{j=1}^{\infty} |m_j| \leq M < \infty. \quad (1.7)$$

For any $\nu = 1, 2, \dots$ and $p_j(x, \xi) \in S_{\lambda, \rho, \delta}^{m_j}$ ($j = 1, 2, \dots, \nu + 1$), there exists

$q_{\nu+1}(x, \xi) \in S_{\lambda, \rho, \delta}^{\bar{m}_{\nu+1}}$ ($\bar{m}_{\nu+1} \equiv m_1 + m_2 + \dots + m_{\nu+1}$) such that

$$q_{\nu+1}(X, D_x) = p_1(X, D_x)p_2(X, D_x) \cdots p_{\nu+1}(X, D_x). \quad (1.8)$$

Furthermore, for any l , there exist a constant A_l and an integer l' such that

$$|q_{\nu+1}|_l^{(\bar{m}_{\nu+1})} \leq (A_l)^{\nu} \prod_{j=1}^{\nu+1} |p_j|_{l'}^{(m_j)}, \quad (1.9)$$

where A_l and l' depend only on M and l , but are independent of ν .

Proof. See Theorem 2.4 in Chapter 7 § 2 of H.Kumano-go [6]. \square

Theorem 1.4.

Let $p_j(x, \xi) \in S_{\lambda, \rho, \delta}^{m_j}$ ($j = 1, 2$). Define $q_{\theta}(x, \xi)$ ($|\theta| \leq 1$) by

$$q_{\theta}(x, \xi) \equiv O_s - \iint e^{-iy \cdot \eta} p_1(x, \xi + \theta \eta) p_2(x + y, \xi) dy d\eta. \quad (1.10)$$

Then $\{q_{\theta}(x, \xi)\}_{|\theta| \leq 1}$ is a bounded set of $S_{\lambda, \rho, \delta}^{m_1+m_2}$. Furthermore, for any l , there exist a constant A'_l and an integer l' independent of θ such that

$$|q_{\theta}|_l^{(m_1+m_2)} \leq A'_l |p_1|_{l'}^{(m_1)} |p_2|_{l'}^{(m_2)}. \quad (1.11)$$

Proof. See Lemma 2.4 in Chapter 2 § 2 or Lemma 2.2 in Chapter 7 §2 of H.Kumano-go [6]. \square

2. The Main Theorem

Theorem 2.1 (The main theorem).

Let $K(t, x, \xi) \in \mathcal{B}^0([0, T]; \mathcal{S}_{\lambda, \rho, \delta}^m)$ ($m > 0, 0 \leq \delta < \rho \leq 1$). Assume that $K(t, x, \xi)$ satisfies the following conditions (a1), (a2) :

(a1) There exist constants $c > 0$ and $m' (0 \leq m' \leq m)$ such that

$$\operatorname{Re} K(t, x, \xi) \leq -c\lambda(\xi)^{m'} \text{ on } [0, T] \times \mathbf{R}_{x, \xi}^{2n}. \quad (2.1)$$

(a2) For any α, β , there exists a constant $C_{\alpha, \beta}$ such that

$$|K_{(\beta)}^{(\alpha)}(t, x, \xi) / \operatorname{Re} K(t, x, \xi)| \leq C_{\alpha, \beta} \lambda(\xi)^{\delta|\beta| - \rho|\alpha|} \text{ on } [0, T] \times \mathbf{R}_{x, \xi}^{2n}. \quad (2.2)$$

Then we have the following (1) – (5) :

(1) Let $\Delta_{t, s} : (T \geq) t \equiv t_0 \geq t_1 \geq \dots \geq t_\nu \geq t_{\nu+1} \equiv s (\geq 0)$ be an arbitrary division of interval $[s, t]$ into subintervals, and let $e^{(t_j - t_{j+1})K(t_{j+1})}(X, D_x)$ be an operator defined by

$$e^{(t_j - t_{j+1})K(t_{j+1})}(X, D_x)u(x) \equiv \iint e^{i(x-x') \cdot \xi} e^{(t_j - t_{j+1})K(t_{j+1}, x, \xi)} u(x') dx' d\xi. \quad (2.3)$$

Then there exists $p(\Delta_{t, s}; x, \xi) \in \mathcal{S}_{\lambda, \rho, \delta}^0$ such that

$$p(\Delta_{t, s}; X, D_x) = e^{(t-t_1)K(t_1)}(X, D_x) e^{(t_1-t_2)K(t_2)}(X, D_x) \dots e^{(t_\nu-s)K(s)}(X, D_x). \quad (2.4)$$

(2) There exist constants C_l, C'_l and an integer l' such that

$$|p(\Delta_{t, s})|_l^{(0)} \leq C_l, \quad (2.5)$$

and

$$\begin{aligned} & |p(\Delta_{t, s}) - p(\Delta'_{t, s})|_l^{(2m)} \\ & \leq C'_l (t-s) \left(|\Delta_{t, s}| + \sup_{|t' - t''| \leq |\Delta_{t, s}|} |K(t') - K(t'')|_{l'}^{(m)} \right). \end{aligned} \quad (2.6)$$

Here, $\Delta_{t, s} : (T \geq) t \equiv t_0 \geq t_1 \geq \dots \geq t_\nu \geq t_{\nu+1} \equiv s (\geq 0)$ is an arbitrary division of interval $[s, t]$ into subintervals, $\Delta'_{t, s}$ is an arbitrary refinement of $\Delta_{t, s}$,

$|\Delta_{t,s}|$ denotes the size of division defined by $|\Delta_{t,s}| \equiv \max_{0 \leq j \leq \nu} |t_j - t_{j+1}|$, and the constants C_l, C'_l and the integer l' are independent of $\nu, \Delta_{t,s}$ and $\Delta'_{t,s}$.

- (3) There exists $p^*(t, s; x, \xi) \in \mathcal{S}_{\lambda, \rho, \delta}^0$ such that $p(\Delta_{t,s}; x, \xi) (\in \mathcal{S}_{\lambda, \rho, \delta}^0)$ converges to $p^*(t, s; x, \xi) (\in \mathcal{S}_{\lambda, \rho, \delta}^0)$ in $\mathcal{S}_{\lambda, \rho, \delta}^{2m}$ as $|\Delta_{t,s}|$ tends to 0.

Furthermore, $p^*(t, s; x, \xi)$ has the following expression:

$$p^*(t, s; x, \xi) = \lim_{|\Delta_{t,s}| \rightarrow 0} O_s - \iint \cdots \iint e^{-i \sum_{j=1}^{\nu} y^j \cdot \eta^j} \times \exp \left(\sum_{j=0}^{\nu} (t_j - t_{j+1}) K(t_{j+1}, x + \bar{y}^j, \xi + \eta^{j+1}) \right) dy^1 d\eta^1 \cdots dy^{\nu} d\eta^{\nu}, \quad (2.7)$$

where $\bar{y}^0 \equiv 0, \bar{y}^j \equiv y^1 + y^2 + \cdots + y^j$, and $\eta^{\nu+1} \equiv 0$.

- (4) For $u \in L^2$, the pseudo-differential operator $U(t, s) \equiv p^*(t, s; X, D_x)$ satisfies the following relation:

$$\begin{aligned} & U(t, s)u(x) \\ &= \lim_{|\Delta_{t,s}| \rightarrow 0} e^{(t-t_1)K(t_1)}(X, D_x) e^{(t_1-t_2)K(t_2)}(X, D_x) \cdots e^{(t_{\nu}-s)K(s)}(X, D_x)u(x) \\ &= \lim_{|\Delta_{t,s}| \rightarrow 0} \iint \cdots \iint \exp \left(\sum_{j=0}^{\nu} i(x^j - x^{j+1}) \cdot \xi^{j+1} + (t_j - t_{j+1})K(t_{j+1}, x^j, \xi^{j+1}) \right) \\ & \quad \times u(x^{\nu+1}) dx^{\nu+1} d\xi^{\nu+1} \cdots dx^1 d\xi^1, \end{aligned} \quad (2.8)$$

in L^2 where $x^0 \equiv x$.

- (5) $U(t, s) \equiv p^*(t, s; X, D_x)$ is the fundamental solution for the operator

$L \equiv \partial_t - K(t, X, D_x)$ such that

$$\begin{cases} LU(t, s) = 0 & \text{on } [s, T] \\ U(s, s) = I & (0 \leq s \leq T). \end{cases} \quad (2.9)$$

Remark.

1° It is sufficient to satisfy the conditions (a1) and (a2) for $|\xi| \geq M$, with a constant $M \geq 0$. In fact, in this case, there exists a sufficiently large $R > 0$ such that the symbol $K_R(t, x, \xi) \equiv K(t, x, \xi) - R$ satisfies (a1) and (a2) for any ξ . Let $U_R(t, s)$ be the fundamental solution of $L_R \equiv \partial_t - K_R(t, X, D_x)$. Then $U(t, s) \equiv e^{(t-s)R}U_R(t, s)$ is the fundamental solution of L .

2° We can replace $(t_j - t_{j+1})K(t_{j+1}, \cdot, \cdot)$ with $\int_{t_{j+1}}^{t_j} K(\tau, \cdot, \cdot) d\tau$. Furthermore, in this case, we can replace (2.6) with

$$|p(\Delta_{t,s}) - p(\Delta'_{t,s})|_l^{(2m)} \leq C'_l(t-s)|\Delta_{t,s}|, \quad (2.6')$$

and the proof of Theorem 2.1 becomes a little easier.

Example.

Consider

$$L \equiv \partial_t + a(t)|x|^{2l}(-\Delta)^m + (-\Delta)^{m'} \quad (0 \leq a(t) \in \mathcal{C}[0, T], m - m' < l).$$

If we set $\rho = 1$, $\delta = (m - m')/l$, $m \rightarrow 2m$ and $m' \rightarrow 2m'$, then the symbol $a(t)|x|^{2l}|\xi|^{2m} + |\xi|^{2m'}$ satisfies the conditions (a1) and (a2). Therefore, we see that these conditions are satisfied not only by the usual parabolic operators, but also by parabolic operators of a degenerate type.

Before we prove Theorem 2.1, we prepare some lemmas:

To begin with, for $T \geq t \geq s \geq 0$, we define $p(t, s; x, \xi)$ by

$$p(t, s; x, \xi) \equiv \exp\left((t - s)K(s, x, \xi)\right). \quad (2.10)$$

The next lemma is a generalization of asymptotic expansion formulas, and an essential part in this paper. Especially, it is important that all constants are independent of $\Delta_{t_0, t_{\nu+1}}$ and ν .

Lemma 2.2 (Key Lemma).

Let $\Delta_{t_0, t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \dots \geq t_\nu \geq t_{\nu+1} (\geq 0)$, $\nu = 1, 2, \dots$, and let N_0 be a fixed positive integer such that $(\rho - \delta)N_0 \geq 2m$. Define $q(\Delta_{t_0, t_1}; x, \xi)$, $q(\Delta_{t_0, t_{\nu+1}}; x, \xi)$, and $r(\Delta_{t_0, t_{\nu+1}}; x, \xi)$ respectively by

$$q(\Delta_{t_0, t_1}; x, \xi) \equiv p(t_0, t_1; x, \xi), \quad (2.11)$$

$$\begin{aligned} q(\Delta_{t_0, t_{\nu+1}}; x, \xi) \equiv & \sum_{|\alpha^1| + |\alpha^2| + \dots + |\alpha^\nu| < N_0} \frac{1}{\alpha^1! \alpha^2! \dots \alpha^\nu!} \\ & \times p_{(\alpha^\nu)}(t_\nu, t_{\nu+1}; x, \xi) \partial_\xi^{\alpha^\nu} \left(p_{(\alpha^{\nu-1})}(t_{\nu-1}, t_\nu; x, \xi) \partial_\xi^{\alpha^{\nu-1}} \left(\right. \right. \\ & \left. \left. \dots p_{(\alpha^2)}(t_2, t_3; x, \xi) \partial_\xi^{\alpha^2} \left(p_{(\alpha^1)}(t_1, t_2; x, \xi) \partial_\xi^{\alpha^1} \left(p(t_0, t_1; x, \xi) \right) \right) \dots \right) \right). \end{aligned} \quad (2.12)$$

and

$$\begin{aligned}
r(\Delta_{t_0, t_{\nu+1}}; x, \xi) &\equiv \sum_{|\alpha^1| + |\alpha^2| + \dots + |\alpha^\nu| = N_0, |\alpha^\nu| \neq 0} \frac{|\alpha^\nu|}{\alpha^1! \alpha^2! \dots \alpha^\nu!} \\
&\times \int_0^1 (1-\theta)^{|\alpha^\nu| - 1} O_s - \iint e^{-iy \cdot \eta} p_{(\alpha^\nu)}(t_\nu, t_{\nu+1}; x + y, \xi) \\
&\times \partial_\xi^{\alpha^\nu} \left(p_{(\alpha_{\nu-1})}(t_{\nu-1}, t_\nu; x, \xi + \theta\eta) \partial_\xi^{\alpha^{\nu-1}} \left(\dots p_{(\alpha^2)}(t_2, t_3; x, \xi + \theta\eta) \right. \right. \\
&\times \left. \left. \partial_\xi^{\alpha^2} \left(p_{(\alpha^1)}(t_1, t_2; x, \xi + \theta\eta) \partial_\xi^{\alpha^1} \left(p(t_0, t_1; x, \xi + \theta\eta) \right) \right) \dots \right) \right) dy d\eta d\theta.
\end{aligned} \tag{2.13}$$

Then it follows that

$$\begin{aligned}
&q(\Delta_{t_0, t_\nu}; X, D_x) p(t_\nu, t_{\nu+1}; X, D_x) \\
&= q(\Delta_{t_0, t_{\nu+1}}; X, D_x) + r(\Delta_{t_0, t_{\nu+1}}; X, D_x).
\end{aligned} \tag{2.14}$$

Furthermore, there exist constants $C_{1,l}, C_{2,l}, C_{3,l}$ such that

$$|q(\Delta_{t_0, t_\nu})|_l^{(0)} \leq C_{1,l}, \tag{2.15}$$

$$\begin{aligned}
&|q(\Delta_{t_0, t_{\nu+1}}) - p(t_0, t_{\nu+1})|_l^{(2m)} \\
&\leq C_{2,l}(t_0 - t_{\nu+1}) \left((t_0 - t_{\nu+1}) + \sup_{t_0 \geq t' \geq t'' \geq t_{\nu+1}} |K(t') - K(t'')|_l^{(m)} \right),
\end{aligned} \tag{2.16}$$

and

$$|r(\Delta_{t_0, t_{\nu+1}})|_l^{(0)} \leq C_{3,l}(t_0 - t_\nu)(t_\nu - t_{\nu+1}), \tag{2.17}$$

for any $\Delta_{t_0, t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \dots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1, 2, \dots$

Proof.

1° For $T \geq t \geq s \geq 0$, we set

$$\eta(t, s; x, \xi) \equiv -(t-s) \operatorname{Re} K(s, x, \xi) (\geq 0). \tag{2.18}$$

Furthermore, for $\Delta_{t_0, t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \dots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1, 2, \dots$, we define $d(\Delta_{t_0, t_\nu}; x, \xi)$ by

$$d(\Delta_{t_0, t_\nu}; x, \xi) \equiv \prod_{j=0}^{\nu-1} p(t_j, t_{j+1}; x, \xi), \tag{2.19}$$

and we set

$$\eta(\Delta_{t_0, t_\nu}; x, \xi) \equiv \sum_{j=0}^{\nu-1} \eta(t_j, t_{j+1}; x, \xi). \quad (2.20)$$

Clearly, we have

$$|d(\Delta_{t_0, t_\nu}; x, \xi)| = \exp\left(-\eta(\Delta_{t_0, t_\nu}; x, \xi)\right). \quad (2.21)$$

2° Define $d_{\alpha, \beta}(\Delta_{t_0, t_\nu}; x, \xi)$ by

$$d_{(\beta)}^{(\alpha)}(\Delta_{t_0, t_\nu}; x, \xi) \equiv d_{\alpha, \beta}(\Delta_{t_0, t_\nu}; x, \xi) d(\Delta_{t_0, t_\nu}; x, \xi). \quad (2.22)$$

Then, by induction, for any α, β ($|\alpha + \beta| \geq 1$) and α', β' , there exists a constant $C_{\alpha, \beta, \alpha', \beta'}$ such that

$$\begin{aligned} |d_{\alpha, \beta}^{(\alpha')}(\Delta_{t_0, t_\nu}; x, \xi)| &\leq C_{\alpha, \beta, \alpha', \beta'} \eta(\Delta_{t_0, t_\nu}; x, \xi) \left(\eta(\Delta_{t_0, t_\nu}; x, \xi) + 1\right)^{|\alpha + \beta| - 1} \\ &\quad \times \lambda(\xi)^{\delta|\beta + \beta'| - \rho|\alpha + \alpha'|}, \end{aligned} \quad (2.23)$$

for any $\Delta_{t_0, t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \dots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1, 2, \dots$

3° Let $\tilde{\alpha}^\nu \equiv (\alpha^1, \dots, \alpha^\nu)$ denote a multi-index of $\mathbf{R}^{\nu n}$. Define $f_{\tilde{\alpha}^\nu}(\Delta_{t_0, t_{\nu+1}}; x, \xi)$ by

$$\begin{aligned} &f_{\tilde{\alpha}^\nu}(\Delta_{t_0, t_{\nu+1}}; x, \xi) d(\Delta_{t_0, t_{\nu+1}}; x, \xi) \\ &\equiv p_{(\alpha^\nu)}(t_\nu, t_{\nu+1}; x, \xi) \partial_\xi^{\alpha^\nu} \left(p_{(\alpha^{\nu-1})}(t_{\nu-1}, t_\nu; x, \xi) \partial_\xi^{\alpha^{\nu-1}} \left(\right. \right. \\ &\quad \left. \left. \dots p_{(\alpha^2)}(t_2, t_3; x, \xi) \partial_\xi^{\alpha^2} \left(p_{(\alpha^1)}(t_1, t_2; x, \xi) \partial_\xi^{\alpha^1} \left(p(t_0, t_1; x, \xi) \right) \right) \dots \right) \right). \end{aligned} \quad (2.24)$$

Then, by induction, for any $N = 1, 2, \dots$ and α, β , there exists a constant $C_{N, \alpha, \beta}$ such that

$$\begin{aligned} |f_{\tilde{\alpha}^\nu}^{(\alpha)}(\Delta_{t_0, t_{\nu+1}}; x, \xi)| &\leq C_{N, \alpha, \beta} \left(\prod_{k=1}^J \eta(t_{j_k}, t_{j_k+1}; x, \xi) \right) \eta(\Delta_{t_0, t_{\nu+1}}; x, \xi) \\ &\quad \times \left(\eta(\Delta_{t_0, t_{\nu+1}}; x, \xi) + 1 \right)^{2(N-1)} \lambda(\xi)^{-(\rho-\delta)N + \delta|\beta| - \rho|\alpha|}, \end{aligned} \quad (2.25)$$

where

$$1 \leq j_1 < j_2 < \dots < j_J \leq \nu, \quad |\alpha^{j_k}| \neq 0 \quad (k = 1, 2, \dots, J),$$

and

$$\sum_{j=1}^{\nu} |\alpha^j| = \sum_{k=1}^J |\alpha^{j_k}| = N,$$

for any $\Delta_{t_0, t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \dots \geq t_{\nu} \geq t_{\nu+1} (\geq 0)$ and $\nu = 1, 2, \dots$

4° For $N = 1, 2, \dots$, define $g_N(\Delta_{t_0, t_{\nu+1}}; x, \xi)$ by

$$g_N(\Delta_{t_0, t_{\nu+1}}; x, \xi) \equiv \sum_{|\alpha^1| + |\alpha^2| + \dots + |\alpha^{\nu}| = N} \frac{1}{\alpha^1! \alpha^2! \dots \alpha^{\nu}!} f_{\tilde{\alpha}^{\nu}}(\Delta_{t_0, t_{\nu+1}}; x, \xi). \quad (2.26)$$

By (2.25), we have

$$\begin{aligned} & |g_N^{(\alpha)}(\Delta_{t_0, t_{\nu+1}}; x, \xi)| \\ & \leq \sum_{J=1}^N \sum_{1 \leq j_1 < j_2 < \dots < j_J \leq \nu} \sum_{\sum_{k=1}^J |\alpha^{j_k}| = N, |\alpha^{j_k}| \neq 0} \frac{1}{\alpha^{j_1}! \alpha^{j_2}! \dots \alpha^{j_J}!} \\ & \quad \times C_{N, \alpha, \beta} \left(\prod_{k=1}^J \eta(t_{j_k}, t_{j_k+1}; x, \xi) \right) \eta(\Delta_{t_0, t_{\nu+1}}; x, \xi) \\ & \quad \times \left(\eta(\Delta_{t_0, t_{\nu+1}}; x, \xi) + 1 \right)^{2(N-1)} \lambda(\xi)^{-(\rho-\delta)N + \delta|\beta| - \rho|\alpha|} \\ & \leq (nN)^N C_{N, \alpha, \beta} \eta(\Delta_{t_0, t_{\nu+1}}; x, \xi) \left(\eta(\Delta_{t_0, t_{\nu+1}}; x, \xi) + 1 \right)^{2(N-1)} \lambda(\xi)^{-(\rho-\delta)N + \delta|\beta| - \rho|\alpha|} \\ & \quad \times \left(\sum_{J=1}^N \sum_{1 \leq j_1 < j_2 < \dots < j_J \leq \nu} \prod_{k=1}^J \eta(t_{j_k}, t_{j_k+1}; x, \xi) \right). \end{aligned} \quad (2.27)$$

Hence, for any $N = 1, 2, \dots$ and α, β , there exists a constant $C'_{N, \alpha, \beta}$ such that

$$\begin{aligned} |g_N^{(\alpha)}(\Delta_{t_0, t_{\nu+1}}; x, \xi)| & \leq C'_{N, \alpha, \beta} \left(\eta(\Delta_{t_0, t_{\nu+1}}; x, \xi) \right)^2 \\ & \quad \times \left(\eta(\Delta_{t_0, t_{\nu+1}}; x, \xi) + 1 \right)^{3(N-1)} \lambda(\xi)^{-(\rho-\delta)N + \delta|\beta| - \rho|\alpha|}, \end{aligned} \quad (2.28)$$

for any $\Delta_{t_0, t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \dots \geq t_{\nu} \geq t_{\nu+1} (\geq 0)$ and $\nu = 1, 2, \dots$

5° Set

$$h_N(\Delta_{t_0, t_{\nu+1}}; x, \xi) \equiv g_N(\Delta_{t_0, t_{\nu+1}}; x, \xi) d(\Delta_{t_0, t_{\nu+1}}; x, \xi). \quad (2.29)$$

Here we note that

$$\sup_{\eta > 0} \eta^k e^{-\eta} < \infty \quad (k = 0, 1, 2, \dots). \quad (2.30)$$

By (2.21), (2.23) and (2.28), there exist constants $C'_{\alpha,\beta}$, $C''_{\alpha,\beta}$, $C''_{N,\alpha,\beta}$, $C'''_{N,\alpha,\beta}$, $C''''_{N,\alpha,\beta}$ such that

$$|d_{(\beta)}^{(\alpha)}(\Delta_{t_0,t_\nu}; x, \xi)| \leq \begin{cases} C'_{\alpha,\beta} \lambda(\xi)^{\delta|\beta|-\rho|\alpha|} \\ C''_{\alpha,\beta} (t_0 - t_\nu) \lambda(\xi)^{m+\delta|\beta|-\rho|\alpha|} \quad (|\alpha + \beta| \geq 1), \end{cases} \quad (2.31)$$

and

$$|h_N^{(\alpha)}(\Delta_{t_0,t_{\nu+1}}; x, \xi)| \leq \begin{cases} C''_{N,\alpha,\beta} \lambda(\xi)^{-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\ C'''_{N,\alpha,\beta} (t_0 - t_{\nu+1}) \lambda(\xi)^{m-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|} \\ C''''_{N,\alpha,\beta} (t_0 - t_{\nu+1})^2 \lambda(\xi)^{2m-(\rho-\delta)N+\delta|\beta|-\rho|\alpha|}, \end{cases} \quad (2.32)$$

for any $\Delta_{t_0,t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \dots \geq t_\nu \geq t_{\nu+1} (\geq 0)$ and $\nu = 1, 2, \dots$

6° Now we note that

$$q(\Delta_{t_0,t_{\nu+1}}; x, \xi) = d(\Delta_{t_0,t_{\nu+1}}; x, \xi) + \sum_{N=1}^{N_0-1} h_N(\Delta_{t_0,t_{\nu+1}}; x, \xi), \quad (2.33)$$

and

$$\begin{aligned} & d(\Delta_{t_0,t_{\nu+1}}; x, \xi) - p(t_0, t_{\nu+1}; x, \xi) \\ &= \sum_{j=0}^{\nu} (t_j - t_{j+1}) \left(K(t_{j+1}, x, \xi) - K(t_{\nu+1}, x, \xi) \right) \\ & \times \int_0^1 \exp \left(\theta \sum_{j=0}^{\nu} (t_j - t_{j+1}) K(t_{j+1}, x, \xi) \right) \exp \left((1 - \theta)(t_0 - t_{\nu+1}) K(t_{\nu+1}, x, \xi) \right) d\theta. \end{aligned} \quad (2.34)$$

By (2.31) and (2.32), we get (2.15) and (2.16). Furthermore, we note that

$$\begin{aligned} r(\Delta_{t_0,t_{\nu+2}}; x, \xi) &= \sum_{0 < |\alpha^{\nu+1}| < N_0} \frac{|\alpha^{\nu+1}|}{\alpha^{\nu+1}!} \int_0^1 (1 - \theta)^{|\alpha^{\nu+1}|-1} \\ & \times O_s - \iint e^{-iy \cdot \eta} h_{N_0-|\alpha^{\nu+1}|}^{(\alpha^{\nu+1})}(\Delta_{t_0,t_{\nu+1}}; x, \xi + \theta\eta) \\ & \times d_{(\alpha^{\nu+1})}(\Delta_{t_{\nu+1},t_{\nu+2}}; x + y, \xi) dy d\eta d\theta \\ & + \sum_{|\alpha^{\nu+1}|=N_0} \frac{|\alpha^{\nu+1}|}{\alpha^{\nu+1}!} \int_0^1 (1 - \theta)^{|\alpha^{\nu+1}|-1} \\ & \times O_s - \iint e^{-iy \cdot \eta} d^{(\alpha^{\nu+1})}(\Delta_{t_0,t_{\nu+1}}; x, \xi + \theta\eta) \\ & \times d_{(\alpha^{\nu+1})}(\Delta_{t_{\nu+1},t_{\nu+2}}; x + y, \xi) dy d\eta d\theta. \end{aligned} \quad (2.35)$$

By (2.31), (2.32) and Theorem 1.4, we get (2.17).

7° By induction, we get (2.14). \square

The idea of the next lemma is found in Fujiwara [3].

Lemma 2.3 (Fujiwara's Skip).

Define $\Upsilon(\Delta_{t_0, t_{\nu+1}}; x, \xi) \in \mathcal{S}_{\lambda, \rho, \delta}^0$ by

$$\begin{aligned} & p(t_0, t_1; X, D_x)p(t_1, t_2; X, D_x) \cdots p(t_{\nu}, t_{\nu+1}; X, D_x) \\ & \equiv q(\Delta_{t_0, t_{\nu+1}}; X, D_x) + \Upsilon(\Delta_{t_0, t_{\nu+1}}; X, D_x). \end{aligned} \quad (2.36)$$

Then it follows that

$$\begin{aligned} \Upsilon(\Delta_{t_0, t_{\nu+1}}; X, D_x) &= \sum^I r(\Delta_{t_0, t_{j_1+1}}; X, D_x)r(\Delta_{t_{j_1+1}, t_{j_2+1}}; X, D_x) \\ & \quad \cdots r(\Delta_{t_{j_{J-1}+1}, t_{j_J+1}}; X, D_x)q(\Delta_{t_{j_J+1}, t_{\nu+1}}; X, D_x), \end{aligned} \quad (2.37)$$

where \sum^I stands for the summation with respect to the sequences of integers (j_1, j_2, \dots, j_J) with the property

$$0 < j_1 < j_1 + 1 < j_2 < j_2 + 1 < \cdots < j_{J-1} < j_{J-1} + 1 < j_J \leq \nu, \quad (2.38)$$

and, in the special case of $j_J = \nu$, we set $q(\Delta_{t_{j_J+1}, t_{\nu+1}}; X, D_x) \equiv I$.

Furthermore, there exists a constant $C_{4,l}$ such that

$$|\Upsilon(\Delta_{t_0, t_{\nu+1}})|_l^{(0)} \leq C_{4,l}(t_0 - t_{\nu+1})^2, \quad (2.39)$$

for any $\Delta_{t_0, t_{\nu+1}} : (T \geq) t_0 \geq t_1 \geq \cdots \geq t_{\nu} \geq t_{\nu+1} (\geq 0)$ and $\nu = 1, 2, \dots$

Proof. Using (2.14) inductively, we get (2.37). Now let A_l, l' be the same constants in Theorem 1.3, and let $C_{1,l}, C_{3,l}$ be the same constants in Lemma 2.2. By (2.15), (2.17) and Theorem 1.3, we have

$$\begin{aligned} |\Upsilon(\Delta_{t_0, t_{\nu+1}})|_l^{(0)} &\leq \sum^I (A_l)^J |r(\Delta_{t_0, t_{j_1+1}})|_{l'}^{(0)} |r(\Delta_{t_{j_1+1}, t_{j_2+1}})|_{l'}^{(0)} \\ & \quad \cdots |r(\Delta_{t_{j_{J-1}+1}, t_{j_J+1}})|_{l'}^{(0)} |q(\Delta_{t_{j_J+1}, t_{\nu+1}})|_{l'}^{(0)} \\ &\leq \sum^I (A_l)^J \left(\prod_{k=1}^J C_{3,l'}(t_0 - t_{\nu+1})(t_{j_k} - t_{j_{k+1}}) \right) C_{1,l'} \\ &\leq C_{1,l'} \left(\prod_{j=0}^{\nu} \left(1 + A_l C_{3,l'}(t_0 - t_{\nu+1})(t_j - t_{j+1}) \right) - 1 \right) \\ &\leq C_{4,l}(t_0 - t_{\nu+1})^2. \quad \square \end{aligned} \quad (2.40)$$

Now we prove Theorem 2.1:

Proof of Theorem 2.1.

1° Define $p(\Delta_{t,s}; x, \xi)$ by

$$p(\Delta_{t,s}; x, \xi) \equiv q(\Delta_{t,s}; x, \xi) + \Upsilon(\Delta_{t,s}; x, \xi). \quad (2.41)$$

Then (1) is clear.

2° By (2.14) and (2.39), we get (2.5). Next, we note that

$$\begin{aligned} & p(\Delta'_{t_j, t_{j+1}}; x, \xi) - p(t_j, t_{j+1}; x, \xi) \\ &= \left(q(\Delta'_{t_j, t_{j+1}}; x, \xi) - p(t_j, t_{j+1}; x, \xi) \right) + \Upsilon(\Delta'_{t_j, t_{j+1}}; x, \xi), \end{aligned} \quad (2.42)$$

Hence, by (2.16) and (2.39), there exists a constant $C_{5,l}$ such that

$$\begin{aligned} & |p(t_j, t_{j+1}) - p(\Delta'_{t_j, t_{j+1}})|_l^{(2m)} \\ & \leq C_{5,l}(t_j - t_{j+1}) \left((t_j - t_{j+1}) + \sup_{t_j \geq t' \geq t'' \geq t_{j+1}} |K(t') - K(t'')|_l^{(m)} \right). \end{aligned} \quad (2.43)$$

Here we can write

$$\begin{aligned} & p(\Delta_{t,s}; X, D_x) - p(\Delta'_{t,s}; X, D_x) = \sum_{j=0}^{\nu} p(\Delta'_{t_0, t_j}; X, D_x) \\ & \circ \left(p(t_j, t_{j+1}; X, D_x) - p(\Delta'_{t_j, t_{j+1}}; X, D_x) \right) \circ p(\Delta_{t_{j+1}, t_{\nu+1}}; X, D_x). \end{aligned} \quad (2.44)$$

By (2.5), (2.43) and Theorem 1.3, we get (2.6).

3° By (2.6) and (2.5), there exists $p^*(t, s; x, \xi) \in S_{\lambda, \rho, \delta}^0$ such that

$$|p^*(t, s)|_l^{(0)} \leq C_l, \quad (2.45)$$

and

$$\begin{aligned} & |p(\Delta_{t,s}) - p^*(t, s)|_l^{(2m)} \\ & \leq C'_l(t-s) \left(|\Delta_{t,s}| + \sup_{|t'-t''| \leq |\Delta_{t,s}|} |K(t') - K(t'')|_l^{(m)} \right). \end{aligned} \quad (2.46)$$

Hence we get (3).

4° By the result of (3), we get (4). See Chapter 3 § 7 in H.Kumano-go [6].

5° Using the results of (2) and (3), it is easy to check (5). \square

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