数理解析研究所講究録
940巻 1996年 110-123

Periodic solutions of integral equations

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1. Introduction

In this paper, we study the behavior of solutions of the integral equations of neutral type

$$x(t)=a(t)+\int_0^t D(t,s,x(s))ds+\int_t^\infty E(t,s,x(s))ds, \quad t\in R^+, \tag{1}$$

and

$$x(t)=p(t)+\int_{-\infty}^t P(t,s,x(s))ds+\int_t^\infty Q(t,s,x(s))ds, \quad t\in R, \tag{2}$$

and their relation to each other. Eq.(2) is a limiting equation of Eq.(1). Conditions on a, p, D, E, P, and Q are given later, but all of them are at least continuous. Many results are obtained for these equations without the third terms of the righthand sides. Excellent up to date collections of such results are found in Corduneanu [4] and Gripenberg-Londen-Staffans [5]. Moreover, some results concerning periodic solutions and attractivity of such equations are obtained in Burton-Furumochi [2]. On the other hand, integral equations of neutral type have been also studied. For example, we can find an integral equation of neutral type in the classical book [3, pp.329-340] of Coddington and Levinson.

The purpose of this paper is to investigate periodicity and convergence of solutions by employing contraction mappings and limiting quations.

In Section 3, we show the existence of a periodic solution of (2) and its attractivity.

2. NOTATIONS AND PRELIMINARY RESULTS

Let R^+ and R denote the intervals $0 \le t < \infty$, and $-\infty < t < \infty$, respectively. Let $a,\ p : R \to R^n$, $D,\ P : \Delta^- \times R^n \to R^n$ and $E,\ Q : \Delta^+ \times R^n \to R^n$ are continuous, where $\Delta^- := \{(t, s) : s \le t\}$ and $\Delta^+ := \{(t, s) : s \ge t\}$. Moreover,

$$p(t+T) = p(t), \text{ and } q(t) := a(t) - p(t) \to 0 \text{ as } t \to \infty, \tag{3}$$

where $T > 0$ is constant,

$$P(t+T, s+T, x) = P(t, s, x), \text{ and } F(t, s, x) := D(t, s, x) - P(t, s, x), \tag{4}$$

$$Q(t+T, s+T, x) = Q(t, s, x), \text{ and } G(t, s, x) := E(t, s, x) - Q(t, s, x), \tag{5}$$

and for any $J > 0$ there are continuous functions $P_J,\ F_J : \Delta^- \to R^+$ and $Q_J,\ G_J : \Delta^+ \to R^+$ such that

$$P_J(t+T, s+T) = P_J(t, s) \quad \text{if} \quad s \le t,$$

$$|P(t, s, x)| \le P_J(t, s) \quad \text{if} \quad s \le t \quad \text{and} \quad |x| \le J,$$

$$|F(t, s, x)| \le F_J(t, s) \quad \text{if} \quad s \le t \quad \text{and} \quad |x| \le J,$$

$$Q_J(t+T, s+T) = Q_J(t, s) \quad \text{if} \quad s \ge t,$$

$$|Q(t, s, x)| \le Q_J(t, s) \quad \text{if} \quad s \ge t \quad \text{and} \quad |x| \le J,$$

$$|G(t,s,x)| \le G_J(t,s) \quad \textit{if} \quad s \ge t \quad \textit{and} \quad |x| \le J,$$

where $|\cdot|$ denotes the Euclidean norm of R^n, and

$$\int_{-\infty}^{t-\tau} P_J(t,s)ds + \int_{t+\tau}^{\infty}(Q_J(t,s)+G_J(t,s))ds \to 0 \tag{6}$$

uniformly for $t \in R$ *as* $\tau \to \infty$,

and

$$\int_0^t F_J(t,s)ds + \int_t^{\infty} G_J(t,s)ds \to 0 \quad \textit{as} \quad t \to \infty. \tag{7}$$

Remark. From Lemma 1 in [6], it is easy to see that Condition (6) with $G_J(t,s) \equiv 0$ is equivalent to the condition that $\int_{-\infty}^{t} P_J(t,s)ds$ and $\int_t^{\infty} Q_J(t,s)ds$ are continuous in t.

Now let $(C, \|\cdot\|)$ be the Banach space of bounded and continuous functions $\xi : R \to R^n$ with the supremum norm. For any $t_0 \in R^+$, let $C(t_0)$ be a set of bounded functions $\xi : R^+ \to R^n$ such that $\xi(t)$ is continuous on R^+ except at t_0 and $\xi(t_0) = \xi(t_0+)$. For any $\xi \in C$, define a map H on C by

$$(H\xi)(t) := p(t) + \int_{-\infty}^{t} P(t,s,\xi(s))ds + \int_t^{\infty} Q(t,s,\xi(s))ds, \quad t \in R.$$

Similarly for any $\xi \in C(t_0)$, define a map H^+ on $C(t_0)$ by

$$(H^+\xi)(t) := a(t) + \int_0^t D(t,s,\xi(s))ds + \int_t^{\infty} E(t,s,\xi(s))ds, \quad t \ge t_0.$$

Moreover, for any $J>0$ let $C_J := \{\xi \in C : \|\xi\| \le J\}$, $C_J(t_0) := \{\xi \in C(t_0) : \|\xi\|_+ \le J\}$, where $\|\cdot\|_+$ denotes the supremum norm on R^+.

First we prepare two basic lemmas.

Lemma 1. *Under the assumptions* (3)-(6), *the following hold.*

(i) *For any* $J>0$ *there is a continuous increasing function* $\delta=\delta_J(\varepsilon) : (0,\infty) \to (0,\infty)$ *with*

$$|(H\xi)(t_1)-(H\xi)(t_2)|<\varepsilon \quad \textit{if} \quad \xi \in C_J \quad \textit{and} \quad |t_1-t_2|<\delta.$$

(ii) *If* (7) *holds, then for any* $t_0 \in R^+$ *and any* $J>0$ *there is a continuous increasing function* $\delta^+=\delta_{t_0,J}(\varepsilon) : (0,\infty) \to (0,\infty)$ *with*

$$|(H^+\xi)(t_1)-(H^+\xi)(t_2)|<\varepsilon \quad \textit{if} \quad \xi \in C_J(t_0) \quad \textit{and} \quad t_0 \le t_1<t_2<t_1+\delta^+.$$

Since this lemma can be proved easily by an elementary method, we omit the proof.

Lemma 2. *Under the assumptions* (3)-(6), *the following hold.*

(i) *If* (7) *holds, and if* (1) *has an* R^+*-bounded solution* $x(t)$ *with an initial time in* R^+, *then for any sequence* $\{s_k\}$ *of non-negative numbers with* $s_k \to \infty$ *as* $k \to \infty$, *the sequence of functions* $\{x_k(t)\}$ *contains a subsequence which converges to an R-bounded solution* $y(t)$ *of the equation*

$$x(t)=p(t+\sigma)+\int_{-\infty}^{t}P(t+\sigma,s+\sigma,x(s))ds+\int_{t}^{\infty}Q(t+\sigma,s+\sigma,x(s))ds, \quad t \in R \quad (2_\sigma)$$

uniformly on any compact subset of R, *where* $x_k(t)$ *is defined by*

$$x_k(t):=\begin{cases} x(0), & t<-s_k, \\ x(t+s_k), & t \ge -s_k, \end{cases} \quad t \in R,$$

σ is a number with $0 \le \sigma < T$, and $y(t)$ satisfies (2_σ) on R.

(ii) *If (2) has an R-bounded solution $x(t)$ with an initial time in R, then for any sequence $\{s_k\}$ with $s_k \to \infty$ as $k \to \infty$, the sequence of functions $\{x_k(t)\}$ contains a subsequence which converges to an R-bounded solution $y(t)$ of (2_σ) uniformly on any compact subset of R, where $x_k(t) := x(t+s_k)$, $t \in R$, σ is a number with $0 \le \sigma < T$, and $y(t)$ satisfies (2_σ) on R. In particular, if (2) has an R-bounded solution $x(t)$ which satisfies (2) on R, then the same conclusion holds for any sequence $\{s_k\}$.*

Proof. (i) Let $t_0 \in R^+$ be the initial time of $x(t)$, and let $x(t)$ denote again the R-extension of the function $x(t)$ obtained by defining $x(t) := x(0)$ for $t<0$. Clearly the set $\{x_k(t)\}$ is uniformly bounded on R. Taking a subsequence if necessary, we may assume that the sequence $\{s_k\}$ is nondecreasing. From Lemma 1, $x(t)$ is uniformly continuous on $[t_0, \infty)$, and since $x_k(t)$ is obtained by an s_k-translation of $x(t)$ to the left, for any $j \in N$, the set $\{x_k(t)\}_{k \ge j}$ is equicontinuous on $[t_0 - s_j, \infty)$, where N denotes the set of positive integers. Thus, taking a subsequence if necessary, we may assume that the sequence $\{x_k(t)\}$ converges to a bounded continuous function $y(t)$ uniformly on any compact subset of R.

Now we show that $y(t)$ satisfies (2_σ) on R for some σ with $0 \le \sigma < T$. For each $k \in N$, let ν_k be an integer with $\nu_k T \le s_k < (\nu_k + 1)T$, and let $\sigma_k := s_k - \nu_k T$. Then, taking a subsequence if necessary, we may assume that $\{\sigma_k\}$ converges to some σ with $0 \le \sigma \le T$. From (1), for $t \ge t_0 - s_k$ we have

$$x_k(t) = p(t+\sigma_k) + q(t+s_k)$$

$$+\int_{-s_k}^{t} P(t+\sigma_k, s+\sigma_k, x_k(s))ds+\int_0^{t+s_k} F(t+s_k, s, x(s))ds \quad (8)$$

$$+\int_t^{\infty} Q(t+\sigma_k, s+\sigma_k, x_k(s))ds+\int_{t+s_k}^{\infty} G(t+s_k, s, x(s))ds.$$

Let $J>0$ be a number with $\|x\|:=\sup\{|x(t)| : t\in R\}\leq J$. From (3) and (7), for any $t\in R$ we obtain

$$\lim_{k\to\infty} q(t+s_k)=0,$$

$$\limsup_{k\to\infty} \left|\int_0^{t+s_k} F(t+s_k, s, x(s))ds\right| \leq \limsup_{k\to\infty} \int_0^{t+s_k} F_J(t+s_k, s)ds=0,$$

and

$$\limsup_{k\to\infty} \left|\int_{t+s_k}^{\infty} G(t+s_k, s, x(s))ds\right| \leq \limsup_{k\to\infty} \int_{t+s_k}^{\infty} G_J(t+s_k, s)ds=0.$$

Now from (6), for any $\varepsilon>0$ there is a $\tau>0$ with

$$\int_{-\infty}^{t-\tau} P_J(t, s)ds+\int_{t+\tau}^{\infty} Q_J(t, s)ds<\varepsilon \quad \text{for all} \quad t\in R.$$

From this, for any $t\in R$ we have

$$\limsup_{k\to\infty} \left|\int_{-s_k}^{t} P(t+\sigma_k, s+\sigma_k, x_k(s))ds-\int_{-\infty}^{t} P(t+\sigma, s+\sigma, y(s))ds\right.$$

$$\left.+\int_t^{\infty} Q(t+\sigma_k, s+\sigma_k, x_k(s))ds-\int_t^{\infty} Q(t+\sigma, s+\sigma, y(s))ds\right|$$

$$\leq \limsup_{k\to\infty} \left|\int_{t-\tau}^{t} (P(t+\sigma_k, s+\sigma_k, x_k(s))-P(t+\sigma, s+\sigma, y(s)))ds\right|$$

$$+\limsup_{k\to\infty}\left(\int_{-\infty}^{t-\tau}P_J(t+\sigma_k, s+\sigma_k)ds+\int_{t+\tau}^{\infty}Q_J(t+\sigma_k, s+\sigma_k, x_k(s))ds\right)$$

$$+\limsup_{k\to\infty}\left|\int_t^{t+\tau}(Q(t+\sigma_k, s+\sigma_k, x_k(s))-Q(t+\sigma, s+\sigma, y(s))ds\right|$$

$$+\int_{-\infty}^{t-\tau}P_J(t+\sigma, s+\sigma, y(s))ds+\int_{t+\tau}^{\infty}Q_J(t+\sigma, s+\sigma)ds<2\varepsilon,$$

which implies $\lim_{k\to\infty}(\int_{-s_k}^{t}P(t+\sigma_k, s+\sigma_k, x_k(s))ds+\int_t^{\infty}Q(t+\sigma_k, s+\sigma_k, x_k(s))ds)$ $=\int_{-\infty}^{t}P(t+\sigma, s+\sigma, y(s))ds+\int_t^{\infty}Q(t+\sigma, s+\sigma, y(s))ds$. Thus, letting $k\to\infty$ in (8), we obtain

$$y(t)=p(t+\sigma)+\int_{-\infty}^{t}P(t+\sigma, s+\sigma, y(s))ds+\int_t^{\infty}Q(t+\sigma, s+\sigma, y(s))ds, \quad t\in R. \quad (9)$$

Since (2_T) is equivalent to (2_0), (9) shows that $y(t)$ is an R-bounded solution of (2_σ) with $0\le\sigma<T$ which satisfies (2_σ) on R.

(ii) This part can be easily proved by a similar method to the one in (i).

3. PERIODIC SOLUTIONS AND ATTRACTIVITY

In this section, we investigate the existence of a T-periodic solution of (2) and its attractivity by employing contraction mappings and limiting equations.

First we note that for any ρ and σ with $0\le\rho$, $\sigma<T$, if (2_ρ) has an R-bounded solution which satisfies (2_ρ) on R, then (2_σ) has an R-bounded solution which satisfies (2_σ) on R. From this fact and Lemma 2, we have the following theorem.

Theorem 1. *If* (3)-(6) *hold, and if* (2) *has a unique R-bounded solution* $x_0(t)$ *which satisfies* (2) *on* R, *then the following hold.*

(i) *The solution* $x_0(t)$ *is T-periodic.*

(ii) *If* (7) *holds, then any* R^+*-bounded solution of* (1) *with any initial time in* R^+ *is asymptotically T-periodic, and approaches* $x_0(t)$ *as* $t \to \infty$.

(iii) *Any R-bounded solution of* (2) *with any initial time in* R *is asymptotically T-periodic, and approaches* $x_0(t)$ *as* $t \to \infty$.

Proof. (i) Let $x_1(t)$ be a function obtained by the T-translation of $x_0(t)$ to the left. Then, clearly $x_1(t)$ is also an R-bounded solution of (2) which satisfies (2) on R. Thus, from the uniqueness of R-bounded solutions which satisfy (2) on R, $x_0(t)$ and $x_1(t)$ must be identical on R, that is, $x_0(t)$ is T-periodic.

(ii) Let $x(t)$ be an R^+-bounded solution of (1) with an initial time in R^+, and let $x_k(t)$ be the sequence of functions as in Lemma 2 with $s_k = kT$. Then, from Lemma 2(i) and the uniqueness of R-bounded solutions which satisfy (2) on R, it is easy to see that $x_k(t)$ converges to $x_0(t)$ uniformly on $[0,T]$. This implies that $x(t)$ is asymptotically T-periodic and its T-periodic part is given by $x_0(t)$.

(iii) From Lemma 2(ii), this part can be easily proved by a similar method to the one in (ii).

Among the assumptions of Theorem 1, the uniqueness of R-bounded solutions of (2) which satisfy (2) on R seems to be most important. Here we give a condition of contraction type which assures the uniqueness of R-bounded solutions of (2) which satisfy (2) on R.

Suppose that $p : \Delta^- \to R^+$, $L_J^- : \Delta^- \to R^+$, and $L_J^+ : \Delta^+ \to R^+$ are continuous functions such that

$$|P(t,s,x)-P(t,s,y)| \le L_J^-(t,s)|x-y| \text{ if } (t,s) \in \Delta^-, \ |x|, \ |y| \le J \qquad (10)$$

and

$$|Q(t,s,x)-Q(t,s,y)|\le L_J^{+}(t,s)|x-y| \text{ if } (t,s)\in\Delta^{+},\ |x|,\ |y|\le J. \tag{11}$$

Then we have the following lemma.

Lemma 3. *In addition to* (10) *and* (11), *if for any* $J>0$

$$\lambda_J:=\sup\{\int_{-\infty}^{t}L_J^{-}(t,s)ds+\int_{t}^{\infty}L_J^{+}(t,s)ds\ :\ t\in R\}<1 \tag{12}$$

holds, then (2) has at most one R-bounded solution which satisfies (2) on R.

Proof. Let $x_i(t)$ $(i=1,2)$ be R-bounded solutions of (2) which satisfy (2) on R with $\|x_i\|\le J$ $(i=1,2)$, and let $z(t):=x_1(t)-x_2(t)$, $t\in R$. Then, from (2) we have

$$z(t)=\int_{-\infty}^{t}(P(t,s,x_1(s))-P(t,s,x_2(s)))ds$$
$$+\int_{t}^{\infty}(Q(t,s,x_1(s))-Q(t,s,x_2(s)))ds,\quad t\in R,$$

which together with (10) and (11) yields

$$\begin{aligned}|z(t)|&\le\int_{-\infty}^{t}L_J^{-}(t,s)|z(s)|ds+\int_{t}^{\infty}L_J^{+}(t,s)|z(s)|ds\\&\le(\int_{-\infty}^{t}L_J^{-}(t,s)ds+\int_{t}^{\infty}L_J^{+}(t,s)ds)\|z\|\le\lambda_J\|z\|,\quad t\in R.\end{aligned} \tag{13}$$

Thus, (12) and (13) imply that $z(t)\equiv 0$ on R.

Using Theorem 1(iii) and Lemma 3, we have the following theorem.

Theorem 2. *In addition to* (3)-(6) *and* (10)-(12), *if*

$$\lambda:=\sup\{\lambda_J : J>0\}<1 \tag{14}$$

holds, then (2) *has a unique T-periodic solution, and it is a unique R-bounded solution which satisfies* (2) *on* R. *Moreover, any R-bounded solution of* (2) *with an initial time* $t_0 \in R$ *and a bounded continuous initial function* $\varphi : (-\infty, t_0) \to R^n$ *approaches the T-periodic solution as* $t \to \infty$.

Proof. First we prove that (2) has a unique T-periodic solution. Let $(P_T, \|\cdot\|)$ be the Banach space of continuous T-periodic functions $\xi : R \to R^n$ with the supremum norm $\|\cdot\|$, and define a map H on P_T by

$$(H\xi)(t):=p(t)+\int_{-\infty}^{t} P(t, s, \xi(s))ds+\int_{t}^{\infty} Q(t, s, \xi(s))ds, \quad t \in R.$$

Then, from (3)-(6), it is easy to see that H maps P_T into P_T. Moreover, for any $\xi_i \in P_T$ with $\|\xi_i\| \le J$ $(i=1,2)$ for some $J>0$, we have

$$|(H\xi_1)(t)-(H\xi_2)(t)|$$

$$\le\int_{-\infty}^{t} L_J^-(t, s)|\xi_1(s)-\xi_2(s)|ds+\int_{t}^{\infty} L_J^+(t, s)|\xi_1(s)-\xi_2(s)|ds$$

$$\le\lambda_J\|\xi_1-\xi_2\|, \quad t \in R,$$

which together with (14) yields $\|H\xi_1-H\xi_2\| \le \lambda\|\xi_1-\xi_2\|$. Thus $H : P_T \to P_T$ is a contraction mapping. Hence H has a unique fixed point in

P_T, which gives a unique T-periodic solution of (2), say $\pi(t)$.

Next, from Lemma 3, $\pi(t)$ is the unique R-bounded solution of (2) which satisfies (2) on R. Thus, the latter part is a direct consequence of Theorem 1(iii).

In Theorem 1(ii), the existence of an R^+-bounded solution of (1) with an initial time in R^+ is assumed. Here we consider a few cases where the existence of R^+-bounded solutions of (1) with $F(t,s,x)\equiv 0$ and $G(t,s,x)\equiv 0$ is assured.

Consider the equation

$$x(t)=a(t)+\int_0^t P(t,s,x(s))ds+\int_t^\infty Q(t,s,x(s))ds, \quad t\in R^+, \tag{15}$$

where $a : R^+ \to R^n$ is bounded continuous, and $P : \Delta^- \times R^n \to R^n$ and $Q : \Delta^+ \times R^n \to R^n$ are continuous and satisfies (10), (11) and (14). Let $(B, \|\cdot\|_+)$ be the Banach space of bounded continuous functions $\xi : R^+ \to R^n$ with the supremum norm $\|\cdot\|_+$, and define a map H on B by

$$(H\xi)(t):=a(t)+\int_0^t P(t,s,x(s))ds+\int_t^\infty Q(t,s,x(s))ds, \quad t\in R^+.$$

Then it is easy to see that H is a contraction mapping from B into B. Thus H has a unique fixed point, which gives a unique R^+-bounded solution of (15) which satisfies (15) on R^+. From this and Theorems 1 and 2, we have the following theorem.

Theorem 3. *Suppose that* (3)-(6), (10), (11) *and* (14) *hold. Then* (15) *has a unique R^+-bounded solution which satisfies* (15) *on R^+ and* (2) *has a unique T-periodic solution. Moreover, any R^+-bounded solution* $x(t)=x(t,t_0,\varphi)$ *of* (15) *approaches the unique T-*

periodic solution of (2) *as* $t \to \infty$, *where* $t_0 \in R^+$ *and* $\varphi : [0,t_0) \to R^n$ *is bounded and continuous.*

Proof. It is easy to see that (15) has a unique R^+-bounded solution which satisfies (15) on R^+, say $\xi(t)$. Let $\xi(t)$ denote again the R-extension of the given $\xi(t)$ obtained by defining $\xi(t) := \xi(0) = a(0)$ for $t<0$, and for any $k \in N$, let $\xi_k(t) = \xi(t+kT)$, $t \in R$. Since Theorem 2 implies that (2) has a unique T-periodic solution, say $\pi(t)$, and it is a unique R-bounded solution of (2) satisfying (2) on R, by Lemma 2(i), it is easily seen that $\xi_k(t)$ converges to $\pi(t)$ uniformly on $[0,T]$ as $k \to \infty$. Thus we obtain $\xi(t)-\pi(t) \to 0$ as $t \to \infty$. The latter part follows directly from Theorem 1(iii).

Next consider (2) under (3)-(6), and suppose that (2) has a unique R-bounded solution satisfying (2) on R, say $\pi(t)$. Then it is T-periodic and it is easy to see that $\pi(t)$ is a solution of the equation

$$x(t)=p(t)+r(t)+\int_0^t P(t,s,x(s))ds+\int_t^\infty Q(t,s,x(s))ds, \quad t \in R^+, \tag{16}$$

where $r(t) := \int_{-\infty}^0 P(t,s,\pi(s))ds$, $t \in R^+$. Moreover, from (6) we have that $r(t)$ is continuous and $r(t) \to 0$ as $t \to \infty$. Thus (16) is a special case of (1) with $q(t)=r(t)$, $F(t,s,x) \equiv 0$ and $G(t,s,x) \equiv 0$. From Theorem 1 and the argument in the proof of Theorem 3, we obtain the following corollary.

Corollary. *Suppose that* (3)-(6) *hold, and that* (2) *has a unique R-bounded solution satisfying* (2) *on* R, *say* $\pi(t)$. *Then it is* T-*periodic and* $\pi(t)$ *is a unique* R^+-*bounded solution of* (16) *which satisfies* (16) *on* R^+, *and any* R^+-*bounded solution* $x(t)=x(t,t_0,\varphi)$

of (16) *approaches* $\pi(t)$ *as* $t \to \infty$, *where* $t_0 \in R^+$ *and* $\varphi : [0, t_0) \to R^n$ *is bounded and continuous.*

Now we show an example.

Example. Consider the scalar linear equation

$$x(t)=p(t)+\alpha\int_{-\infty}^{t} e^{s-t}(\cos\ t)x(s)ds+\beta\int_{t}^{\infty} e^{t-s}(\sin\ t)x(s)ds, \quad t\in R, \quad (17)$$

where $p : R \to R$ is continuous 2π-periodic, and α and β are constants with $|\alpha|+|\beta|<1$. Eq. (17) is a special case of (2) with $n=1$, $P(t,s,x)=\alpha e^{s-t}(\cos\ t)x$ and $Q(t,s,x)=\beta e^{t-s}(\sin\ t)x$. Thus, (3)-(6), (10) with $L_J^-(t,s)=|\alpha|e^{s-t}$, (11) with $L_J^+(t,s)=|\beta|e^{t-s}$, and (14) with $\lambda=|\alpha|+|\beta|$ hold. Thus, from Theorem 2, (17) has a unique R-bounded solution satisfying (17) on R, say $\pi(t)$, and it is 2π-periodic, and any R-bounded solution of (17) with an initial time $t_0 \in R$ and a bounded continuous initial function $\varphi : (-\infty, t_0) \to R^n$ approaches $\pi(t)$ as $t \to \infty$.

On the other hand, $\pi(t)$ is a unique R^+-bounded solution of the equation

$$\begin{aligned} x(t)=p(t)&+\alpha\int_{-\infty}^{0} e^{s-t}(\cos\ t)\pi(s)ds+\alpha\int_{0}^{t} e^{s-t}(\cos\ t)x(s)ds \\ &+\beta\int_{t}^{\infty} e^{t-s}(\sin\ t)x(s)ds, \end{aligned} \qquad t\in R^+ \quad (18)$$

which satisfies (18) on R^+. Moreover, from Corollary, any R^+-bounded solution $x(t)=x(t,t_0,\varphi)$ of (15) approaches the 2π-periodic solution $\pi(t)$ of (18) as $t \to \infty$, where $t_0 \in R^+$ and $\varphi : [0, t_0) \to R^n$ is bounded and continuous.

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