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Higher microlocalization and propagation of analytic singularities

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1 Introduction

1. In this paper we want to report on some results obtained in the study of propagation of analytic singularities for pseudodifferential operators with characteristics of highly variable multiplicity. It is typical for such operators that microlocal singularities will not propagate along uniquely defined curves or leaves associated with the characteristic surface of the operator, but will rather split along families of curves or leaves. It is often possible to study propagation phenomena of this type with the aid of the following important result:

Theorem 1.1. *(Kawai-Kashiwara, cf. [6]) Let $p(x,D)$ be an analytic pseudodifferential operator defined in a conic neighborhood W of (x^0,ξ^0), let $\psi : W \to R$ be a real analytic function such that $\psi(x^0,\xi^0)=0$, and assume that p is microhyperbolic at (x^0,ξ^0) in the direction $(-(\partial\psi/\partial\xi)(x^0,\xi^0),\ (\partial\psi/\partial x)(x^0,\xi^0))$. Let u be a hyperfunction such that $pu=0$ on W (in the sense that $WF_Apu\cap W=\emptyset$) and assume that*

$$WF_Au\cap\{(x,\xi)\in W;\psi(x,\xi)<0\}=\emptyset.$$

Then it follows that $(x^0,\xi^0)\notin WF_Au$.

Many proofs,variants and extensions of this result have been considered in the literature: [3], [6], [7], [18], [24] and [25].

2. One of the main difficulties in the study of operators with characteristics of variable multiplicity comes from the fact that the characteristic variety of such operators is not smooth. Indeed, near a singular point of the variety a number of arguments and of constructions performed in the smooth case will break down. In the analytic category these difficulties can often be understood best from the point of view of higher (analytic)

microlocalization. One of the main reasons why this is so is that at the level of higher microlocalization one can often stay away from the singular parts of the characteristic variety. As a consequence some suitable higher order wave front sets will propagate in a way similar to what happens for the classical wave front set in the smooth case. It seems worthwhile to try to combine the advantages of the two approaches described above, the one with results of type of the theorem of Kawai-Kashiwara, the other with methods of higher microlocalization, by proving results of the type of theorem 1.1. for higher order wave front sets. Our main result below, cf. theorem 2.1. later on, is a result of this flavor.

3. Second analytic microlocalization has been first considered by M.Kashiwara.(Cf. [5].) At present there are at least three variants (which are not completely equivalent, since they refer formally to different situations) of higher analytic microlocalization in the literature (Cf. [5], [11], [12], [14], [19]). Here we shall use the theory developed in [14].

Before we can state our main result, we want to recall the definition of higher order analytic wave front set (or of "higher analytic spectrum") as introduced in [14]. We state it in euclidean coordinates. Actually, all definitions which we give seem to have an invariant meaning, but invariant definitions have been established up to now only in the case of first and second microlocalization. Since we shall not use later on wave front sets of order higher than three and since the wave front sets introduced in [14] for second microlocalization are equivalent with the ones considered in the involutice case by [5], [11], [12], [19] (for the equivalence, cf. [2], [14], [15]), we restrict ourselves in all what follows to the case of tri-microlocalization.

4. Let us at first consider a sequence of linear subspaces M_j, $j = 0, 1, 2, 3$, in R^n such that $M_0 = R^n$, $M_j \subset M_{j-1}$, $M_j \neq M_{j-1}$, $M_3 = \{0\}$. Denote by $\Pi_j : R^n \to M_j$ the orthogonal projection on M_j and by $\dot{M}_j = M_j \ominus M_{j+1}$ the orthogonal complement of M_{j+1} in M_j. Further, we consider U open in R^n, $x^0 \in U$, $\xi^j \in \dot{M}_j$, $j = 0, 1, 2$, and $u \in D'(U)$. A function $f : (0, \infty) \to (0, \infty)$ will be called sublinear if $\forall \varepsilon$, $\exists$ c such that

$$f(t) \leq \varepsilon t + c.$$

Definition 1.2. *We shall say that $(x^0, \xi^0, \xi^1, \xi^2)$ is not in the (semi-isotropic) analytic 3-wave front set of u and write*

$$(x^0, \xi^0, \xi^1, \xi^2) \notin WF^3_{A,s} u,$$

if we can find open conic neighborhoods $G^j \subset M_j$ of ξ^j, $\varepsilon, c, c', \beta > 0$, sublinear functions $f_j : (0, \infty) \to (0, \infty)$, $j = 1, 2$, and a bounded sequence $\{u_i\} \subset E'(U)$ such that

$$u = u_i, \text{for } |x - x^0| < \varepsilon, \text{ and all } i$$

$$|\hat{u}_i(\xi)| \le c(ci/|\Pi_2\, \xi|)^i \text{ if } i = 1,2,\dots, \Pi_j\, \xi \in G^j, j = 0,1,2,$$

$$|\Pi_j\, \xi| \ge f_j(|\Pi_{j-1}\xi|)\,,\ j = 1,2, \text{ and } |\Pi_2\, \xi| \ge c'|\Pi_1\, \xi|^{\beta+1}/|\Pi_0\xi|^{\beta}.$$

Remark 1.3. *a) For given (ξ^0, ξ^1, ξ^2) as above we shall say that $A \subset R^n$ is a tri-neighborhood of (ξ^0, ξ^1, ξ^2) if there are open cones $G^j \subset M_j$, $j = 0,1,2$, so that $\{\xi; \Pi_j\xi \in G^j, j = 0,1,2\} \subset A$. A set W is called a tri-neighborhood of $(z^0, \xi^0, \xi^1, \xi^2)$ if it contains a set of form $U \times G$ where U is a neighborhood of z^0 in R^n and if G is a tri-neighborhood of (ξ^0, ξ^1, ξ^2).*
b) For a discussion of the terminology we refer to [14].

The main reason why theories of higher order wave front sets are useful in the analytic category is that the $(k-1)$-wave front set will propagate if the k-wave front set is "void". For a geometrically invariant formulation of what we mean by this in the case $k = 2$, cf. e.g. [14], propositon 5.2.3. For the case $k = 3$, and denoting by WF^2_A the second analytic wave front set, we state here the following result (cf. [14], theorem 2.1.12)

Theorem 1.4. *Let $U \subset R^n$ be open and consider $x^1, x^2 \in U$ with $[x^1, x^2] \subset U$, $x^1 - x^2 \in M_2$. Here we denote by $[x^1, x^2]$ the segment with endpoints x^1 and x^2. Consider ξ^0, ξ^1 and assume that for any $\eta \in \dot{M}_2$ and any $x \in [x^1, x^2]$ it follows that*

$$(x, \xi^0, \xi^1, \eta) \notin WF^3_{A,s}u. \tag{1.1}$$

Moreover, assume that $(x^1, \xi^0, \xi^1) \notin WF^2_A u$. Then it follows that

$$(x, \xi^0, \xi^1) \notin WF^2_A u, \text{ whatever } x \in [x^1, x^2] \text{ is.}$$

It also follows from this that $(x, \xi^0, \xi^1) \notin WF^2_A u$, for all x in the connected component of $x^1 + M_2 \cap U$ which contains x^1.

2 Statement of the main result

1. We start from a classical analytic symbol p of order μ defined on $U \times G$ where U is a neighborhood of $0 \in R^{n+1}$ and G is an open cone in R^{n+1}. Let p_μ be its principal part. We assume that p_μ vanishes of some order s on an anlytic homogenous regular involutive variety Σ in T^*U which contains the point $(0, \lambda^0)$, $\lambda^0 = (0, \dots, 0, 1)$. It will be no loss of generality in applications to assume that $\Sigma = \{(z, \lambda); \lambda' = 0\}$ for some group of variables of type $\lambda' = (\lambda_0, \lambda_1 \dots, \lambda_d)$, $\lambda = (\lambda', \lambda_{d+1}, \dots, \lambda_n)$. Let us also compute the localization

$p_{\mu,1}$ of p_μ along Σ. It is in general a function on the normal bundle to Σ, but in our special coordinates above, we may just write for λ in a conic neighborhood of λ^0 that

$$p_{\mu,1}(z,\lambda) = \sum_{|\alpha| = s} (\partial/\partial\lambda')^\alpha p_\mu(z,0,\lambda_{d+1},\dots,\lambda_n)\lambda'^\alpha/\alpha!.$$

In particular it is clear from this that $p_{\mu,1}$ is positively homogenous of order μ in the variables λ, and, in addition, homogenous of order s in the variables λ'. Consider $d' < d$, denote $\lambda'' = (\lambda_0, \lambda_1, \dots, \lambda_{d'})$ and fix $\lambda^1 \neq 0$ with $\lambda^{1\prime\prime} = 0$ and $\lambda^1_i = 0$ for $i > d$. Also assume that $p_{\mu,1}$ vanishes of some order m on $\{(z,\lambda); \lambda'' = 0\}$. We denote by $p_{\mu,2}$ the localization of $p_{\mu,1}$ along $\lambda'' = 0$. It is thus given by the relation

$$p_{\mu,2}(z,\lambda) = \sum_{|\beta| = m} (\partial/\partial\lambda'')^\beta p_{\mu,1}(z,0,\lambda_{d'+1},\dots,\lambda_n)\lambda''^\beta/\beta!.$$

It follows that, in addition to the homogeneities inherited from $p_{\mu,1}$, $p_{\mu,2}$ is homogeneous of order m in λ''. It is possible to give an invariant meaning also to these conditions in terms of the bi-homogeneous and bi-symplectic structures of the normal bundle to Σ; we refer to [11] or [14] for details. We have not studied the invariant meaning for the statements which follow hereafter. It is clear that $p_{\mu,2}$ is of form $\sum_{|\gamma|=m} a_\gamma(z,\lambda_{d'+1},\dots\lambda_n)\lambda''^\gamma$ with $a_\gamma(z,\lambda_{d'+1},\dots\lambda_n)$ positively homogeneous of order $\mu - m$ in λ and homogeneous of order $s - m$ in λ'. We shall now write the variables λ'' as $\lambda'' = (\tau, \zeta'')$, where $\tau = \lambda_0 \in R$. Similarily, $\lambda' = (\tau, \zeta')$, $\lambda, = (\tau, \zeta)$. We also fix $\lambda^2 \neq 0$ in R^{n+1} with $\lambda^2_0 = 0$, $\lambda^2_i = 0$ for $i > d'$. We moreover assume that $p_{\mu,2}$ satisfies the following conditions:
a) the coefficient of τ^m in $p_{\mu,2}$ does not vanish at $(z = 0, \zeta^{2\prime\prime}, \lambda^1_{d'+1}, \dots, \lambda^1_d, \lambda^0_{d+1}, \dots, \lambda^0_n)$.
b) $p_{\mu,2}$ vanishes of order m at $(z = 0, \lambda^{2\prime\prime}, \lambda^1_{d'+1}, \dots, \lambda^1_d, \lambda^0_{d+1}, \dots, \lambda^0_n)$.
c) $p_{\mu,2}$ is micro-hyperbolic with respect to $t = 0$ at $(z = 0, \lambda^0, \lambda^1, \lambda^2)$. By this we mean that there is a real neighborhood U' of $z = 0$, a real tri-neighborhood G' of (ξ^0, ξ^1, ξ^2), and $c > 0$ so that $p_{\mu,2}(z,\tau,\xi) = 0$, $z \in U'$, $\xi \in G'$ together with $|\tau| \leq c|\xi''|$ implies $\mathrm{Im}\ \tau \leq 0$.

Note that by assumption a), the coefficient of τ in $p_{\mu,2}$ is elliptic in the two-microlocal calculus near $(0, \lambda^0, \lambda^1)$. Two-microlocally near $(0, \lambda^0, \lambda^1)$ it is therefore no loss of generality to assume (if we compose everything with the inverse of the coefficient of τ) that we have

$$p_\mu(z,\lambda) = \tau^m + \sum_{|\alpha|+j=m, j<m} a_{\alpha,j}(z,\lambda_{d'+1},\dots,\lambda_n)\zeta''^\alpha\tau^j + O(|\lambda''|^{m+1}/|\lambda'|) + O(|\lambda'|^{m+1}/|\lambda|),$$

with coefficients $a_{\alpha,j}$ which are positively homogeneous of order zero in λ and λ'. This is, actually, two-microlocally, the model on which we work. In regions where $p_{\mu,2}$ is tri-microlocally elliptic, it will have at most the order of magnitude $|\lambda''|^m$. It can therefore dominate the remainder term $O(|\lambda'|^{m+1}/|\lambda|)$ only in regions of form $|\lambda''| \geq c|\lambda'|^{1+\beta}/|\lambda|^{\beta}$ with $\beta < 1/m$. This is the justification why we restrict our attention to such regions in the definition of $WF^3_{A,s}$.

We can now state the following result:

Theorem 2.1. *Assume that under the above assumptions u is a distribution defined in a neighborhood of U and that it satisfies the following conditions for some tri-neighborhood W of* $(0, \lambda^0, \lambda^1, \lambda^2)$*:*

$$WF^3_A p(z, D)u \cap W = \emptyset, \tag{2.2}$$

$$WF^3_A u \cap W \cap \{t < 0\} = \emptyset. \tag{2.3}$$

Then it follows that $(0, \lambda^0, \lambda^1, \lambda^2) \notin WF^3_A u$.

Remark 2.2. *Although we have stated theorem 2.1. for the case of tri-microlocalization, the argument works as well for the case of standard wave front sets, respectively for the case of second microlocalization. In particular, one thus obtains a new proof for theorem 1.1. As far as the case of two-microlocalization is concerned, I was told by prof. N. Tose that he is also aware of the fact that a result of the type of theorem 2.1. is true. Note that the result presented here, as well as its analogue for the case of two-microlocalization, refer to a highly involutive setting.*

Remark 2.3. *The proof of theorem 2.1. depends on characterizations in terms of duality of microlocal smoothness (cf. [14]). It is similar in some sense to the argument given in [13] to prove microlocal smoothness in the standard Cauchy problem, but there are a number of additional technical complications. The proof in the case considered in [13] was centered around the decomposition (3.6) considered later on and on contour integral formulas related to that decomposition. Since we think that these formulas are the most interesting part of the argument, we shall describe in the next section how they have to be adapted to fit the present needs. For brevity, arguments will be kept at a formal level and we shall not even make clear how microhyperbolicity is used. Details will be given elsewhere.*

3 Contour integration formulas

1. To describe the main idea, assume first that $p(z, D)$ is a linear partial differential operator with analytic coefficients defined in a neighborhood U of $0 \in R^{n+1}$. We assume

in fact that $p(z,D) = D_t^m + \sum_{|\alpha|+j\leq m, j<m} a_{\alpha,j}(z) D_t^j D_x^\alpha$, with $a_{\alpha,j}(z)$ defined and analytic for $\{z \in C^{n+1}, |z| < \varepsilon'\}$. ¿From the Cauchy-Kowalewska theorem it follows that there is ε and a map

$$T: \begin{array}{c} A(z \in C^{n+1}, |z| < \varepsilon') \\ \times \\ \prod_{j=0}^{m-1} A(x \in C^n; |x| < \varepsilon') \end{array} \longrightarrow A(z \in C^{n+1}; |z| < \varepsilon)$$

which associates with $g \in A(z \in C^{n+1}; |z| < \varepsilon')$, $g_j \in A(x \in C^n; |x| < \varepsilon')$, $j = 0, ..., m-1$, the solution h of the Cauchy problem

$$p(z,D)h = g \text{ on } z \in R^{n+1}, |z| < \varepsilon, \tag{3.4}$$

$$(i\partial/\partial t)^j h_{|x_n=0_+} = g_j, j = 0, ..., m-1. \tag{3.5}$$

Let

$${}^tT: A'(z \in C^{n+1}; |z| < \varepsilon) \longrightarrow \begin{array}{c} A'(z \in C^{n+1}; |z| < \varepsilon') \\ \times \\ \prod_{j=0}^{m-1} A'(x \in C^n; |x| < \varepsilon') \end{array}$$

be the dual map (which acts thus between spaces of analytic functionals, denoted A'). Explicitly, if $v \in A'(z \in C^{n+1}; |z| < \varepsilon)$ is given, then v is related to

$${}^tTv = \begin{pmatrix} w \\ w_i, i \leq m-1 \end{pmatrix}$$

by the fact that

$$v =^t p(z,D)w + \sum_{j=0}^{m-1} (-i\partial/\partial t)^j \delta_t \otimes w_j, \tag{3.6}$$

where δ_t is the Dirac distribution in the variable t at $t = 0$, and ${}^tp$ is the operator transposed to p. (3.6) is of course a relation in analytic functionals. It follows that

$$v(h) = w(g) + \sum_j w_j(g_j) \tag{3.7}$$

if (3.4), (3.5) and (3.6) hold simultaneously, and the same relation will also hold for non-analytic h, g, g_j, provided $v(h)$, $w(g)$ and $w_j(g_j)$ have a natural meaning. The interesting thing is now that from (3.6) we obtain rather explicit information in terms of contour integration formulas for the Fourier-Borel transform of w and of the w_j. That this is so, is of course well-known for operators with constant coefficients and is related to the well-established analogy between the Cauchy-Kowalewska and the Weierstrass preparation theorems. (Cf. e.g. [8].) For operators with variable coefficients, essentially the same kind of formulas have been established in [13]. Let us briefly recall these formulas from

[13]. To do so, we fix a constant c so that the symbol $p(z,\lambda) = \tau^m + \sum_{\alpha,j} a_{\alpha,j}(z)\tau^j\zeta^\alpha$ can be inverted in the symbol algebra of symbols of formal analytic pseudodifferential operators in $\{|z| < \varepsilon\} \times \{\lambda \in C^{n+1}, |\tau| > c(1+|\zeta|)\}$. Denote by $\sum q_j$ the formal inverse to $p(z,\lambda)$, i.e. assume that in the symbol algebra $p \circ \sum q_j \sim 1$. Also denote by $\Lambda(\lambda)$ the counterclockwise oriented contour $\Lambda(\lambda) = \{\sigma \in C; |\sigma| = (c+1)(1+|\lambda|)\}$, so that if $\sigma \in \Lambda(\lambda)$, then it makes sense to consider $\sum q_j(z, \sigma+\tau, \zeta)$. We then define for $v \in A'(z \in C^{n+1}; |z| < \varepsilon)$ the map $v \to S(v)$ by

$$S(v)(\lambda) = \frac{1}{2\pi i}\int_{\Lambda(\lambda)} v[e^{-i\langle z, \lambda+\sigma N\rangle}\frac{1}{\sigma}\sum_{j\le c'|\lambda|} q_j(z, -\lambda-\sigma N)]d\sigma, \tag{3.8}$$

where $N = (1,0,\ldots,0)$ and c' is a suitable constant. It is proved in [13] that

$$S(\sum_{j=0}^{m-1}(-i\partial/\partial t)^j\delta_t \otimes w_j) = 0$$

and that (if c' is suitable,) there is $d > 0$ so that,

$$S({}^tp(z,D)w)(\lambda) - \hat{w}(\lambda) = O(e^{-d|\lambda|}),$$

where $\hat{w}$ denotes the Fourier-Borel transform of w. ¿From (3.6) we therefore obtain that $\hat{w}(\lambda) \sim S(v)(\lambda)$, i.e. we can essentially calculate $\hat{w}$ by an explicit contour integration formula in terms of v. This is interesting in combination with (3.7), in that we can use it to check the regularity of h there by duality if we have information on the regularity of the g and g_j. (A related situation appears in [13].) It is important to note that the integrand in (3.8) is meromorphic in σ and that therefore the value of $S(v)(\lambda)$ can be calculated from the residua of the integrand as a function in σ. The possible residua of this integrand are of course located at $\sigma = 0$ respectively at $\tau+\sigma = -\tau_j(z,\zeta)$, where $\tau_j(z,\zeta)$ are the roots of $\tau \to p_m(z,\tau,\zeta) = 0$.

2. Written in this way, the argument in [13] was based in an essential way on the fact that one already had a decomposition of type (3.6) available. We shall now describe how one can modify the preceding argument if one wants to work in a situation when one does not have (3.6). Although we need a tri-microlocal variant of the calculations, we shall present them in a setting of the type which one will encounter in first microlocalization. In fact, apart from technical details, the formulas which we obtain are the same in first or higher involutive microlocalization and the idea can perhaps be better understood if we consider the first nontrivial case. We shall then assume in the sequel that p is a classical analytic symbol of form

$$p(z,\lambda) = \tau^m + \sum_{i<m} a_i(z,\zeta)\tau^i$$

where the a_i are analytic symbols of order $m-i$ defined in a conic neighborhood $U \times G$ of $(0,\xi^0)$, $\xi^0 = (0,\dots,0,1)$, U a neighborhood of 0 in C^n, $G \subset C^n$ an open cone which contains ξ^0. We assume that the principal symbols of all a_i vanishes at $(0,\xi^0)$, which is the case if p is obtained as a result of an application of the Weierstrass preparation theorem for analytic symbols. (Cf. [16].) As for the equation which one wants to study, it will be of form $pu = 0$ on W where W is some suitable conic neighborhood of $(0,\lambda^0)$, $\lambda^0 = (0,\dots,0,1)$, W subset of $(U \cap R^{n+1}) \times R \times (G \cap R^{n+1})$. It might also be useful to observe at this moment that although the symbol $p(z,\lambda)$ has a natural meaning on a set of form $U \times C \times G$, it will be an analytic symbol only on sets of type $U \times \{\lambda \in C^{n+1}; \zeta \in G, |\tau| < c''|\zeta|\}$ for arbitrarily fixed c''. (In particular, we shall have to assume that W itself lies in a region of form $U \times \{|\tau| < c''|\zeta|\} \times G$). Note here however that if we fix ε small and $G' \subset\subset G$ and if c is large enough, we can find a formal analytic symbol $\sum q_j$, the q_j of order $-m-j$, defined for $\{z; |z| < \varepsilon\}$, $\zeta \in G'$, $|\tau| > c|\zeta|$, and such that $p \circ \sum_j q_j \sim 1$ in the calculus of formal analytic symbols. For given v we can now again define $S(v)(\lambda)$ by the formula (3.8), provided we restrict our attention to points λ for which the ζ-component stays in G'. We shall then fix some suitable c'' and denote by $\Gamma = \{\lambda \in R^{n+1}; \zeta \in G', |\tau| < c''|\zeta|\}$. Since $S(v)(\lambda)$ would essentially give $\hat{w}$ in the relation (3.6) if that relation would make sense, and since ${}^tp(z,D)w(y)$ would then approximatively be equal to $\int_{R^{n+1}} exp(i\langle y,\lambda\rangle)\ {}^tp(y,\lambda)S(v)(\lambda)\,d\lambda$, it is now a reasonable idea to try to understand what the expression

$$v(y) - \int_\Gamma e^{i\langle y,\lambda\rangle}\ {}^tp(y,\lambda)S(v)(\lambda)\,d\lambda$$

gives in the case at hand. Let us then denote by $I(y)$ the expression

$$I(y) = \frac{1}{2\pi i}\int_\Gamma \int_{\Lambda(\lambda)} \int e^{-i\langle z,\lambda+\sigma N\rangle}(1/\sigma){}^tp(y,\lambda) \sum_{j\le c'|\lambda|} q_j(z,-\lambda-\sigma N)v(z)\,dz\,d\sigma\,d\lambda,$$

where now we have supposed, to simplify the meaning of our expressions, that v is a function. To study this, we shall replace ${}^tp(y,\lambda)$ by ${}^tp(y,\lambda+\sigma N)+{}^tp(y,\lambda)-{}^tp(y,\lambda+\sigma N)$. It is convenient to set $I(y) = (2\pi i)(II(y)+III(y))$, where II, respectively III, are defined by

$$II(y) = \int_\Gamma \int_{\Lambda(\lambda)} \int e^{-i\langle z,\lambda+\sigma N\rangle}\frac{{}^tp(y,\lambda+\sigma N) - {}^t p(y,\lambda)}{\sigma} \sum_{j\le c'|\lambda|} q_j(z,-\lambda-\sigma N)v(z)\,dz\,d\sigma\,d\lambda,$$

and

$$III(y) = \int_\Gamma \int_{\Lambda(\lambda)} \int e^{-i\langle z,\lambda+\sigma N\rangle}\frac{{}^tp(y,\lambda+\sigma N)}{\sigma} \sum_{j\le c'|\lambda|} q_j(z,-\lambda-\sigma N)v(z)\,dz\,d\sigma\,d\lambda.$$

Two remarks are now important. The first (which we use in the study of $II(y)$) is that ${}^tp(y,\lambda+\sigma N)-{}^tp(y,\lambda)$ vanishes at $\sigma = 0$, so that $({}^tp(y,\lambda+\sigma N)-{}^tp(y,\lambda))/\sigma$ is a polynomial in σ and that therefore the residuum at $\sigma = 0$ has disappeared. The second remark (to be used in the study of $III(y)$), refers to the general theory of pseudodifferential operators and is classical. It says that if q_i, $i = 1, 2$, are two symbols of analytic pseudodifferntial operators, then we can at first define ${}^t q_2 v$ modulo a real analytic function by the formula $F({}^t q_2 v)(\lambda) = \int exp(-i\langle z, \lambda\rangle) q(z, -\lambda) v(z)\, dz.$($F$ is the Fourier transformation.) It follows that modulo a real analytic funtion, the composition $(q_1 \circ {}^t q_2) v(y)$ is formally given by

$$\int\int e^{i\langle y - z, \lambda\rangle} q_1(y, \lambda) q_2(z, -\lambda) v(z)\, dz\, d\lambda.$$

To apply our first remark, we shall write

$$\frac{{}^t p(y, \lambda + \sigma N) -{}^t p(y, \lambda)}{\sigma} = \sum_{r+k\leq m, k\geq 1} a_{rk}(y, \zeta)\tau^r(\tau + \sigma)^k$$

and make the change of variables $\sigma \to \tau + \sigma = \nu$ in the integral with respect to σ in the definition of II. Since the residuum at $\sigma = 0$ has disappeared, it follows that

$$II(y) = \int_\Gamma e^{i\langle y, \lambda\rangle} \sum_{r,k} a_{rk}(y, \zeta)\tau^r b_k(\zeta)\, d\lambda,$$

where we have denoted

$$b_k(\zeta) = \int_{\Lambda'(\zeta)} \int_{R^{n+1}} e^{-i\langle x, \zeta\rangle - it\nu} \sum_{j\leq c''|\lambda|} q_j(z, -\nu, -\zeta)\nu^k v(z)\, dz\, d\nu,$$

and where $\Lambda'(\zeta)$ is the counterclock-wisely oriented boundary of any rectangle in the complex ν-plane which contains all roots $\tau_j(z, \zeta)$ of the equation $p_m(z, \tau, \zeta) = 0$ and which is such that $\nu \in \Lambda'(\zeta)$ implies that the distance of ν to the set of root $\{\tau_j(z, \zeta), j = 1, \ldots, m\}$ is of the order or magnitute $\tilde{c}|\zeta|$ for some constant $\tilde{c}$. If we denote by $a(y, \zeta) = \sum_{r+k\leq m, r\geq 1} a_{rk}(y, \zeta)\tau^r b_k(\zeta)$, then we will have

$$II(y) = \int_\Gamma e^{i\langle y, \lambda\rangle} a(y, \lambda)\, d\lambda.$$

This shows that in some sense the Fourier transform of $II(y)$ is on Γ a polynomial in τ of order at most $m - 1$: it is the closest we can come to an expression of form $\sum_{0\leq j<m} D_t^j \delta_t \otimes w_j$. Unfortunately, the estimates of the Fourier transform of II become bad near the boundary of Γ. This is related to the fact that our assumption in theorem 2.1. is not strong enough to give full control on the Cauchy data of u at $t = 0$.

We now want to make a few comments on how to study the term III. We shall in fact apply Taylor expansion in the variable z to write

$$q_j(z, -\lambda - \sigma N) = \sum_{|\alpha|<[c''|\lambda|]-j} q_{j(\alpha)}(y, -\lambda - \sigma N)(z - y)^\alpha/\alpha! +$$

$$\sum_{|\alpha|=[c''|\lambda|]-j+1} R_{j\alpha}(z,y,\lambda+\sigma N)(z-y)^\alpha/\alpha!,$$

where $[c''|\lambda|]$ is the integer part of $c''|\lambda|$, the $R_{j\alpha}$ are remainder terms in Taylor's formula and where $q_{j(\alpha)}$ is $(\partial/\partial z)^\alpha q_j$. We may thus assume that the $R_{j\alpha}$ are analytic in (z,y,λ,σ) and that they satisfy estimates of form :

$$|(\frac{\partial}{\partial\lambda})^\beta R_{j\alpha}(z,y,\lambda,\sigma)| \le \tilde{c}^{|\alpha|+|\beta|+j+1}\alpha!\beta!j!(1+|\lambda+\sigma N|)^{-m-j-|\beta|},$$

with a constant which does not depend on (z,y,λ,σ), if σ is in $\Lambda(\lambda)$. Also observe, as is standard in this context, that

$$e^{i\langle y-z,\lambda\rangle}(z-y)^\alpha = (\frac{1}{i}\frac{\partial}{\partial\lambda})^\alpha e^{i\langle y-z,\lambda\rangle},$$

so that

$$III(y) = \int\int\int e^{i\langle y-z,\lambda+\sigma N\rangle - i\langle y,\sigma N\rangle}\frac{v(z)}{\sigma}$$

$$\{\sum_{|\alpha|+j<[c''|\lambda|]}(\frac{1}{i}\frac{\partial}{\partial\lambda})^\alpha[{}^tp(y,\lambda+\sigma N)q_{j(\alpha)}(y,-\lambda-\sigma N)/\alpha!]+$$

$$+\sum_{|\alpha|+j=[c'|\lambda|]+1}(\frac{1}{i}\frac{\partial}{\partial\lambda})^\alpha[{}^tp(y,\lambda+\sigma N)R_{j\alpha}(z,y,\lambda+\sigma N)]/\alpha!\}\,dz\,d\sigma\,d\lambda.$$

If c' is suitable, it will now follow that

$$\sum_{|\alpha|+j<[c''|\lambda|]}(\frac{1}{i}(\frac{\partial}{\partial\lambda})^\alpha[{}^tp(y,\lambda+\sigma N)q_{j(\alpha)}(y,-\lambda-\sigma N)/\alpha!] = 1+O(e^{-d|\lambda+\sigma N|})$$

for some small positiv constant d. In a similar way also

$$\sum_{|\alpha|+j=[c'|\lambda|]+1}(\frac{1}{i}\frac{\partial}{\partial\lambda})^\alpha[{}^tp(y,\lambda+\sigma N)R_{j\alpha}(z,y,\lambda+\sigma N)]$$

is an $O(\exp(-d|\lambda+\sigma N|))$. We can conclude that

$$III(y) = \int_\Gamma\int\int_{\Lambda(\lambda)} e^{i\langle y-z,\lambda+\sigma N\rangle}\frac{1}{\sigma}v(z)\,d\sigma\,dz\,d\lambda+$$

$$+\int_\Gamma\int\int_{\Lambda(\lambda)} e^{i\langle y-z,\lambda+\sigma N\rangle}\frac{1}{\sigma}O(e^{-d|\lambda+\sigma N|})v(z)\,d\sigma\,dz\,d\lambda.$$

Here we note that the first integral in the preceding expression is equal to

$$2\pi i\int_\Gamma\int e^{i\langle y-z,\lambda+\sigma N\rangle}\frac{v(z)}{\sigma}\,dz\,d\lambda = 2\pi iv - 2\pi i\int_{R^{n+1}\backslash\Gamma}\int e^{i\langle y-z,\lambda+\sigma N\rangle}\frac{v(z)}{\sigma}\,dz\,d\lambda.$$

Since $|\lambda+\sigma N|\sim|\zeta|$ on $\Lambda(\lambda)$, we obtain from all the above that

$$v = {}^t p(y,D)F^{-1}(S(v)) + \int e^{i\langle y,\lambda\rangle}a(y,\lambda)\,d\lambda+$$

$$\int_{R^{n+1}\backslash\Gamma}\int e^{i\langle y-z,\lambda+\sigma N\rangle}\frac{v(z)}{\sigma}\,dz\,d\lambda + \int_\Gamma\int e^{i\langle y-z,\lambda\rangle}O(e^{-\tilde{d}|\zeta|})v(z)\,dz\,d\lambda.$$

It is essentially this the expression which we use in the proof to replace the expression (3.6). (Technically speaking a number of additional transformations must be performed to compensate for the bad behaviour of expressions like $S(v)(\lambda)$ near the boundary of Γ.)

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