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Extreme points of an intersection of operator intervals

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Though the title of my talk at the RIMS meeting was *Extreme points of an intersection of matrix intervals*, here the problem is treated in the Hilbert space setting with title *Extreme points of an intersection of operator intervals.* The detail with full proof will appear in the **Proceedings of International Mathematics Conference '94, Kaohsiung, Taiwan**, World Scienctific, 1996.

1. Introduction

Let $\mathcal{H}$ be a (complex) Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and $\mathcal{B}(\mathcal{H})$ the space of (bounded linear) operators on $\mathcal{H}$. When $dim(\mathcal{H})=n<\infty$, with respect to a suitably chosen orthonormal basis, $\mathcal{B}(\mathcal{H})$ is identified with the space of $n\times n$ matrices.

Recall that an operator A is *selfadjoint* if $A^{*}=A$ where A^{*} is the *adjoint* of A, that is, $\langle A^{*}x,y\rangle=\langle x,Ay\rangle$ $(x,y\in \mathcal{H})$. Selfadjointness of A is characaterized by that $\langle x,Ax\rangle$ is real for all $x\in \mathcal{H}$. Recall further that A is *positive (semidefinite)* if $\langle x,Ax\rangle\geq 0$ for all $x\in \mathcal{H}$. Order relation $A\geq B$ for a pair of selfadjoint operators A,B is defined as $A-B$ is positive. Therefore $A\geq 0$ means positivity of A. We write $A>B$ to mean that $A\geq B$ and $A-B$ is invertible, which is equivalent to $A-B\geq\varepsilon I$ for some $\varepsilon>0$ where I is the identity operator.

Given $A_{1},A_{2},\ldots,A_{m}\geq 0$, let us observe the set

$$\Delta(A_{1},A_{2},\ldots,A_{m})\stackrel{def}{=}\{X\mid 0\leq X\leq A_{j}\ j=1,2,\ldots,m\}.$$

In particular, $\Delta(A)$ for $A\geq 0$ is the *operator interval* $\{X\mid 0\leq X\leq A\}$, so that $\Delta(A_{1},A_{2},\ldots,A_{m})$ is an intersection of operator intervals;

$$\Delta(A_{1},A_{2},\ldots,A_{m})=\bigcap_{j=1}^{m}\Delta(A_{j}).$$

Since $\Delta\equiv\Delta(A_{1},A_{2},\ldots,A_{m})$ is a convex set, compact with respect to the weak operator topology, according to the Krein-Milman theorem it has plenty many extreme points. Recall that $X\in\Delta$ is *extreme* if $X=(Y+Z)/2$ for $Y,Z\in\Delta$ is possible only when $Y=Z=X$. Let us denote by $ex\Delta\equiv ex\Delta(A_{1},A_{2},\ldots,A_{m})$ the set of extreme points of Δ.

Our aim is a detailed study of the extreme points of Δ. In Section 3 we give several necessary and sufficient conditins for $X\in\Delta$ to be an extreme point. In Section 4 we present labelling of extreme points and in Section 5, under the assumption $dim(\mathcal{H})<\infty$ and $m=2$, we give a complete parametrization of extreme points of Δ. The final part, Section 6, contains an algorithm of construction of an extreme point, associated with arbitrarily given $X\in\Delta$. In the preliminary part, Section 2, we recall various properties of two basic operations, necessary for our study; parallel sum and short.

This paper enlarges and extends the content of an unpublished manuscript [Ao 89] , a part of which has been made public in [P 91].

2. Preliminaries

Given $A, B > 0$, the operator

$$A : B \stackrel{def}{=} (A^{-1} + B^{-1})^{-1}$$

is called the *parallel sum* of A and B. This notion was first introduced in [AD 69] as mathematical description of the impedance of parallel connection of electrical networks. From the standpoint of quadratic forms the following variational description is more useful (see [AT 75]) and shows that parallel addition corresponds to the so-called *inf-convolution* of two positive quadratic forms (see [M 88]);

$$\langle x, (A : B)x\rangle = \inf\{\langle y, Ay\rangle + \langle z, Bz\rangle \ ; \ y + z = x\} \quad (x \in \mathcal{H}). \tag{2.1}$$

Via (2.1) we can extend the definiton of parallel sum to any pair of positive operators. This extended operation enjoyes the following properties (see [AT 75], [NA 76], [PS 76] and [EL 89]);

$$A,\ B \geq A : B = B : A, \tag{2.2}$$

$$(A : B) : C = A : (B : C), \tag{2.3}$$

$$(\lambda A) : (\mu A) = \frac{\lambda\mu}{\lambda + \mu} A \quad (\lambda, \mu > 0), \tag{2.4}$$

$$A_1 \geq A_2,\ B_1 \geq B_2 \Longrightarrow A_1 : B_1 \geq A_2 : B_2, \tag{2.5}$$

and more generally

$$A_k \downarrow A,\ B_k \downarrow B \Longrightarrow A_k : B_k \downarrow A : B, \tag{2.6}$$

where $A_k \downarrow A$ say, means tht A_k decreasingly converges to A in strong operator-topology as $k \to \infty$, and

$$(S^*AS) : (S^*BS) = S^*(A : B)S \quad \text{for invertible } S \in \mathcal{B}(\mathcal{H}). \tag{2.7}$$

In view of commutativity (2.2) and associativity (2.3) there is no confusion to use

$$\prod_{j=1}^{m} : A_j \ \equiv \ (\ldots((A_1 : A_2) : A_3) \cdots : A_m).$$

Another notion we need is short (operation). For $A, X \geq 0$ the sequence $(kX) : A$ increases as $k \to \infty$. Following [Ao 76] let us define $[X]A$ as its strong limit;

$$[X]A \stackrel{def}{=} \lim_{k\to\infty} (kX) : A. \tag{2.8}$$

By (2.5) the following are clear from definition (2.8):

$$A \geq [X]A, \tag{2.9}$$

$$A_1 \geq A_2 \geq 0,\ X_1 \geq X_2 \Longrightarrow [X_1]A_1 \geq [X_2]A_2, \tag{2.10}$$

and by (2.3) and (2.4)

$$\alpha X_1 \geq X_2 \geq \beta X_1 \text{ for some } \alpha, \beta > 0 \Longrightarrow [X_1]A = [X_2]A. \tag{2.11}$$

Every positive operator X admits a unique *positive square-root* $X^{1/2}$, that is, $X^{1/2} \geq 0$ and $(X^{1/2})^2 = X$. For general $X \in \mathcal{B}(\mathcal{H})$ the positive square root of X^*X is called the *modulus* of X and denoted by $|X|$. When X is selfadjoint, $|X|$ commutes with X.

We use $ran(X)$ and $ker(X)$ to denote the *range* and the *kernel* of $X \in \mathcal{B}(\mathcal{H})$ respectively. Then obviously

$$ran(X)^{\perp} = ker(X^*) \quad \text{and} \quad ker(X)^{\perp} = \overline{ran(X^*)}, \tag{2.12}$$

where $\{\ \}^{\perp}$ denotes the orthogonal complement. For $X \geq 0$ we have

$$ker(X) = ker(X^{1/2}) \quad \text{and} \quad ran(X) \subset ran(X^{1/2}) \subset \overline{ran(X)}. \tag{2.13}$$

When $dim(\mathcal{H}) < \infty$, every subspace is closed, so that positive A is invertible if $ran(A) = \mathcal{H}$, or equivalently $ker(A) = \{0\}$.

For $A \geq 0$ and selfadjoint $X \in \mathcal{B}(\mathcal{H})$

$$A \geq X \geq -A \iff X = A^{1/2}CA^{1/2}, \tag{2.14}$$

where C is a (selfadjoint) *contraction*, that is, $I \geq C \geq -I$. If in addition $ran(C) \subset \overline{ran(A)}$ is required, C is uniquely determined. In a similar way $A \geq X \geq 0$ is characterized by $I \geq C \geq 0$ in (2.14).

Range inclusion is characterized by an operator inequality (see [D 66]); for $X, Y \in \mathcal{B}(\mathcal{H})$

$$ran(X) \supset ran(Y) \iff \alpha XX^* \geq YY^* \quad \text{for some} \quad \alpha \geq 0. \tag{2.15}$$

A consequence of (2.14) is that the operation $A \longmapsto [X]A$ is determined only by the range space $\mathcal{L} = ran(X^{1/2})$. Therefore $[X]A$ will be called the *short* of A to the operator range $\mathcal{L}$.

If $0 \leq C, D \leq I$ and $ran(C), ran(D) \subset \overline{ran(A)}$ then we have

$$(A^{1/2}CA^{1/2}) : (A^{1/2}DA^{1/2}) = A^{1/2}(C : D)A^{1/2}. \tag{2.16}$$

(see [PS 76]).

If $ran(X)$ is closed, then by (2.13) $[X]A = [P]A$ where P is the projection to $ran(X) = ran(X^{1/2})$. The shorted operator $[P]A$ is written in the block operator matrix with respect to the decompositon $\mathcal{H} = ran(P) \oplus ker(P)$ in the following form (see [A 71]):

$$[P]A = \begin{bmatrix} A_{11} - A_{12} \cdot A_{22}^{-1} \cdot A_{21} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.17}$$

for

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \geq 0.$$

Here we remark that positivity of A ensures that there is an operator K from $ker(P)$ to $ran(P)$ such that $\|K\| \leq 1$ and $A_{12} = A_{11}^{1/2} K A_{22}^{1/2}$, and $A_{12} \cdot A_{22}^{-1} \cdot A_{21}$ should be understood as

$$A_{12} \cdot A_{22}^{-1} \cdot A_{21} \stackrel{def}{=} A_{11}^{1/2} K K^* A_{11}^{1/2}.$$

The shorted operator $[P]A$ for a projection P admits the following maximum characterization, originally due to M. Krein (see [AT 75]);

$$[P]A = \max\{Y | 0 \leq Y \leq A, ran(Y) \subset ran(P)\}. \tag{2.18}$$

For general $X \geq 0$ we have

$$[X]A = A^{1/2} Q A^{1/2}, \tag{2.19}$$

where Q is the projection to the closure of $A^{-1/2} ran(X) \equiv \{x \in \mathcal{H} \; | A^{1/2} x \in ran(X)\}$ (see [P 78] and [K 84]). Conversely for any projection Q with $ran(Q) \subset \overline{ran(A)}$

$$A^{1/2} Q A^{1/2} = [X]A \quad \text{with } X = A^{1/2} Q A^{1/2}. \tag{2.20}$$

It is known (see [Ao 76], [P 78] and [EL 86]) that

$$[X]A = A \iff A = \sup\{Y \; ; \; 0 \leq Y \leq A \text{ and } Y \leq \lambda X \text{ for some } \lambda = \lambda(Y) \geq 0\} \tag{2.21}$$

Sum and intersection of operator ranges are well determined (see [FW 71]): for $A, B \geq 0$

$$ran(A) + ran(B) = ran(A^2 + B^2)^{1/2}, \tag{2.22}$$

and

$$ran(A) \cap ran(B) = ran(A^2 : B^2)^{1/2}. \tag{2.23}$$

The range of a short to a closed subspace is also well determined (see [AT 75]): for projection P

$$ran([P]A)^{1/2} = ran(P) \cap ran(A^{1/2}). \tag{2.24}$$

It follows from (2.8) and (2.24) that for $A, B \geq 0$

$$ran(A^{1/2}) \cap ran(B^{1/2}) = \{0\} \iff A : B = 0. \tag{2.25}$$

3. Characterizations

It is well known that $ex\Delta(I)$ for identity operator I consists of all projections (see [Sa 71, p.12]). Remark that a projection X is characterized as a selfadjint idempotiont, that is, $X(I-X)=0$, and also that when $I \geq X \geq 0$ then $X(I-X) = X:(I-X)$. By definition (2.8) $[X]I$ for every $X \geq 0$ coincides with the projection to $\overline{ran(X)}$. These thogether can be formulated as various characterizations of $ex\Delta(I)$ in the following Lemma (see [AT 88] and [EL 86]):

Lemma 1.

$$\begin{aligned} ex\Delta(I) &= \{P \mid projection\ P\} \\ &= \{[X]I \mid X \geq 0\} \\ &= \{X \in \Delta(I) \mid X:(I-X)=0\}. \end{aligned}$$

An easy consequence is that for a projection P

$$ex\Delta(P) = \{Q \mid projection\ Q \leq P\}.$$

Given $A \geq 0$ let P be the projection to $\overline{ran(A)}$. Then by (2.14) the affine map $X \longmapsto A^{1/2}XA^{1/2}$ transforms bijectively $\Delta(P)$ to $\Delta(A)$, so that the affine structure of $ex\Delta(A)$ can be copied from Lemma 1 and its consequence via (2.16), by using (2.19) and (2.20). Therefore we have a complete answer for the case $m=1$ (cf. [AT 88], [EL 86], [AMT 92] and [P 92]).

Theorem 2. *For* $A \geq 0$

$$\begin{aligned} ex\Delta(A) &= \{A^{1/2}PA^{1/2} \mid projection\ P\ such\ that\ ran(P) \subset \overline{ran(A)}\} \\ &= \{[X]A \mid X \geq 0\} \\ &= \{X \in \Delta(A) \mid X:(A-X)=0\}. \end{aligned}$$

Remark that, in general, not every extreme point of $\Delta(A)$ is of the form $[P]A$ for a projection P (see [P 91] and [GM 94]).

The case $m > 1$ is more delicate, but sitll can be derived from Lemma 1 (cf. [Ao 89] and [P 91])

Theorem 3. *Let* $A_j \geq 0$ $(j=1,2,\ldots,m)$ *and* $X \in \Delta(A_1,\ldots,A_m)$. *Then the following conditions on* X *are mutually equivalent.*

(i) $X \in ex\Delta(A_1,\ldots,A_m)$.

(ii) $X : \{\prod_{j=1}^m : (A_j - X)\} = 0$.

(iii) $ran(X^{1/2}) \cap \bigcap_{j=1}^m ran(A_j - X)^{1/2} = 0$.

Also each of (i), (ii) *and* (iii) *is equivalent to any of the following.*

(i') $X \in ex\Delta([X]A_1,\ldots,[X]A_m)$.

(ii') $X : \{\prod_{j=1}^{m} : ([X]A_j - X)\} = 0.$

(iii') $ran(X^{1/2}) \cap \bigcap_{j=1}^{m} ran([X]A_j - X)^{1/2} = 0.$

Theorem 4. *Let $A_j \geq 0$ $(j = 1, 2, \ldots, m)$ and $X \in \Delta(A_1, \ldots, A_m)$. Then each of the following conditions implies any one (and all) in Theorem 3.*

(iv) $\prod_{j=1}^{m} : ([X]A_j - X) = 0.$

(v) $ker(X) + \sum_{j=1}^{m} ker(A_j - X) = \mathcal{H}.$

(vi) *There are mutually commuting (not necesarily selfadjoint) idempotents $Q_j \in \mathcal{B}(\mathcal{H})$ $(j = 0, 1, 2, \ldots, m)$ such that*

$$Q_i Q_j = \delta_{ij} Q_j \ (i, j = 0, 1, 2, \ldots, m), \quad \sum_{j=0}^{m} Q_j = I, \quad and \quad X = \sum_{j=1}^{m} A_j Q_j.$$

When $dim(\mathcal{H}) < \infty$, each of (iv), (v) *and* (v) *is equivalent to any one (and all) in Theorem 3.*

4. Labelling

All characteerizations of extremality in Theorem 3 and Theorem 4 are of qualitative nature except (vi). Representation (vi), however, has a defect in using non-selfadjoint summands. In fact, there is no guarantee of selfajointness of $A_j Q_j$ $(j = 1, 2, \ldots, m)$.

In the finite dimensional case, however, the following theorem gives a labelling of every extreme point X of $\Delta(A_1, A_2, \ldots, A_m)$ in terms of those in $\Delta(Y_j)$ $(j = 1, 2, \ldots, m)$ where Y_j $(j = 1, 2, \ldots, m)$ are defined recursively from X (cf. [Ao 89] and [P 91]).

Theorem 5. *Let $A_j \geq 0$ $(j = 1, 2, \ldots, m)$ and $X \in \Delta(A_1, \ldots, A_m)$. Then if there are $X_k \geq 0$ $(k = 1, 2, \ldots, m)$ such that*

$$X = \sum_{j=1}^{m} X_j \quad and \quad X_k \in ex\Delta(A_k - \sum_{j=1}^{k-1} X_j) \quad (k = 1, 2, \ldots, m), \tag{4.1}$$

where $\sum_{j=1}^{k-1} X_j \equiv 0$ for $k = 1$, then X is an extreme point of $\Delta(A_1, \ldots, A_m)$.

Those X_k $(k = 1, 2, \ldots, m)$, satisfying (4.1) *and the additional condition that for some $1 > \varepsilon > 0$*

$$(1 - \varepsilon)(A_k - \sum_{j=1}^{k} X_j) \geq \sum_{j=k+1}^{m} X_j \equiv X - \sum_{j=1}^{k} X_j \quad (k = 1, 2, \ldots, m) \tag{4.2}$$

are unique for X if exist.

When $dim(\mathcal{H}) < \infty$, existence of those X_k $(k = 1, 2, \ldots, m)$, satisfying (4.1) *and* (4.2), *is always guaranteed for every $X \in ex\Delta(A_1, A_2, \ldots, A_m)$.*

5. Parametrization

When $dim(\mathcal{H}) < \infty$ and $m = 2$, in Theorem 5 X_1 is considered as a free parameter for $X \in ex\Delta(A_1, A_2)$ because $X_2 = X - X_1$. But the requirements $(1 - \varepsilon)(A_1 - X_1) \geq X - X_1$ for some $0 < \varepsilon < 1$ and $X_2 \in ex\Delta(A_2 - X_1)$ are still restrictive. In this section, using a result from the theory of indefintie inner product spaces, we shall present a more transparent parametrization of $ex\Delta(A_1, A_2)$ along an idea of [Ao 93].

Let $dim(\mathcal{H}) < \infty$ and $m = 2$. For brevity let us write

$$A \stackrel{def}{=} A_1 \quad \text{and} \quad B \stackrel{def}{=} A_2.$$

According to Theorem 4 extremity of $X \in \Delta(A, B)$ is characterized by

$$([X]A - X) : ([X]B - X) = 0.$$

When restricted to $ran(X) = ran(X^{1/2})$, each of $[X]A$, $[X]B$ and X is positive invertible by (2.26). Therefore assuming A, $B > 0$ we shall consider parametrization of all invertible extreme points of $\Delta(A, B)$. In this case requirement for extremity of X becomes

$$(A - X) : (B - X) = 0. \tag{5.1}$$

Let $A, B > 0$, and

$$C \stackrel{def}{=} \frac{1}{2}(A - B).$$

If $A \geq B$ or $A \leq B$ then $\Delta(A, B)$ reduces to either $\Delta(B)$ or $\Delta(A)$, and a parametrization of $\Delta(A, B)$ is already known by Theorem 2. Therefore we shall assume that C is is indefinite, that is, $C \not\geq 0$, and $C \not\leq 0$. The space $\mathcal{H}$ is written as an orthogonal sum

$$\mathcal{H} = ran(C) \oplus ker(C). \tag{5.2}$$

Since

$$ran(C) = ran(|C| - C) \oplus ran(|C| + C),$$

we can also write

$$\mathcal{H} = ran(|C| - C) \oplus ran(|C| + C) \oplus ker(C). \tag{5.3}$$

Let

$$n_+ = dim(|C| - C) \quad \text{and} \quad n_- = dim(|C| + C).$$

Then $n_\pm > 0$ by indefiniteness of C. The triple $\{n_+, n_-, n_0\}$ with $n_0 \stackrel{def}{=} dim\ ker(C)$ is usually called the *inertia* of selfajoint C. Fix an invertible operator $V \in \mathcal{B}(\mathcal{H})$ such that with respect to decompositons (5.2) and (5.3)

$$V = \begin{bmatrix} V_{11} & 0 \\ 0 & I_0 \end{bmatrix} \quad \text{and} \quad C = V^* \cdot \begin{bmatrix} I_+ & 0 & 0 \\ 0 & -I_- & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot V, \tag{5.4}$$

where I_0 and $I_\pm$ are the identity matrices of order n_0 and $n_\pm$ respectively.

Theorem 5. *Let dim($\mathcal{H}$) $< \infty$ and let A, $B > 0$, and let C and V be as in* (5.4). *Then every invertible extreme point X of $\Delta(A,B)$ is uniquely written in the form*

$$X = \frac{1}{2}(A+B) - V^* \cdot \begin{bmatrix} D(K) & 0 \\ 0 & 0 \end{bmatrix} \cdot V, \tag{5.5}$$

where K is an $n_- \times n_+$ matrix with $K^*K < I_+$ and

$$D(K) \stackrel{def}{=} \begin{bmatrix} (I_+ + K^*K)(I_+ - K^*K)^{-1} & 2(I_+ - K^*K)^{-1}K^* \\ 2K(I_+ - K^*K)^{-1} & (I_- + KK^*)(I_- - KK^*)^{-1} \end{bmatrix}. \tag{5.6}$$

Conversely each $n_- \times n_+$ matrix K with $K^*K < I_+$ gives rise to an invertible extreme point X of $\Delta(A,B)$ by (5.5) *and* (5.6).

6. Construction

Let $A_j \geq 0$ $(j = 1, 2, \ldots, m)$. Oviously 0 is an extreme point, which is $\leq X$ for all $X \in \Delta(A_1, A_2, \ldots, A_m)$. In this section along an idea of [Ao 93] we shall present an algorithm of obtaining an extreme point $\tilde{X}$ such that $X \leq \tilde{X}$ for given $X \in \Delta(A_1, A_2, \ldots, A_m)$.

Theorem 6. *Let $X \in \Delta(A_1, \ldots, A_m)$. Starting with $X_0 \stackrel{def}{=} X$, define successibly*

$$X_{k+1} \stackrel{def}{=} X_k + \prod_{j=1}^{k} : (A_j - X_k) \quad (k = 1, 2, \ldots). \tag{6.1}$$

Then $\{X_k \mid k = 1, 2, \ldots\}$ is an increasing sequence in $\Delta(A_1, \ldots, A_m)$ and its (strong) limit $X_\infty \stackrel{def}{=} \lim_{k\to\infty} X_k$ is an extreme point of $\Delta(A_1, \ldots, A_m)$ such that

$$X \leq X_\infty \quad \text{and} \quad \prod_{j=1}^{m} : (A_j - X_\infty) = 0. \tag{6.2}$$

Remark that the algorithm in Theorem 6 produces all extreme points X of $\Delta(A_1, A_2, \ldots, A_m)$ such that

$$\prod_{j=1}^{m} : (A_j - X) = 0,$$

because for such X it is immediate to see that all X_k coincide with X for all k, and $X_\infty = X$.

Theorem 7. *Let* $\dim(\mathcal{H}) < \infty$ *and* $X \in \Delta(A_1, A_2, \ldots, A_m)$. *Starting with* $X_0 \overset{def}{=} X$, *define successibly*

$$X_{k+1} \overset{def}{=} X_k + \prod_{j=1}^{k} : ([X]A_j - X_k) \quad (k = 1, 2, \ldots). \tag{6.4}$$

Then $\{X_k \mid k = 1, 2, \ldots\}$ *is an increasing sequence in* $\Delta(A_1, \ldots, A_m)$ *and its limit* $X_\infty \overset{def}{=} \lim_{k\to\infty} X_k$ *is an extreme point of* $\Delta(A_1, \ldots, A_m)$ *such that*

$$X \leq X_\infty = [X]X_\infty \quad \text{and} \quad \prod_{j=1}^{m} : ([X_\infty]A_j - X_\infty) = 0. \tag{6.5}$$

Conversely all extreme points are obtained in this way.

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