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THE STUDY OF KNASTER - KURATOWSKI - MAZURKIEWICZ THEORY AND APPLICATIONS TO ABSTRACT ECONOMICS

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Abstract

In this paper, we present some recent results of the Knaster - Kuratowski - Mazurkiewicz theory and their applications. In particular, we focus our attention on the following topics:

(1) The K-K-M Theory.
(2) Topological Intersection Theorems.
(3) Fixed Points and Maximal Elements.
(4) Equilibria of Abstract Economies in Topological Vector Spaces.
(5) Equilibria of Abstract Economies in Frechet Spaces.
(6) Equilibria of Abstract Economies in Finite Dimensional Spaces.
(7) Equilibria of Abstract Economies in Topological Spaces.
(8) Random Equilibria of Abstract Economies.

Finally, we give the outline how the K-K-M theory is used to study the existence of equilibria for abstract economies from the point of view of fixed point theorems.

1. The K-K-M Theory

Since Knaster, Kuratowski and Mazurkiewicz established so-called the K-K-M principle in 1929, many applications of K-K-M principle have been developed.Today the field related to the study of the classical K-K-M principle and its applications is often called KKM Theory. In order to make a clear presentation, we begin with the classical K-K-M principle and all proofs are omitted for saving spaces. The interested readers can find all details from Yuan [65-70] and related references therein.

Let $\mathcal{F}(X)$ and 2^X denote the family of all non-empty finite subsets of X and the family of all subsets of X. Let $N = \{0, 1, \cdots, n\}$ and $\Delta_J = co\{e_j : j \in J\}$.

The K-K-M Principle (1929) in Finite Dimensional Spaces. Let $C_0, \cdots, C_n$ be closed subsets of the standard n-dimensional simplex Δ_N and let $\{e_0, \cdots, e_n\}$ be the set of its vertices. If for each $J \in \mathcal{F}(N)$, $\Delta_J \subset \cup_{j\in J} C_j$. Then $\cap_{i=0}^{n} C_i \neq \emptyset$.

The first dual form of the K-K-M Principle was given by Sperner in 1928, the following one was formulated by Fan in 1968:

Theorem 1.1 (Fan [1968], see also Sperner [1928]). Let $\{C_i\}_{i\in N}$ be a closed covering of Δ_N such that $\Delta_{N\setminus\{i\}} \subset C_i$ for all $i \in N$. Then $\cap_{i\in N} C_i \neq \emptyset$.

In 1973, Shapley gave a generalization of the K-K-M principle:

Theorem 1.2 (Shapley [1973]). Let $\{C_S\}_{S\in\mathcal{F}(N)}$ be a family of closed subsets of Δ_N such that $\Delta_T \subset \cup_{S\subset T} C_S$ holds for each $T \in \mathcal{F}(N)$. Then there exists a balanced family $\mathcal{B}$ of $\mathcal{F}(N)$ for which $\cap_{S\in\mathcal{B}} C_S \neq \emptyset$.

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Definition 1.1. A subfamily $\mathcal{B}$ of $\mathcal{F}(N)$ is called *balanced* if $m_N \in co\{m_S : S \in \mathcal{B}\}$, where $m_S = \frac{\Sigma_{j\in S} e_j}{\#S}$ for each $S \in \mathcal{F}(N)$ and m_S is also called *barycenter* of S.

In 1988, Ichiishi gave another 'dual form' of Shapley's result above into the following alternative version of Theorem 1.2:

Theorem 1.2′ (Ichiishi [1988]). Let $\{C_S\}_{S\in\mathcal{F}(N)}$ be a closed covering of Δ_N such that $\Delta_T \subset \cup_{S\supset N\setminus T} C_S$ for each $T \in \mathcal{F}(N)$. Then there exists a balanced family $\mathcal{B}$ for which $\cap_{S\in\mathcal{B}} C_S \neq \emptyset$.

Let $S = S_1\times,\cdots,\times S_m$, where S_i is a simplex in which the coordinate of whose points are indexed by the number of N_k, that is, for $k = 1,\cdots,m$, $S_k = \Delta_{N_k}$, or say, the S_k is the collection of all real functions x^k defined on N_k which satisfy: $x^k(i) \geq 0$ for all $i \in N_k$ and $\Sigma_{i\in N_k} x^k(i) = 1$.

Theorem 1.3 (Peleg [1969]). Let $C_i^k, i \in N_k, k = 1,\cdots,m$ be closed subsets of S such that for each $Q \subset N_k$, where $k = 1,2,\cdots,m$, $\cup_{j\in Q} C_j^k \supset \{x : x \in S \text{ and } x^k(i) = 0 \text{ for all } i \in N_k \setminus Q\}$, i.e., $\Delta_{N_1} \times \cdots \Delta_{N_{k-1}} \times \Delta_Q \times \Delta_{N_{k+1}} \cdots \times \Delta_{N_m} \subset \cup_{j\in Q} C_j^k$. Then $\cap_{k=1}^m \cap_{i\in N_k} C_i^k \neq \emptyset$.

Let Δ^N be a simplex. We denote by $\mathcal{F}$ the family of all faces of Δ^N.

Definition 1.2. A set-valued mapping $B : \mathcal{F} \to 2^{\Delta^N}$ is called a *Shapley*-mapping if for each $\tau \in \mathcal{F}$,

$$\tau \cup \subset_{\tau\supset\rho\in\mathcal{F}} B(\rho).$$

In 1987, Shih and Tan gave another 'dual form' of Shapley's Theorem above:

Theorem 1.4 (Shih and Tan [1987]). If $B : \mathcal{F} \to 2^{\Delta}$ is a Shapley-map with each $B(\rho)$ an open subset of Δ. Then there exists a balanced set $\mathcal{D}$ of faces of Δ such that

$$\cap_{\tau\in\mathcal{D}} B(\tau) \neq \emptyset.$$

This results shows that both K-K-M principle and Shapley theorem hold if the word 'closed' is replaced by the word 'open'.

Theorem 1.5 (The K-K-M Principle). Let $C_0,\cdots,C_n$ be **open** or closed subsets of the standard n-dimensional simplex Δ_N and let $\{e_0,\cdots,e_n\}$ be the set of its vertices. If for each $J \in \mathcal{F}(N)$, $\Delta_J \subset \cup_{j\in J} C_j$. Then $\cap_{i=0}^n C_i \neq \emptyset$. Here: $N = \{0,1,\cdots,n\}$ and $\Delta_J = co\{e_j : j \in J\}$.

A number of generalizations of above results and their applications are given by many authors, e.g., Fan, Told, Shih and Tan, Ichiishi and Idzik, Border, Lassonde, Gwinner, Granas, Park, Takahashi, Tarafdar $\cdots$ and so on, we only mention a few names here.

In 1987, Horvath proved the following lemma:

Theorem (Horvath (1987). Let X be a topological space. For each non-empty subset J of $\{0,1,\cdots,n\}$, let F_J be a non-empty contractible subset of X with $F_J \subset F_{J'}$ whenever $\emptyset \neq J \subset J' \subset \{0,1,\cdots,n\}$. Then there exists a continuous function $f : \Delta_N \to X$ such that $f(\Delta_J) \subset F_J$ for each non-empty subset J of $\{0,1,\cdots,n\}$.

In 1988, by Horvath's lemma above, Bardaro and Ceppitelli first introduced a new kind of spaces (which is among topological spaces and vector topological spaces) called *H-spaces*. The definition as follows:

Definition 1.3. A pair $(X, \{\Gamma_A\})$ (also called an H-structure) is said to be an H-space (also called c-space according to Horvath (1991)) if X is a topological space and $\{\Gamma_A\}_{A \in \mathcal{F}(X)}$ a given family of non-empty contractible subsets Γ_A of X such that $\Gamma_A \subset F_B$ whenever $A \subset B$.

Then a number of finite intersection theorems have been given by Bardaro and Ceppitelli (1988), Horvath (1991), Kim, Ding and Tan (1990), Tarafdar (1990), Chang and Zhang (1991), Chang and Ma (1992), Park (1993) Zhou et al. In particular, we have the following:

Theorem 1.6. Let $C_0, C_1, \cdots, C_n$ be non-empty closed (resp., open) subsets of a topological space X such that $\cup_{j \in J} C_j$ is contractible for each $J \subset \{0, 1, \cdots, n\}$.

Then $\cap_{j=0}^{n} C_j \neq \emptyset$.

In what follows, we shall discuss the K-K-M principle in infinite vector topological spaces.

Throughout this section, we shall denote by TVS the topological vector space.

Definition 1.4. Let X be a non-empty subset of a real vector space E. A set-valued mapping $G : X \to 2^E$ is called a *Knaster-Kuratowski-Mazurkiewicz mapping* or simply a KKM-mapping if $co\{x_1, x_2, \cdots, x_n\} \subset \cup_{i=1}^{n} G(x_i)$ for each $x_1, x_2, \cdots, x_n \in X$.

Theorem 1.7 (K-K-M-Fan Theorem (Fan [1961]). Let E be a vector space, X be an arbitrary subset of E and $G : X \to 2^E$ a KKM-map with finitely closed values[1]. Then the family $\{G(x) : x \in X\}$ of sets has the finite intersection property, i.e., $\cap_{x \in A} G(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$.

This result was further generalized by Brezis, Nirenberg and Stampacchia (1972), Dugundji and Granas (1978), Tarafdar and Thompson (1978), Yuan (1993), and many others in different ways in the setting of TVS.

In 1961, it was Fan, who first generalized the classic K-K-M principle to infinite vector topological space and established an elementary but very basic '*geometric lemma*' for set-valued mapping:

Fan's Geometric Lemma (1961). Let X be a non-empty convex subset of TVS E and $A \subset X \times X$ such that

(1) the set $\{x \in X : (x, y) \in A\}$ is closed for each $y \in X$;

(2) the set $\{y \in X : (x, y) \notin A\}$ is convex or empty for each $x \in X$;

(3) $(x, x) \in A$ for all $x \in X$.

Then $\exists x_0 \in X$ such that $\{x_0\} \times X \subset A$.

By introducing the concept '*generalized HKKM* map', we have the following characterization of a generalized HKKM mapping in topological spaces:

Theorem 1.8. Let X be a non-empty set and Y a compact topological space. Let $G : X \to 2^Y$ be transfer closed valued on X. Then the intersect $\cap_{x \in X} G(x)$ is non-empty if and only if the mapping clG is a generalized HKKM mapping.

Definition 1.5. Let X be a non-empty set and Y a topological space. A mapping $G : X \to 2^Y$ is said to be a *generalized HKKM* mapping (in short, GHKKM) if for

[1] A subset A in E is finitely closed if its intersection with each finite dimensional linear subspaces $L \subset E$ is closed in the Euclidean topology of L.

each finite subset $A = \{x_1, \cdots, x_n\}$ of X, there exist a corresponding finite subset $B = \{y_1, y_2 \cdots, y_n\}$ (y_i's need not be distinct here) in Y and a family $\{\Gamma_C\}_{C \in \mathcal{F}(B)}$ of non-empty contractible subsets of Y such that $\Gamma_C \subset \Gamma_{C'}$ whenever $C \subset C' \in \mathcal{F}(Y)$ such that

$$\Gamma_{\{y_{i_j} : j \in J\}} \subset \cup_{j=1}^{s} G(x_{i_j})$$

for $\emptyset \neq J \subset \{0, 1, \cdots, n\}$.

2. Topological Interesection Theorems

The study of 'Topological Intersection Theorems' is motivated by the fact that

(a): Many existence questions in mathematics can be reduced to the *Intersection Problem*:

Let Y be a non-empty set, X an index set and $\{\Phi(x) : x \in X\}$ a family of non-empty subsets of Y. Now the question is that when the family has non-empty intersection, i.e., $\cap_{x \in X} \Phi(x) \neq \emptyset$?

(b): Applications to Economics, e.g., the topological characterization of Market equilibria and social choice (see Chichilnisky [1993], Takahashi [1994], Yuan [1994] and references therein.

Let X and Y be two topological spaces and let $\Phi : X \to 2^Y$ be a set-valued mapping. Then $F^* : Y \to 2^X$ is defined by $F^*(y) = \{x \in X : y \notin F(x)\}$ for each $y \in Y$ is called the dual of F.

In 1993, Kindler established the following characterization for a set-valued mapping which has finite intersection property:

Theorem 2.1 (Kindler [1993]). Let X ad Y be two non-empty sets and $F : X \to 2^Y$ with non-empty values. Then the following are equivalent:

(a) $\cap_{x \in X} F(x) \neq \emptyset$.

(b) there exist topologies on X and Y such that

(i) Y is compact; (ii) F is upper semicontinuous with closed values; (iii) the set $\cap_{x \in A} F(x)$ is connected or empty for each $A \in \mathcal{F}(X)$; and (iv) the set $\cap_{y \in B} F^*(y)$ is connected or empty for each $B \in 2^Y$.

Theorem 2.2 (Kindler [1993]). Let X and Y be two non-empty sets and $F : X \to 2^Y$ with non-empty values. Then the following are equivalent:

(a) $\cap_{x \in A} F(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$.

(b) there exist topologies on X and Y such that

(i) F is lower semicontinuous with open values; (ii) the set $\cap_{x \in A} F(x)$ is connected or empty for each $A \in \mathcal{F}(X)$; and (iii) for each closed subset $B \subset Y$, the set $\cap_{y \in B} F^*(y)$ is empty or connected.[2]

As an immediate consequence of Theorem 2.1, we have

Theorem 2.3. Let X and Y be topological spaces and $F : X \to 2^Y$ be a set-valued mapping with non-empty values such that

(i) the graph of F is closed in $X \times Y$;

(ii) Y is compact;

(iii) for each $A \in \mathcal{F}(X)$, the set $\cap_{x \in A} F(x)$ is non-empty or connected;

(iv) for each $B \in 2^Y$, the set $\cap_{y \in B} F^*(y)$ is empty or connected.

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[2] The condition (iii) is equivalent to the following (iii)': the set $\{x \in X : F(x) \subset F(x_1) \cup F(x_2)\}$ is connected for each $x_1, x_2 \in X$.

Then the family $\{F(x) : x \in X\}$ has the finite intersection property, i.e., $\cap_{x\in A}F(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$.

Clearly the condition (i) implies the following (i)′:

(i)′ $F(x)$ and $F^{-1}(y) := \{x \in X : y \in F(x)\}$ are closed for each $x \in X$ and $y \in Y$.

Motivated by the study of minimax theorems for separately upper (or lower) semicontinuous functions, Kindler asked the following question:

Question (Kindler [1993]): *Does the conclusion of Theorem 3 above remain true, if the conditions (i) is replaced by the condition* (i)′ ?

Recently we have proved the following results which partially answer Kindler's question in the affirmative.

Theorem 2.4 (Yuan [1994]). Let X and Y be both topological spaces. Suppose that $F : X \to 2^Y$ is a set-valued mapping with non-empty compact values such that
(1) for each $x, y \in X$, there exists a continuous mapping $u : [0,1] \to X$ with $f(0) = x$, $f(1) = y$ and $F((u(t)) \subset F(u(t_1)) \cup F(u(t_2))$ for each $t \in [t_1, t_2] \subset [0,1]$;
(2) for each $A \in \mathcal{F}(X)$, if the set $\cap_{x\in A}F(x)$ is non-empty, then $\cap_{x\in A}F(x)$ is connected;
(3) for each $y \in Y$, the set $F^{-1}(y) = \{x \in X : y \in F(x)\}$ is closed in X.
Then $\cap_{x\in X}F(x) \neq \emptyset$.

Let X be a non-empty convex subset of a vector space E, we shall denote by $[x_1, x_2]$ the line segment $\{tx_1 + (1-t)x_2 : t \in [0,1]\}$, equipped with the Euclidean topology. A function $f : X \to R \cup \{-\infty, +\infty\}$ is said to be segment upper (respectively, lower) semicontinuous if the function $t \to f(tx_1 + (1-t)x_2)$ is upper (respectively, lower) semicontinuous on $[0,1]$ for each given $x_1, x_2 \in X$.

We also have the following:

Theorem 2.5. Let X be a convex of a vector space E and Y be a topological space. Suppose that $F : X \to 2^Y$ is a set-valued mapping with non-empty compact values such that
(1) for each $x, y \in X$, $F(z) \subset F(x) \cup F(y)$ for each $z \in [x, y]$;
(2)* for each $A \in \mathcal{F}(X)$, if the set $\cap_{x\in A}F(x)$ is non-empty, then $\cap_{x\in A}F(x)$ is connected;
(3) for each $y \in Y$, the set $F^{-1}(y)$ is closed in the line segment $[x_1, x_2]$ for each $x_1, x_2 \in X$.
Then $\cap_{x\in X}F(x) \neq \emptyset$.

Theorem 2.6. Let X and Y be two topological spaces and $\Psi, \Phi : X \to 2^Y$ be both set-valued mappings with non-empty values. Suppose the following conditions are satisfied:
(1) for each $x \in X$, $\Psi(x) \subset \Phi(x)$ and $\cap_{x\in A}\Phi(x)$ is empty or closed (respectively, open) and connected values for each $A \in \mathcal{F}(X)$;
(2) $\cap_{y\in B}\Phi^*(y)$ is connected or empty for each $B \in 2^Y$;
(3) the set $\Psi^{-1}(y)$ is open for each $y \in Y$;
(4) for each $A \in \mathcal{F}(X)$, if $\cap_{x\in A}\Phi(x) \neq \emptyset$ implies that $\cap_{x\in A}\Psi(x) \neq \emptyset$.
Then the family $\{\Phi(x) : x \in X\}$ has the finite intersection property, i.e., $\cap_{x\in A}\Phi(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$.[3]

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[3] The condition (2) is equivlent to the following: The set $\{x \in X : \Phi(x) \subset \Phi(x_1) \cup \Phi(x_2)\}$ is connected

The following example shows that the conclusion of Theorems 2.6 does not hold if we withdraw the condition (4):

Example 2.1. Let $X = [0, 2\pi)$ and $Y = \{e^{i\theta} : 0 \leq \theta < 2\pi\}$. Define $F, G : X \to 2^Y$ by

$$F(\theta) = \{e^{i\psi} : \theta - 1 < \psi < \theta + 1\}$$

for each $\theta \in X$ and

$$G(\theta) = \begin{cases} \{e^{i\psi} : \quad -1 \leq \psi \leq \frac{2\pi}{3} + 1\} & \text{if } \theta \in [0, \frac{2\pi}{3}]; \\ \{e^{i\psi} : \frac{2\pi}{3} - 1 \leq \psi \leq \frac{4\pi}{3} + 1\} & \text{if } \theta \in (\frac{2\pi}{3}, \frac{4\pi}{3}]; \\ \{e^{i\psi} : \quad \frac{4\pi}{3} - 1 \leq \psi \leq 1\} & \text{if } \theta \in (\frac{4\pi}{3}, 2\pi) \end{cases}$$

for each $\theta \in X$. Then it is easy to verify that

(1) $\Psi(\theta) \subset \Phi(\theta)$ and $\cap_{\theta \in A} \Phi(\theta)$ is closed and connected or empty for each $A \in \mathcal{F}(X)$;
(2) $\Psi^{-1}(\psi) = \{\theta : \psi - 1 < \theta < \psi + 1\}$, which is open in X for each $\psi \in Y$;
(3) the set $\{\theta \in X : \Phi(\theta) \subset \Phi(\theta_1) \cup \Phi(\theta_2)\}$ is connected.

But the condition (4) of Theorem 6 does not holds, so that the family $\{\Phi(\theta) : \theta \in X\}$ does not have the finite intersection property, e.g., $\Phi(\frac{\pi}{6}) \cap \Phi(\frac{2\pi}{3} + \frac{\pi}{6})) \cap G(\frac{4\pi}{3} + \frac{\pi}{6}) = \emptyset$.

Before we close this section, we should mention the following topological intersection theorem given by Chichilnisky in 1981 (for survey article, see Chichilnisky [1993] in Bull. AMS.) which also includes K-K-M principle, Caratheodory, Helly, Brouwer fixed point theorem as special cases.

Theorem 2.7 (Chichilnisky [1993]). Let $\{U_\alpha\}_{\alpha \in \Gamma}$ be an excisive family of $k \geq 2$ sets. Then the following are equivalent:
(a) the set $\cap_{i \in A} U_i$ is non-empty and acyclic for each $A \in 2^\Gamma \setminus \{\emptyset\}$.
(b) the set $\cup_{i \in A} U_i$ is acyclic for each $A \in 2^\Gamma \setminus \{\emptyset\}$.

Note: Let $\{U_i\}_{i \in \Gamma}$ be a family of space X. Then $\{U_i\}_{i \in \Gamma}$ is said to be **excisive family** provided

$$\cup_{i \in \Gamma} U_i = \cup_{i \in \Gamma} (int_{U_\Gamma} U_i),$$

where $int_{U_\Gamma}(U_i)$ denotes the relative interior of U_i in the set $U_\Gamma = \cup_{i \in \Gamma} U_i$.

As an application of Theorem 2.7, we have

Theorem 2.8 (Horvath [1987]). Let $C_0, C_1, \cdots, C_n$ be non-empty closed (resp., open) subsets of a topological space X such that $\cup_{j \in J} C_j$ is contractible for each $J \subset \{0, 1, \cdots, n\}$.
Then $\cap_{j=0}^n C_j \neq \emptyset$.

In 1961, it was Fan, who first generalized the classic KKM principle to infinite vector topological space and established an elementary but very basic '*geometric lemma*' for set-valued mapping:

The F-K-K-M Theorem (Fan [1961]). Let E be a vector space, X be an arbitrary subset of E and $G : X \to 2^E$ a KKM mappin with finitely closed values[4]. Then the

for each $x_1, x_2 \in X$.

[4]A subset A in E is finitely closed if its intersection with each finite dimensional linear subspaces $L \subset E$ is closed in the Euclidean topology of L.

family $\{G(x) : x \in X\}$ of sets has the finite intersection property, i.e., $\cap_{x\in A}G(x) \neq \emptyset$ for each $A \in \mathcal{F}(X)$.

As an application of F-KKM theorem, Fan proved the following intersection theorem in TVS and this result could be regarded as the equivalent form of the existence of maximal elements for set-valued mappings which have open inverse values:

Theorem 2.9 (Fan [1961]). Let X be a non-empty compact convex subset of a TVS and $F : X \to 2^X$ with non-empty closed values such that
(i) the set $X\setminus F^{-1}(y)$ is convex (where, $F^{-1}(y) = \{x \in X : y \in F(x)\}$) for each $y \in X$; and
(ii) $x \in F(x)$ for each $x \in X$.
Then $\cap_{x\in X}F(x) \neq \emptyset$.

3. Fixed Points and Maximal Elements

Theorem 3.1. (Fan (1961) - Browder (1968) Fixed Point Theorem). Let X be a non-empty compact convex subset of a topological space E and $F : X \to 2^X$ be such that
(a) for each $x \in X$, $F(x)$ is non-empty convex; and
(b) the set $F^{-1}(y) := \{x \in X : y \in F(x)\}$ is open in X for each $y \in X$.
Then F has one fixed point.

Definition 3.1. Let X and Y be two topological spaces and a mapping $F : X \to 2^Y \cup \{\emptyset\}$ is said to be
(i): F is transfer open inverse valued on X if for each $y \in Y$ and $x \in X$ with $x \in F^{-1}(y) = \{x \in X : y \in F(x)\}$, there exist some $y' \in Y$ and a non-empty open neighborhood $N(x)$ of x in X such that $N(x) \subset F^{-1}(y')$.

It is clear that $F : X \to 2^Y \cup \{\emptyset\}$ is transfer open inverse valued on X if and only if the mapping $G : Y \to 2^X \cup \{\emptyset\}$ defined by $G(y) = X \setminus F^{-1}(y)$ for each $y \in Y$ is transfer closed valued.

(ii) a point $x \in X$ is said to be a maximal element of the mapping F provided $F(x) = \emptyset$.
The following example shows that a transfer open inverse valued mapping may be not open inverse valued.

Example 3.1. Let $X = [0,1]$ and define a mapping $F : X \to 2^{[0,1]}$ by

$$F(x) = \begin{cases} [x,1], & \text{if } x \text{ is rational} \\ [0,1], & \text{if } x \text{ is irrational} \end{cases}$$

The Exampel 3.1 shows that following Theorem 3.2 is really a genralization of Fan-Browder fixed point theorem above.

Theorem 3.2 (Yuan [1993]). Let X be a non-empty compact convex subset of a topological space E and $F : X \to 2^X$ a mapping with non-empty values such that
(a) for each $x \in X$, $F(x)$ is non-empty convex; and
(b) F is transfer open inverse valued.
Then F has a fixed point.

Theorem 3.3 (Maximal Element). Let X be a non-empty compact convex subset of a topological space E and $F : X \to 2^X$ a mapping such that

(a) for each $x \in X$, $F(x)$ is non-empty convex;
(b) F is transfer open inverse valued; and
(c) $x \notin F(x)$.
Then there exists $x \in X$ such that $F(x) = \emptyset$.

4. Equilibria of Abstract Economies

In this section, we shall discuss some existence results of equilibria for abstract economies in the settings of

(1) in Topological Vector Spaces;
(2) in Locally Convex Spaces;
(3) in Frechet Spaces; and
(4) in Finite Diemensional Spaces.

For the backgroun of study for general equilibria of abstract economies (resp., generalized games), the interested readers are referred to references such as Hildenbrand and Sonnenschein [29], Mas-Colell and Zame [45], Yuan [69-70] and reference therein. Now we need following definitions and notions.

Definition 4.1. Let X be a topological space, Y be a non-empty subset of a vector space E, $\theta : X \to E$ be a map and $\phi : X \to 2^Y$ be a correspondence. Then

(1) ϕ is said to be of class $\mathcal{U}_\theta$ if (a) for each $x \in X$, $\theta(x) \notin \phi(x)$ and (b) ϕ is upper semicontinuous with closed and convex values in Y;
(2) ϕ_x is an $\mathcal{U}_\theta$-majorant of ϕ at x if there is a open neighborhood $N(x)$ of x in X and $\phi_x : N(x) \to 2^Y$ such that (a) for each $z \in N(x)$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin \phi_x(z)$ and (b) ϕ_x is upper semicontinuous with closed and convex values;
(3) ϕ is said to be $\mathcal{U}_\theta$-majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists a $\mathcal{U}_\theta$-majorant ϕ_x of ϕ at x.

Definition 4.2. Let X be a topological space, Y be a non-empty subset of a vector space E, $\theta : X \to E$ be a map and $\phi : X \to 2^Y$ be a correspondence. Then

(1) ϕ is said to be of class $\mathcal{L}_\theta$ if for every $x \in X$, $co\phi(x) \subset Y$ and $\theta(x) \notin co\phi(x)$ and for each $y \in Y$, $\phi^{-1}(y) = \{x \in X : y \in \phi(x)\}$ is compactly open in X;
(2) a correspondence $\phi_x : X \to 2^Y$ is said to be an $\mathcal{L}_\theta$–majorant of ϕ at $x \in X$ if there exists an open neighborhood N_x of x in X such that (a): for each $z \in N_x$, $\phi(z) \subset \phi_x(z)$ and $\theta(z) \notin co\phi_x(z)$; (b): for each $z \in X$, $co\phi_x(z) \subset Y$; and (c): for each $y \in Y$, $\phi_x^{-1}(y)$ is compactly open in X;
(3) ϕ is $\mathcal{L}_\theta$–majorized if for each $x \in X$ with $\phi(x) \neq \emptyset$, there exists an $\mathcal{L}_\theta$–majorant of ϕ at x in X.

Definition 4.3. A *generalized game* (*an abstract economy*) is a family of quadruples $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ where I is a (finite or infinite) set of players (agents) such that for each $i \in I$,

X_i is a non-empty subset of a topological space;
$A_i, B_i : X \to 2^{X_i}$ are constraint mappings;
$P_i : X \to 2^{X_i}$ is a preference mapping.
An *equilibrium* of Γ is a point $\hat{x} \in X$ such that for each $i \in I$,
$\hat{x}_i = \pi_i(\hat{x}) \in \overline{B_i}(\hat{x})$ and $A_i(\hat{x}) \bigcap P_i(\hat{x}) = \emptyset$.

Theorem 4.0. Let X be a non-empty compact convex subset of a topological vector space. Suppose the mapping $P : X \to 2^X$ is $\mathcal{L}_{I_X}$-majorized. Then there exists an $\hat{x} \in X$ such that $P(\hat{x}) = \emptyset$, i.e., $\hat{x}$ is a maximal element of P.

Theorem 4.1 (One-Person Game). Let X be a non-empty compact and convex subset of a topological vector space. Let $P : X \to 2^X$ be an $\mathcal{L}$-majorized and $A, B : X \to 2^X$ be such that

(a) for each $x \in X$, $A(x) \neq \emptyset$ and $coA(x) \subset B(x)$;

(b) for each $y \in X$, $A^{-1}(y)$ is compactly open in X.

Then there exists an equilibrium point $\hat{x} \in X$, i.e.,

$\hat{x} \in \overline{B}(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) \neq \emptyset$.

Theorem 4.2 (Qualitative Game). Let $\Gamma = (X_i, P_i)_{i \in I}$ be a qualitative game. Suppose the following conditions are satisfied for each $i \in I$:

(a) X_i is a non-empty compact convex subset of a topological vector space E_i; (b) $P_i : X \to 2^{X_i}$ is F-majorized;

(c) $\cup_{i \in I}\{x \in X : P_i(x) \neq \emptyset\} = \cup_{i \in I} int_X\{x \in X : P_i(x) \neq \emptyset\}$.

Then Γ has an equilibrium point in X.

Theorem 4.3 (Abstract economy). Let $\Gamma = (X_i; A_i, B_i; P_i)_{i \in I}$ be an abstract economy. Suppose that the following conditions are satisfied for each $i \in I$:

(i) X_i is a non-empty compact convex subset of a topological vector space;

(ii) for each $x \in X$, $A_i(x)$ is non-empty, $A_i(x) \subset coB_i(x)$;

(iii) for each $y \in X_i$, $A_i^{-1}(y)$ is compactly open in X;

(iv) for each $i \in I$, $A_i \cap P_i$ is F-majorized;

(v) $E_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is open in X.

Then Γ has an equilibrium $\hat{x}$ in X.

Defintion 4.4. Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy. Γ is said to have *approximate equilibria* if for each for non-empty open convex neighborhood V_i of the topological vector space E_i for each $i \in I$, the abstract economy game $\Gamma' = (X_i, A_i, B_{V_i}, P_i)_{i \in I}$ has an equilibrium point, i.e., there exists a point $x = (x_i)_{i \in I} \in X$ such that

$A_i(x) \cap P_i(x) = \emptyset$ and $x_i \in \overline{B_{V_i}}(x)$,

where the mapping $B_{V_i} : X \to 2^{X_i}$ is defined by

$$B_{V_i}(x) = (B_i(x) + V_i) \cap X_i$$

for each $x \in X$ and for each $i \in I$.

Theorem 4.4 (Approximate Equilibria). Let $\Gamma = (X_i, A_i, B_i, P_i)_{i \in I}$ be an abstract economy such that $X = \prod_{i \in I} X_i$. Suppose that the following conditions are satisfied for each $i \in I$:

(1) X_i is a nonempty compact and convex subset of a topological vector space E_i;

(2) A_i is lower semicontinuous with non-empty values and for each $x \in X$, $co(A_i(x)) \subset \overline{B_i(x)}$;

(3) $A_i \cap P_i$ is F-majorized;

(4) $E^i = \{x \in X : (A_i \cap P_i)(x) \neq\}$ is open in X.

Then Γ has *approximate equilibria*, i.e., for give any $V = \Pi_{i \in I} V_i$ where for each $i \in I$, V_i is a convex open neighborhood of zero in E_i, there exists a point $\hat{x}_V = (\hat{x}_{V_i})_{i \in I} \in K$ such that for each $i \in I$,

$$\hat{x}_{V_i} \in \overline{B_{V_i}}(\hat{x}_V)$$

and

$$A_i(\hat{x}_V) \cap P_i(\hat{x}_V) = .$$

where $B_{V_i}(x) = (B_i(x) + V_i) \cap X$ for each $i \in I$ and for each $x \in X$.

Theorem 4.5. Let $\Gamma = (X_i, A_i, B_i, P_i)_{i\in I}$ be an abstract economy such that $X = \prod_{i\in I} X_i$. Suppose the following conditions are satisfied for each $i \in I$:

(1) X_i is a nonempty compact convex subset of a locally convex topological vector space E_i;

(2) A_i is lower semicontinuous such that for each $x \in X$, $A_i(x)$ is nonempty and $co(A_i(x)) \subset \overline{B_i(x)}$;

(3) $A_i \cap P_i$ is F-majorized;

(4) the set $E^i = \{x \in X : (A_i \cap P_i)(x) \neq\}$ is open in X.

Then Γ has an equilibrium point in X.

Theorem 4.6. Let X be a non-empty convex subset of a locally convex topological vector space and D be a non-empty compact subset of X. Let $P : X \to 2^D$ be $\mathcal{U}$-majorized and $A : X \to 2^D$ be upper semicontinuous with closed and convex values.

Then there exists a point $\hat{x} \in coD$ such that either $\hat{x} \in A(\hat{x})$ and $P(\hat{x}) = \emptyset$ or $\hat{x} \notin A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$.

Theorem 4.7. Let $\Gamma = (X_i, P_i)_{i\in I}$ be a qualitative game such that for each $i \in I$,

(a) X_i is a non-empty convex subset of a locally convex topological vector space E_i and D_i is a non-empty compact subset of X_i;

(b) the set $E^i = \{x \in X : P_i(x) \neq \emptyset\}$ is open in X;

(c) $P_i : E^i \to 2^{D_i}$ is $\mathcal{U}$-majorized;

(d) there exists a non-empty compact and convex subset F_i of D_i such that $F_i \cap P_i(x) \neq \emptyset$ for each $x \in E^i$.

Then there exists a point $x \in X$ such that $P_i(x_i) = \emptyset$ for all $i \in I$.

Theorem 4.8. Let $\Gamma = (X_i; A_i, B_i; P_i)_{i\in I}$ be a generalized game (an abstract economy) where I is any (countable or uncountable) number of players (agents) such that for each $i \in I$:

(i) X_i is a non-empty compact and convex subset of a locally convex topological vector space E_i;

(ii) each $x \in X$, $A_i(x)$ is non-empty, $A_i(x) \subset B_i(x)$ and $B_i(x)$ is convex;

(iii) the set $E^i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is paracompact (which is satisfied if X_i is metrizable) and open in X;

(iv) the mapping $A_i \cap P_i : X \to 2^{X_i}$ is $\mathcal{U}$-majorized.

Then Γ has a equilibria point.

Example 4.1. Let $X = [0, 1]$ and define $A, B, P : X \to 2^X$ by

$$A(x) = B(x) = \begin{cases} [\frac{1}{2}, 1], \text{if } x \in [0, 1/2); \\ [0, 1], \text{if } x = 1/2; \\ [0, \frac{1}{2}], \text{if } x \in (1/2, 1]. \end{cases}$$

and

$$P(x) = \begin{cases} \{\frac{x}{4}\}, \text{if } x \in [1/2, 1]; \\ \emptyset \quad, \text{if } x \in [0, 1/2). \end{cases}$$

It is easy to see that A and P are both upper semicontinuous with closed and convex values and $x \notin P(x)$ for each $x \in X$; thus $A \cap P$ is $\mathcal{U}$-majorized. Note that the subset

$E = \{x \in X : A(x) \cap P(x) \neq \emptyset\} = [1/2, 1]$ is closed in $[0,1]$ and A, B and P satisfy the hypotheses (i), (ii), (iv) but not (iii) of Theorem 4.8. However, at the unique fixed point $1/2$ of the correspondence A, we have $A(\frac{1}{2}) \cap P(\frac{1}{2}) = [0,1] \cap \{1/8\} \neq \emptyset$. Thus the generalized game $([0,1]; A, B; P)$ has no equilibrium point.

5. Equilibria in Frechet Spaces

In this section, we shall give several existence theorems of equilibria for generalized games in Frechet spaces in which the strategy sets are compact and convex.

Theorem 5.1. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ and $X = \Pi_{i \in I} X_i$, where I is any set. Suppose that for each $i \in I$, the following conditions are satisfied:

(i) X_i is a non-empty compact and convex subset of a Frechet space E_i;

(ii) A_i is lower semicontinuous with non-empty closed convex values;

(iii) for each $x \in X$, $\pi_i(x) \notin A_i(x) \cap P_i(x)$;

(iv) the set $U_i := \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is closed in X.

(v) the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is closed and convex.

Then Γ has an equilibrium point.

Theorem 5.2. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be a generalized game, where I is any set. Suppose for each $i \in I$, the following conditions are satisfied:

(i) X_i is a non-empty compact and convex subset of a Frechet space E_i;

(ii) A_i is upper semicontinuous with non-empty closed convex values;

(iii) the set $U_i = \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is paracompact and open in X;

(iv) the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is closed and convex.

Then there exists $x^* \in X$ such that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

In the case of finite dimensional spaces, we have the following:

Theorem 5.1′. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$. and $X = \Pi_{i \in I} X_i$, where I is any (countable or uncountable) set. Suppose that for each $i \in I$, the following conditions are satisfied:

(i) X_i is a non-empty compact and convex subset of a finite dimensional space E_i;

(ii) A_i is lower semicontinuous with non-empty convex values (but not necessarily closed);

(iii) for each $x \in X$, $\pi_i(x) \notin A_i(x) \cap P_i(x)$;

(iv) the set $U_i := \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is closed in X.

(v) $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is convex (but not necessarily closed).

Then there exists $x^* \in X$ such that for each $i \in I$, $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

Theorem 5.2′. Let $\Gamma = (X_i, A_i, P_i)_{i \in I}$ be a generalized game where I is any set. Suppose for each $i \in I$, the following conditions are satisfied:

(i) X_i is a non-empty compact and convex subset of a finite dimensional vector space E_i;

(ii) A_i is upper semicontinuous with non-empty closed convex values;

(iii) the set $U_i := \{x \in X : A_i(x) \cap P_i(x) \neq \emptyset\}$ is paracompact and open in X;

(iv) the mapping $A_i \cap P_i$ is lower semicontinuous on U_i such that for each $x \in U_i$, $A_i(x) \cap P_i(x)$ is convex (but not necessarily closed).

Then there exists $x^* \in X$ such that for each $i \in I$, either $\pi_i(x^*) \in A_i(x^*) \cap P_i(x^*)$ or $\pi_i(x^*) \in A_i(x^*)$ and $A_i(x^*) \cap P_i(x^*) = \emptyset$.

6. Equilibria in Finite Dimensional Spaces

In this section, some results for lower semicontinuous maps and fixed point theorems are obtained and applied to achieve existence theorems of equilibrium points of a generalized game and of a qualitative game in finite dimensional spaces.

We now first introduce some notations. Let E be a vector space and $A \subset E$. We shall denote by $aff(A)$ the affine span of A and $ri(A)$ the relative interior of A in $aff(A)$. The subset A is said to be finite dimensional if A is contained in a finite dimensional subspace of E.

Theorem 6.1. Let S be a topological space and $F_1,\ F_2 : S \to 2^{\mathbb{R}^n}$ be lower semicontinuous at $x_0 \in S$ such that F_1 is open and convex-valued. Then $F_1 \cap F_2$ is also lower semicontinuous at x_0.

The following example shows that Theorem 6.1 fails to hold if $\mathbb{R}^n$ is replaced by an infinite dimensional Banach space:

Example 6.1. (Lechicki and Spakowski). Let $Y = l^\infty$, the Banach space of all bounded sequences $x = (x_n)_{n=1}^\infty$ of real numbers with $\|x\|_\infty = \sup_{n\in N} |x_n| < \infty$ and $S = [0,1]$. Define $G_1, G_2 : S \to 2^Y$ by

$$G_1(t) = \{x \in Y : x_1 \geq t, x_k \leq k - t\}$$

for $k \geq 2$ and

$$G_2(t) = \{x \in Y : x_1 \leq 1 - t, x_k \leq k(1 - x_1 - t) \text{ and } x_k \leq k + x_1/k - t/k\}$$

for $k \geq 2$. Then G_1 and G_2 are both lower semicontinuous at 0 with closed convex values and, $int_Y(G_1(0) \cap G_2(0)) \neq \emptyset$. But $G_1 \cap G_2$ is not lower semicontinuous at 0.

Theorem 6.2. Let S be a topological space, X be a non-empty subset of a finite dimensional topological vector space and $F_1, F_2 : S \to 2^{aff(X)}$ be lower semicontinuous at $x \in S$. If $F_1(x)$ and $F_2(x)$ are convex and $ri(F_1(x)) \cap F_2(x) \neq \emptyset$ or $F_1(x) \cap ri(F_2(x)) \neq \emptyset$ whenever $F_1(x) \cap F_2(x) \neq \emptyset$, then $F_1 \cap F_2$ is also lower semicontinuous at x.

Theorem 6.3. Let I be a non-empty countable set. For each $i \in I$, let C_i be a non-empty compact convex subset of a finite dimensional topological vector space E_i and $F_i : C := \Pi_{j\in I} C_j \to 2^{E_i}$ be lower semicontinuous such that

(a) $F_i(x)$ is convex for each $x \in C$;

(b) $F_i(x) \cap ri(C_i) \neq \emptyset$ for each $x \in C$;

(c) $F_i(C) \subset aff(C_i)$.

Then the map $F := \Pi_{i\in I} F_i$ has a fixed point in C.

Corollary 6.4. Let X be a non-empty subset of a finite dimensional topological vector space E, C be a non-empty compact convex subset of X and $F : X \to 2^E$ be lower semicontinuous such that $F(C) \subset aff(C)$ and for each $x \in C$, $F(x)$ is convex and $F(x) \cap ri(C) \neq \emptyset$.

Then F has a fixed point in X.

By Corollary 6.4, we now have the existence theorem of equilibria for generalized games:

Theorem 6.5. Let $\mathcal{G} = (X_i; A_i, B_i; P_i)_{i\in I}$ be a generalized game, where I is countable. Suppose that for each $i \in I$, the following properties hold:

(1) X_i is a non-empty subset of a finite dimensional topological vector space E_i.

(2) $A_i, B_i : X = \Pi_{i\in I}X_i \to 2^{X_i}$ are lower semicontinuous such that $coA_i(x) \subset B_i(x)$ for each $x \in X$.

(3) $P_i : X \to 2^{X_i}$ is lower semicontinuous on D_i, where $D_i = \{x \in X : P_i(x) \cap A_i(x) \neq \emptyset\}$ is closed in X.

(4) There exists a non-empty finite dimensional compact convex subset C_i of X_i such that

(a) for each $x \in C := \Pi_{j\in I}C_j$, $P_i(x)$ is open in $aff(X_i)$;

(b) for each $x \in C$, $coA_i(x) \cap ri(C_i) \neq \emptyset$;

(c) for each $x \in C \cap D_i$, $coP_i(x) \cap coA_i(x) \cap C_i \neq \emptyset$ and (d) $coA_i(x) \subset aff(C_i)$.

(5) For each $x \in C$, $\pi_i(x) \notin coP_i(x)$.

Then $\mathcal{G}$ has an equilibrium point x^* in C.

The following example shows that the condition (3) of Theorem 6.5 could not withdrawn.

Example 6.2. Define $A : [0,1] \to 2^{[0,1]}$ by $A(x) = [0, 1-x]$ for each $x \in [0,1]$. Clearly, A is continuous with closed convex values. Now define $A_1 : [0,1] \to 2^{[0,1]}$ by

$$A_1(x) := \begin{cases} [0, 1-x], & \text{if } x \in (0,1]; \\ \{1\}, & \text{if } x = 0 \end{cases}$$

is also lower semicontinuous. We also define $A_2 : [0,1] \to 2^{[0,1]}$ by

$$A_2(x) = \begin{cases} co([0,1-x] \cup \{\frac{1}{2}\}), & \text{if } x \in (0,1] \\ co(\{1\} \cup \{\frac{1}{2}\}), & \text{if } x = 0, \end{cases}$$

then A_2 is also lower semicontinuous and in fact

$$A_2(x) = \begin{cases} [0, 1-x], & \text{if } x \in (0,1/2]; \\ [0,1/2], & \text{if } x \in (1/2,1); \\ [1/2,1], & \text{if } x = 0; \\ [0,1/2], & \text{if } x = 1. \end{cases}$$

We now define $P : [0,1] \to 2^{[0,1]}$ by $P(x) = [0,x)$ for each $x \in [0,1]$ then for any $y \in [0,1]$, we have $P^{-1}(y) = (y,1]$. Since $\{x \in [0,1], P(x) \cap A_2(x) \neq \emptyset\} = (0,1]$ and the fixed point set of A_2 is $(0,1/2]$, but for each $x \in (0,1/2]$, $A_2(x) \cap P(x) \neq \emptyset$.

Finally we shall derive the following existence theorem of an equilibrium point for a qualitative game.

Theorem 6.6. Let $\mathcal{G} = (X_i; P_i)_{i\in I}$ be a qualitative game where I is countable. Suppose that for each $i \in I$, the following properties hold

(1) X_i is a non-empty subset of a topological vector space E_i.

(2) $P_i : X = \Pi_{j\in I}X_j \to 2^{X_i}$ is lower semicontinuous on D_i, where $D_i = \{x \in X : P_i(x) \neq \emptyset\}$ is closed in X.

(3) There exists a non-empty finite dimensional compact convex subset C_i of X_i such that

(a) for each $x \in C := \Pi_{j\in I}C_j$, $P_i(x) \subset aff(C_i)$;

(b) for each $x \in C \cap D_i$, $coP_i(x) \cap ri(C_i) \neq \emptyset$;

(c) for each $x \in C$, $\pi_i(x) \notin coP_i(x)$.
Then $\mathcal{G}$ has an equilibrium point x^* in C.

Before concluding this section, we would like to note that Theorem 6.5 is a non-compact form of Theorem 5.1′.

7. Equilibria in Topological Spaces

In this section, we shall present some existence results of equilibria for abstarct economics without linear structures. First we have:

Theorem 7.1 (Maximal Element). Let X be a contractible subset of a Hausdorff compact space Y. Suppose that $B_0, B : Y \to 2^X \cup \{\emptyset\}$ are such that
(a) for each $x \in X$, $B_0(x) \subset B(x)$ and $B_0^{-1}(x)$ is open;
(b) for each open set S in Y, the set $\cap_{y \in S} B(y)$ is empty or contractible; and
(c) for any $x \in X$, $x \notin B(x)$.
Then there exist $y_0 \in Y$ such that $B_0(y_0) = \emptyset$.

Theorem 7.2. Let I be an arbitrary set and let $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ be families of compact topological spaces and contractible topological spaces respectively. Suppose there exists a family of mappings $\{P_i : X \to 2^{Y_i} \cup \{\emptyset\}, i \in I\}$ such that for each $i \in I$,
(i) $Y_i \subset X_i$;
(ii) there exist two mappings $A_i, B_i : X \to 2^{Y_i} \cup \{\emptyset\}$ such that
(ii)$_a$: for each $x \in X$, $A_i(x) \subset B_i(x)$ and $\{z \in X : P_i(z) \neq \emptyset\} \subset \{z \in X : A_i(z) \neq \emptyset\}$;
(ii)$_b$: $A_i^{-1}(y)$ is open in X for each $y \in Y_i$ and
(ii)$_c$: for each open subset U of X, the set $\cap_{z \in U} B_i(z)$ is empty or contractible;
(iii) for each $x \in X$, $x_i \notin B_i(x)$.
Then there exists $x_0 \in X$ such that $P_i(x_0) = \emptyset$ for each $i \in I$.

Definition 7.1. Let X and Y be two topological spaces. Let $P : X \to 2^Y \cup \{\emptyset\}$ be a set-valued mapping and $\theta : X \to Y$ be a single-valued mapping. Then two mappings $A, B : X \to 2^{Y_i} \cup \{\emptyset\}$ is said to be an H$_\theta$-pair of the mapping P provided

(a) for each $x \in X$, $\theta(x) \notin B(x)$ and $A(x) \subset B(x)$; and

(b) for each $y \in Y$, $A^{-1}(y)$ is open in X and the set $\{x \in X : P(x) \neq \emptyset\} \subset \{x \in X : A(x) \neq \emptyset\}$.

Theorem 7.3. Let I be an arbitrary set and let $\{X_i : i \in I\}$ and $\{Y_i : i \in I\}$ be families of compact topological spaces and contractible topological spaces respectively. Suppose there exist families $\{P_i, A_i, B_i : X \to 2^{Y_i} \cup \{\emptyset\}, i \in I\}$ such that for each $i \in I$:
(i) $Y_i \subset X_i$;
(ii) for each $x \in X$, $A_i(x) \subset B_i(x)$;
(iii) for each $x \in X$, $A_i(x)$ is non-empty and the set $A_i^{-1}(y)$ is open in X for each $y \in Y_i$;
(iv) the mapping $A_i \cap P_i$ has an H-pair mappings $\psi_i, \phi_i : X \to 2^{Y_i} \cup \{\emptyset\}$ such that the set $\cap_{x \in U_1} A_i(x) \cap \cap_{x \in U_2} \phi_i(x)$ is empty or contractible for any subsets U_1 and U_2 of X[5].
Then there exists $\hat{x} \in X$ such that $A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset$ and $\hat{x}_i \in \overline{B_i}(\hat{x})$ for each $i \in I$.

8. Random Equilibria

[5]This property is automatically satisfied if both A_i and ϕ_i are convex-valued provided X and Y_i are convex subset of topological vector spaces.

I - the set of players and (Ω, Σ) - a measurable space. A *random generalized game (or a random abstract economy)* is a collection $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ such that for each $i \in I$,

X_i is a non-empty subset of a TVS;

$A_i, B_i : \Omega \times X \to 2^{X_i}$ are random constraint mappings;

$P_i : \Omega \times X \to 2^{X_i}$ is a preference mapping

(which are interpreted as for each player (or agent), the associated constraint and preferences A_i, B_i and P_i have stochastic actions).

Definition 8.1. A *random equilibrium* of Γ is a single-valued measurable mapping $\Omega \to X$ such that for each $i \in I$, $\pi_i(\psi(\omega)) \in \overline{B_i}(\omega, \psi(\omega))$ and $A_i(\omega, \psi(\omega)) \cap P_i(\omega, \psi(\omega)) = \emptyset$ for all $\omega \in \Omega$. Here, π_i is the projection from X onto X_i.

Theorem 8.1. Let (Ω, Σ) be a measurable space, Σ be a Suslin family and $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ be a random generalized game and $X = \Pi_{i \in I} X_i$. suppose that I is countable and $Dom(A_i \cap P_i) \in \Omega \otimes \mathcal{B}(X)$, and $Graph\overline{B_i} \in \Sigma \otimes \mathcal{B}(X \times X_i)$ for each $i \in I$. Suppose that the following conditions are satisfied:

(i) for each $i \in I$, X_i is a non-empty convex Suslin and compact subset of a locally convex Hausdorff topological vector space;

(ii) for each $i \in I$ and for each $(\omega, x) \in \Omega \times X$, $A_i(\omega, x)$ is non-empty, $coA_i(\omega, x) \subset B_i(\omega, x)$;

(iii) for each $i \in I$ and for any given $\omega \in \Omega$, $A(\omega, \cdot) : X \to 2^{X_i}$ is lower semicontinuous;

(iv) for each $i \in I$ and $\omega \in \Omega$, $A_i(\omega, \cdot) \cap P_i(\omega, \cdot)$ is L_C-majorized;

(v) for each $i \in I$, $E_i(\omega) = \{x \in X : A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is open in X for each $\omega \in \Omega$.

Then Γ has a random equilibrium.

Theorem 8.2. Let (Ω, Σ) be a measurable space with Σ a Suslin family and $\Gamma = (\Omega; X_i, A_i, B_i, P_i)_{i \in I}$ be a random generalized game, where I is a countable and $Dom(A_i \cap P_i) \in \Sigma \otimes \mathcal{B}(X)$, $Graph\overline{B_i} \in \Sigma \otimes \mathcal{B}(X \times X_i)$ for each $i \in I$. Suppose that the following conditions are satisfied:

(a) for each $i \in I$, X_i is a non-empty convex Suslin and compact subset in a locally convex Hausdorff topological vector space E_i;

(b) for each $i \in I$ and for each fixed $\omega \in \Omega$, $B_i(\omega, \cdot)$ is compact and upper semicontinuous with non-empty compact and convex values, and for each $(\omega, x) \in \Omega \times X$, $A_i(\omega, x) \subset B_i(\omega, x)$;

(c) for each $i \in I$ and for each fixed $\omega \in \Omega$, $P_i(\omega, \cdot)$ is lower semicontinuous and L_C-majorized;

(d) for each $i \in I$ and $\omega \in \Omega$, $E^i(\omega) = \{x \in X : A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is open in X.

Then Γ has a random equilibrium.

Theorem 8.3. Let (Ω, Σ) be a measurable space with Σ a Suslin family and $\Gamma = (\Omega; X_i; A_i, B_i; P_i)_{i \in I}$ be a random generalized game such that I is countable and $Dom(A_i \cap P_i) \in \Sigma \otimes \mathcal{B}(X)$ and $Graph\overline{B_i} \in \Sigma \otimes \mathcal{B}(X \times X_i)$ for each $i \in I$. Suppose that the following conditions are satisfied:

(i) for each $i \in I$, X_i is a non-empty compact and convex Suslin subset in a locally convex Hausdorff topological vector space E_i;

(ii) for each $i \in I$, for each $(\omega, x) \in \Omega \times X$, $A_i(\omega, x)$ is non-empty, $A_i(\omega, x) \subset B_i(\omega, x)$ and $B_i(\omega, x)$ is convex;

(iii) for each $i \in I$, for each $\omega \in \Omega$, the set $E^i(\omega) = \{x \in X : A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is open and paracompact in X;

(iv) for each fixed $\omega \in \Omega$, the mapping $A_i(\omega,\cdot) \cap P_i(\omega,\cdot) : X \to 2^{X_i}$ is $\mathcal{U}$-majorized on $E^i(\omega)$.
Then Γ has a random equilibrium.

Theorem 8.4. Let $\Gamma = (\Omega; X_i; A_i; P_i)_{i\in I}$ be a random generalized game with Σ a Suslin family and I be countable. Suppose for each $i \in I$, the following conditions are satisfied:

(i) $Dom(A_i \cap P_i) \in \Sigma \otimes \mathcal{B}(X)$ and $GraphA_i \in \Sigma \otimes \mathcal{B}(X \times X_i)$;

(ii) X_i is a non-empty compact and convex subset of a Frechet space E_i;

(iii) $A_i(\omega,\cdot)$ is lower semicontinuous with non-empty closed convex values for each fixed $\omega \in \Omega$;

(iv) for each $(\omega, x) \in \Omega \times X$, $\pi_i(x) \notin A_i(\omega, x) \cap P_i(\omega, x)$;

(v) the set $U_i(\omega) := \{x \in X : A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is closed in X for each fixed $\omega \in \Omega$;

(vi) the mapping $A_i(\omega,\cdot) \cap P_i(\omega,\cdot)$ is lower semicontinuous on $U_i(\omega)$ such that for each $x \in U_i(\omega)$, $A_i(\omega, x) \cap P_i(\omega, x)$ is closed and convex.

Then the random generalized game Γ has a random equilibrium.

Theorem 8.5. Let $\Gamma = (\Omega; X_i; A_i; P_i)_{i\in I}$ be a random generalized game with Σ a Suslin family and I be countable. Suppose for each $i \in I$, the following conditions are satisfied:

(i) $Dom(A_i \cap P_i) \in \Sigma \otimes \mathcal{B}(X)$ and $GraphA_i \in \Sigma \otimes \mathcal{B}(X \times X_i)$;

(ii) X_i is a non-empty compact and convex subset of a Frechet space E_i;

(iii) for each fixed $\omega \in \Omega$, $A_i(\omega,\cdot)$ is continuous with non-empty compact convex values;

(iv) the set $U_i(\omega) = \{x \in X : A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is either open or closed in X for each fixed $\omega \in \Omega$;

(v) the mapping $A_i(\omega,\cdot) \cap P_i(\omega,\cdot)$ is lower semicontinuous on $U_i(\omega)$ such that for each $x \in U_i(\omega)$, $A_i(\omega, x) \cap P_i(\omega, x)$ is closed and convex for each fixed $\omega \in \Omega$.

Then the random generalized game Γ has a random equilibrium.

Theorem 8.6. Let $\Gamma = (\Omega; X_i; A_i; P_i)_{i\in I}$ be a random generalized game with Σ a Suslin family and I be countable. Suppose for each $i \in I$, the following conditions are satisfied:

(i) $Dom(A_i \cap P_i) \in \Sigma \otimes \mathcal{B}(X)$ and $GraphA_i \in \Sigma \otimes \mathcal{B}(X \times X_i)$;

(ii) X_i is a non-empty compact and convex subset of a finite dimensional space E_i;

(iii) $A_i(\omega,\cdot)$ is continuous with non-empty compact convex values for each fixed $\omega \in \Omega$;

(iv) the set $U_i(\omega) := \{x \in X : A_i(\omega, x) \cap P_i(\omega, x) \neq \emptyset\}$ is either open or closed in X for each fixed $\omega \in \Omega$;

(v) the mapping $A_i(\omega,\cdot) \cap P_i(\omega,\cdot)$ is lower semicontinuous on $U_i(\omega)$ such that for each $x \in U_i(\omega)$, $A_i(\omega, x) \cap P_i(\omega, x)$ is convex (but not necessarily closed) for each fixed $\omega \in \Omega$.

Then the random generalized game Γ has a random equilibrium.

9. Conclusion Remarks

At the end of this paper, we would like to point that the the study of general existence of equilibria for abstact economics and related topics as applications of K-K-M theory can illustrated by the following diagram and related notions which are not appear in this paper can be found from [61-70] (in particular, from [69] or [70]) and references therein.

The Knaster - Kuratowski - Mazurkiewicz Principle

$\Uparrow$

The Fan's Geometry Lemma

$\Uparrow$

The Fan - Browder Fixed Point Theorem

$\Uparrow$

The Maximal Element Theorem of Ky Fan Mappings

$\Uparrow$

Selection Theorem of Ky Fan Majorized Mappings

$\Uparrow$

The Maximal Element Theorem of Ky Fan Majorized Mappings

$\Uparrow$

The One-Person Game

$\Uparrow$

The Qualitative Game

$\Uparrow$

The Abstract Economy (resp., Generalized Game) $\Longleftarrow$ Nash - Equilibria

$\Uparrow$

Approximate Equilibria

$\nearrow\!\nearrow$ $\qquad$ $\nwarrow\!\nwarrow$

Lower Semicontinuous Mappings $\qquad$ Upper Semicontinuous Mappings.

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