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A note on Sturm-type comparison theorems on a half-open interval

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1. Introduction and statement of the results

In this note, we investigate comparison theorems of Sturm-type on a half-open interval $[a, \omega)$, $\omega \leq \infty$. We consider two differential equations

$$(p(t)x')' + q(t)x = 0, \qquad a \leq t < \omega, \tag{1.1}$$

$$(P(t)y')' + Q(t)y = 0, \qquad a \leq t < \omega, \tag{1.2}$$

where $p(t)$, $q(t)$, $P(t)$, and $Q(t)$ are continuous functions on $[a, \omega)$, and

$$p(t) \geq P(t) > 0 \quad \text{and} \quad Q(t) \geq q(t) \quad \text{on } [a, \omega).$$

In this case, (1.2) is called a Sturm majorant for (1.1) on $[a, \omega)$ and (1.1) is called a Sturm minorant for (1.2).

Sturm's comparison theorem can be stated as folows: (See, e.g., [2, Chap.11, Theorem 3.1].)

Theorem A. *Let $x(t) \not\equiv 0$ be a solution of* (1.1) *and let $x(t)$ has exactly n (≥ 1) zeros $t = t_1 < t_2 < \cdots < t_n$ in $(a, b]$, $b < \omega$. Let $y(t)$ be a solution of* (1.2). *If either $x(a) = 0$ or $x(a) \neq 0$, $y(a) \neq 0$, and*

$$\frac{p(a)x'(a)}{x(a)} \geq \frac{P(a)y'(a)}{y(a)},$$

then $y(t)$ has one of the following properties:

(i) *$y(t)$ has at least n zeros in (a, t_n);*

(ii) *$y(t)$ is a constant multiple of $x(t)$ on $[a, t_n]$ and $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ on $[a, t_n]$.*

Let $x(t) > 0$ in (t_n, ω) in Theorem A. In this case, it seems interesting to ask the question whether a solution $y(t)$ of (1.2) has at least one zero in (t_n, ω) or not?

Assume that (1.1) is nonoscillatory at $t = \omega$. It is well known [2, Chap.11, Theorem 6.4] that (1.1) has a principal solution $x_0(t)$ which is essentially unique (up to a constant factor) such that

$$\int^{\omega} \frac{ds}{p(s)[x_0(s)]^2} = \infty$$

and for any solution $x_1(t)$ linearly independent of $x_0(t)$,

$$\lim_{t \to \omega} \frac{x_0(t)}{x_1(t)} = 0.$$

The solution $x_1(t)$ is called a nonprincipal solution.

Our main results are the following.

Theorem 1. *Assume that* (1.1) *is nonoscillatory at* $t = \omega$. *Let* $x_0(t)$ *be a principal solution of* (1.1) *satisfying* $x_0(t) > 0$ *in* (a, ω). *Let* $y(t)$ *be a solution of* (1.2). *If either* $x_0(a) = 0$ *or* $x_0(a) \neq 0$, $y(a) \neq 0$, *and*

$$\frac{p(a)x_0'(a)}{x_0(a)} \geq \frac{P(a)y'(a)}{y(a)}, \tag{1.3}$$

then $y(t)$ *has one of the following properties*:

(i) $y(t)$ *has at least one zero in* (a, ω);

(ii) $y(t)$ *is a constant multiple of* $x_0(t)$ *on* $[a, \omega)$ *and* $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ *on* $[a, \omega)$.

Theorem 2. *Assume that* (1.1) *is nonoscillatory at* $t = \omega$. *Let* $x_0(t)$ *be a principal solution of* (1.1) *and let* $x(t)$ *has exactly* n (≥ 1) *zeros in* (a, ω). *Let* $y(t)$ *be a solution of* (1.2). *If either* $x_0(a) = 0$ *or* $x_0(a) \neq 0$, $y(a) \neq 0$, *and* (1.3) *holds, then* $y(t)$ *has one of the following properties*:

(i) $y(t)$ *has at least* $n + 1$ *zeros in* (a, ω);

(ii) $y(t)$ *is a constant multiple of* $x_0(t)$ *on* $[a, \omega)$ *and* $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ *on* $[a, \omega)$.

Remark. For other results concerning comparison theorems of Sturm-type on a half-open interval, we refer to [4] and [5].

When $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[a, \omega)$, as a consequence of Theorems 1 and A, we have the following.

Corollary 1. *Assume that* (1.1) *is nonoscillatory at* $t = \omega$. *Let* $x_0(t)$ *be a principal solution of* (1.1) *and let* t_0 $(\geq a)$ *be the largest zero, i.e.,* $x_0(t_0) = 0$ *and* $x_0(t) > 0$ *in* (t_0, ω). *Then we have the following properties*:

(i) *every nonprincipal solution has exactly one zero in* (t_0, ω);
(ii) *every solution of* (1.1) *has exactly one zero on* $[t_0, \omega)$.

Equation (1.1) is said to be disconjugate on an interval J if every solution of (1.1) has at most one zero on J. (See [1] and [2].) By Corollary 1, we obtain a criterion for (1.1) to be disconjugate.

Corollary 2. *Assume that* (1.1) *is nonoscillatory at* $t = \omega$. *Let* $x_0(t)$ *be a principal solution of* (1.1) *and let* t_0 $(\geq a)$ *be the largest zero. Then* (1.1) *is disconjugate on* $[t_1, \omega)$ *if and only if* $t_0 \leq t_1$.

Finally, we give a comparison theorem for disconjugacy.

Corollary 3. *Assume that* (1.2) *is nonoscillatory at* $t = \omega$. *(Then* (1.1) *is nonoscillatory at* $t = \omega$.*) Let* $x_0(t)$ *and* $y_0(t)$ *be principal solutions of* (1.1) *and* (1.2), *respectively. Let* t_0 *and* t_1 $(t_0,\ t_1 \geq a)$ *be the largest zeros of* $x_0(t)$ *and* $y_0(t)$, *respectively. Then, we have either* (i) $t_0 < t_1$ *or* (ii) $t_0 = t_1$ *and* $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ *on* $[t_0, \omega)$. *In particular, if* (1.2) *is disconjugate on an interval* J, *then* (1.1) *is disconjugate on* J.

Remark. The comparison theorems for disconjugacy have been shown in [1] by different methods.

2. Proofs of Theorems

We prepare the following lemmas.

Lemma 1. *Assume that* $q(t) \leq 0$ *on* $[a, \omega)$ *in* (1.1). *Then* (1.1) *is nonoscillatory at* $t = \omega$ *and a principal solution* $x_0(t)$ *of* (1.1) *satisfies* $x_0(t) > 0$ *and* $x_0'(t) \leq 0$ *on* $[a, \omega)$.

Lemma 2. *Assume that* (1.1) *is nonoscillatory at* $t = \omega$. *Let* $x_0(t)$ *be a principal solution of* (1.1) *and let* $y(t)$ *be a solution of* (1.2) *satisfying* $y(t) > 0$ *on* $[T, \omega)$, $T \geq a$. *Then* $x_0(t) > 0$ *on* $[T, \omega)$ *and*

$$\frac{p(t)x_0'(t)}{x_0(t)} \leq \frac{P(t)y'(t)}{y(t)} \quad \text{on } [T, \omega).$$

Lemmas 1 and 2 are shown in [2, Chap.11, Corollary 6.4] and [2, Chap.11, Corollary 6.5], respectively. However, for the sake of the completeness, we give (slight simple) proofs of them.

Proof of Lemma 1. Let $x_i(t)$, $i = 1, 2$, be solutions of (1.1) determined by $x_i(a) = 1$ and $x_i'(a) = i$. It is easy to see that $(p(t)x_i'(t))' \geq 0$ and $x_i(t) > 0$ on $[a, \omega)$, $i = 1, 2$. Since $x_1(t)$ and $x_2(t)$ are linearly independent, either $x_1(t)$ or $x_2(t)$ is a nonprincipal solution. Without loss of generality, we may assume that $x_1(t)$ is a nonprincipal solution. By [2, Chap.11, Corollary 6.3],

$$x_0(t) = x_1(t) \int_t^\omega \frac{ds}{p(s)[x_1(s)]^2}, \quad a \leq t < \omega,$$

is well defined and a principal solution of (1.1). We see that $x_0(t) > 0$ on $[a, \omega)$. We obtain

$$x_0'(t) = x_1'(t) \int_t^\omega \frac{ds}{p(s)[x_1(s)]^2} - \frac{1}{p(t)x_1(t)}, \quad a \leq t < \omega.$$

Since $p(t)x_1'(t)$ is nondecreasing and $x_1(t)$ is positive,

$$p(t)x_0'(t) \leq \int_t^\omega \frac{x_1'(s)}{[x_1(s)]^2} ds - \frac{1}{x_1(t)} = -\lim_{s\to\omega} \frac{1}{x_1(s)} \leq 0, \quad a \leq t < \omega.$$

Thus, we have $x_0'(t) \leq 0$ on $[a, \omega)$. $\square$

Proof of Lemma 2. Let

$$u(t) = \exp\left(\int_T^t \frac{P(s)y'(s)}{p(s)y(s)} ds\right), \quad T \leq t < \omega.$$

Then $u(t) > 0$ on $[T, \omega)$ and satisfies

$$\frac{p(t)u'(t)}{u(t)} = \frac{P(t)y'(t)}{y(t)} \quad \text{and} \quad (p(t)u')' + Q_0(t)u = 0 \quad \text{for } T \leq t < \omega, \tag{2.1}$$

where

$$Q_0(t) = Q(t) + \left(\frac{1}{P(t)} - \frac{1}{p(t)}\right)\left(\frac{P(t)y'(t)}{y(t)}\right)^2, \quad T \leq t < \omega.$$

Let $z(t) = x_0(t)/u(t)$ on $[T, \omega)$. Then $z(t)$ is a solution of

$$(p(t)[u(t)]^2 z')' + [u(t)]^2 (q(t) - Q_0(t)) z = 0, \quad T \leq t < \omega. \tag{2.2}$$

Since $x_0(t)$ is a principal solution, we have

$$\int^\omega \frac{ds}{p(s)[x_0(s)]^2} = \int^\omega \frac{ds}{p(s)[u(s)]^2[z(s)]^2} = \infty.$$

Thus $z(t)$ is a principal solution of (2.2). We note that $Q_0(t) \geq Q(t) \geq q(t)$ on $[T, \omega)$. Then, by Lemma 1, we have $z(t) > 0$ and $z'(t) \leq 0$ on $[T, \omega)$, which implies $x_0(t) > 0$ on $[T, \omega)$. From the left side of (2.1) and

$$\frac{x'(t)}{x(t)} = \frac{u'(t)}{u(t)} + \frac{z'(t)}{z(t)}, \quad T \leq t < \omega,$$

we conclude that

$$\frac{p(t)x'(t)}{x(t)} \leq \frac{p(t)u'(t)}{u(t)} = \frac{P(t)y'(t)}{y(t)}, \quad T \leq t < \omega.$$

□

Proof of Theorem 1. Assume that $y(t) > 0$ in (a, ω). By Picone's identity [3], we have

$$\frac{d}{dt}\left[\frac{x_0}{y}(px_0'y - Px_0y')\right] = (Q-q)x_0^2 + (p-P)x_0'^2 + \frac{P(x_0'y - x_0y')^2}{y^2}. \tag{2.3}$$

We observe that if $x_0(a) = 0$ then

$$\lim_{t\to a} \frac{x_0(t)}{y(t)}(p(t)x_0'(t)y(t) - P(t)x_0(t)y'(t)) = -P(a)x_0(a)y'(a)\lim_{t\to a}\frac{x_0(t)}{y(t)} = 0,$$

and that if $x_0(a) \neq 0$, $y(a) \neq 0$, and (1.3) holds, then

$$\lim_{t\to a} \frac{x_0(t)}{y(t)}(p(t)x_0'(t)y(t) - P(t)x_0(t)y(t)') = [x_0(a)]^2\left(\frac{p(a)x_0'(a)}{x_0(a)} - \frac{P(a)y'(a)}{y(a)}\right) \geq 0.$$

Therefore, integrating (2.3) over $[\tau, t]$ and letting $\tau \to a$, it follows that

$$[x_0(t)]^2\left(\frac{p(t)x_0'(t)}{x_0(t)} - \frac{P(t)y'(t)}{y(t)}\right) \geq \int_a^t \left[(Q-q)x_0^2 + (p-P)x_0'^2 + \frac{P(x_0'y - x_0y')^2}{y^2}\right] ds$$

for $a < t < \omega$. From Lemma 2, we have

$$\int_a^t \left[(Q-q)x_0^2 + (p-P)x_0'^2 + \frac{P(x_0'y - x_0y')^2}{y^2}\right] ds \leq 0, \quad a < t < \omega,$$

which implies that $q(t) \equiv Q(t)$, $p(t) \equiv P(t)$, and $x_0(t)y'(t) \equiv x_0'(t)y(t)$ on $[a, \omega)$. Hence, $y(t)$ is a constant multiple of $x_0(t)$ on $[a, \omega)$. This completes the proof of Theorem 1. □

Proof of Theorem 2. Let $t = t_1 < t_2 < \cdots < t_n$ be zeros of $x_0(t)$ in (a, ω). We note that $y(t)$ satisfies either (i) or (ii) in Theorem A on $[a, t_n]$.

By applying Theorem 1 on $[t_n, \omega)$, we have either $y(t)$ has at least one zero in (t_n, ω) or $y(t)$ is a multiple constant of $x_0(t)$ on $[t_n, \omega)$ and $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[t_n, \omega)$. In the former case, $y(t)$ has at least $n+1$ zeros in (a, ω). In the latter case, since $y(t_n) = 0$, we have either $y(t)$ has at least $n+1$ zeros in (a, ω) or $y(t)$ is a multiple constant of $x_0(t)$ on $[a, \omega)$ and $p(t) \equiv P(t)$ and $q(t) \equiv Q(t)$ on $[a, \omega)$. This completes the proof of Theorem 2. □

References

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