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ASYMPTOTIC STABILITY CONDITION FOR LINEAR DIFFERENTIAL-DIFFERENCE EQUATION WITH N DELAYS

TADAYUKI HARA , RINKO MIYAZAKI AND TAKAYASU MORII

(大阪府大・工　原 惟行　宮崎倫子　森居崇泰)

DEPARTMENT OF MATHEMATICAL SCIENCES,

OSAKA PREFECTURE UNIVERSITY, SAKAI 593, JAPAN

1. Introduction. In this paper we give some new necessary and sufficient conditions for the uniform asymptotic stability of the zero solution of the linear differential-difference equation with N delays

$$x'(t)=A\sum_{k=1}^{N}x(t-\tau_k) \tag{1.1}$$

where A is a 2×2 constant matrix.

Among many authors investigating the stability of delay differential equations, Stépán[1] has shown that the zero solution of the scalar delay differential equation with two delays

$$x'(t)=-a\left(x(t-\tau_1)+x(t-\tau_2)\right), \tag{1.2}$$

where $a>0,\ \tau_1,\tau_2\geq 0,\ \tau_1+\tau_2>0$, is uniformly asymptotically stable if and only if

$$2a(\tau_1+\tau_2)\cos\left(\frac{\tau_1-\tau_2}{\tau_1+\tau_2}\frac{\pi}{2}\right)<\pi. \tag{1.3}$$

Also in [2], the first author has recently shown that the zero solution of the linear delay differential equation with a positive constant delay

$$x'(t)=-\rho R(\theta)x(t-\tau), \tag{1.4}$$

where ρ is a real constant and $R(\theta)$ represents a 2×2 matrix

$$\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

with $|\theta| < \frac{\pi}{2}$, is uniformly asymptotically stable if and only if

$$0 < \rho\tau < \frac{\pi}{2} - |\theta|. \tag{1.5}$$

The purpose of this paper is to obtain some new results by merging above results together and increasing delays to N. With regard to N delays, we consider the case $\{\tau_k\}$ is an arithmetic sequence, that is,

$$\tau_k = \tau + (k-1)l \quad \text{with} \quad \tau \geq 0 \quad \text{and} \quad l > 0 \quad \text{for } k = 1, 2, \cdots, N.$$

This idea came up since two delays as in (1.2) always form an arithmetic sequence. Also, by the transfomation $x(t) = Py(t)$ with an appropriate regular matrix P, we can rewrite (1.1) as

$$y'(t) = P^{-1}AP \sum_{k=1}^{N} y(t - \tau_k).$$

Consequently, we consider the equation (1.1) where the matrix A is either of the following two matrices:

(I) the case matrix A has real eigenvalues a_1 and a_2,

$$A = -\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}$$

where a_1, a_2, and b are real numbers.

(II) the case matrix A has complex eigenvalues $\rho(\cos\theta \pm \sin\theta)$,

$$A = -\rho R(\theta) = -\rho \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where ρ is a real number and $|\theta| < \dfrac{\pi}{2}$.

For the case (I), we have

Theorem 1.1. *The zero solution of* (1.1) *is uniformly asymptotically stable if and only if*

$$a_1, a_2 > 0$$

and

$$\frac{a(\tau_1+\tau_N)}{2}\frac{\sin\left(\dfrac{Nl}{\tau_1+\tau_N}\dfrac{\pi}{2}\right)}{\sin\left(\dfrac{l}{\tau_1+\tau_N}\dfrac{\pi}{2}\right)}<\frac{\pi}{2},$$

where $a=\max\{a_1,a_2\}$.

For the case (II), we have

Theorem 1.2. *The zero solution of* (1.1) *is uniformly asymptotically stable if and only if*

$$\rho>0 \tag{1.6}$$

and

$$\frac{\rho(\tau_1+\tau_N)}{2}\frac{\sin\left(\dfrac{Nl}{\tau_1+\tau_N}\left(\dfrac{\pi}{2}-|\theta|\right)\right)}{\sin\left(\dfrac{l}{\tau_1+\tau_N}\left(\dfrac{\pi}{2}-|\theta|\right)\right)}<\frac{\pi}{2}-|\theta|. \tag{1.7}$$

If $\theta=0$ and $N=2$, the condition (1.7) coincides with (1.3). If $N=1$, the condition (1.7) also coincides with (1.5). The proof of Theorem 1.2, very similar to the proof of Theorem 1.1, will be only given in the next section.

2. Proof. First, the following proposition stands.

Proposition 2.1.

$$\frac{\rho(\tau_1+\tau_N)}{2}\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right)<\frac{\pi}{2}-|\theta| \tag{2.1}$$

and (1.7) *are equivalent.*

Proof. On the formula concerning with the sum of cosines

$$\sum_{k=1}^{N}\cos\left(x+(k-1)y\right)=\frac{\cos\left(x+\dfrac{1}{2}(N-1)y\right)\sin\dfrac{1}{2}Ny}{\sin\dfrac{1}{2}y},$$

if

$$x = -\frac{(N-1)l}{\tau_1 + \tau_N}\left(\frac{\pi}{2} - |\theta|\right), \qquad y = \frac{2l}{\tau_1 + \tau_N}\left(\frac{\pi}{2} - |\theta|\right),$$

then we have the fraction part of sines in (1.7) and the summation part of cosines in (2.1). Thus, (1.7) and (2.1) are equivalent. □

In the proof of Theorem 1.2 we use condition (2.1) instead of (1.7).

The theorem is proved by using the fact that the zero solution of (1.1) is uniformly asymptotically stable if and only if all the roots of the characteristic equation of (1.1)

$$D(\lambda) = \det\left[\lambda I + \rho R(\theta)\sum_{k=1}^{N} e^{-\lambda\tau_k}\right] = 0 \tag{2.2}$$

lie in the left half of the complex plane, that is, the real part of any characteristic root of (2.2) is negative. Thus, we investigate the characteristic roots of (2.2) to prove the theorem.

Proof of Theorem 1.2. (sufficiency) Let

$$p_+(\lambda) = \lambda + \rho e^{i\theta}\sum_{k=1}^{N} e^{-\lambda\tau_k} \tag{2.3}$$

and

$$p_-(\lambda) = \lambda + \rho e^{-i\theta}\sum_{k=1}^{N} e^{-\lambda\tau_k}. \tag{2.4}$$

The characteristic equation of (1.1) is

$$\begin{aligned}
D(\lambda) &= \begin{vmatrix} \lambda + \rho\cos\theta\sum_{k=1}^{N} e^{-\lambda\tau_k} & -\rho\sin\theta\sum_{k=1}^{N} e^{-\lambda\tau_k} \\ \rho\sin\theta\sum_{k=1}^{N} e^{-\lambda\tau_k} & \lambda + \rho\cos\theta\sum_{k=1}^{N} e^{-\lambda\tau_k} \end{vmatrix} \\
&= \left(\lambda + \rho\cos\theta\sum_{k=1}^{N} e^{-\lambda\tau_k}\right)^2 + \left(\rho\sin\theta\sum_{k=1}^{N} e^{-\lambda\tau_k}\right)^2 \\
&= (\lambda + \rho\cos\theta\sum_{k=1}^{N} e^{-\lambda\tau_k} + i\rho\sin\theta\sum_{k=1}^{N} e^{-\lambda\tau_k})(\lambda + \rho\cos\theta\sum_{k=1}^{N} e^{-\lambda\tau_k} - i\rho\sin\theta\sum_{k=1}^{N} e^{-\lambda\tau_k}) \\
&= (\lambda + \rho(\cos\theta + i\sin\theta)\sum_{k=1}^{N} e^{-\lambda\tau_k})(\lambda + \rho(\cos\theta - i\sin\theta)\sum_{k=1}^{N} e^{-\lambda\tau_k}) \\
&= (\lambda + \rho e^{i\theta}\sum_{k=1}^{N} e^{-\lambda\tau_k})(\lambda + \rho e^{-i\theta}\sum_{k=1}^{N} e^{-\lambda\tau_k})
\end{aligned}$$

$$= p_+(\lambda)p_-(\lambda)$$
$$= 0.$$

When λ is a complex root and $\bar{\lambda}$ is a complex conjugate of λ, the relation

$$p_+(\lambda) = \overline{p_-(\bar{\lambda})} \tag{2.5}$$

stands. Also, to satisfy (2.2), $p_+(\lambda) = 0$ or $p_-(\lambda) = 0$, that is,

$$p_+(\lambda) = \lambda + \rho e^{i\theta} \sum_{k=1}^{N} e^{-\lambda\tau_k} = 0, \qquad \text{or} \qquad p_-(\lambda) = \lambda + \rho e^{-i\theta} \sum_{k=1}^{N} e^{-\lambda\tau_k} = 0.$$

On above equations, when $\tau_1 + \tau_2 + \cdots + \tau_N = 0$, that is, $\tau_k = 0$ for $k = 1, 2, \cdots, N$, $\lambda + N\rho e^{\pm i\theta} = 0$. Then, by (1.6) and $|\theta| < \frac{\pi}{2}$, we have

$$Re\ \lambda = Re\ \{-N\rho(\cos\theta \pm i\sin\theta)\} = -N\rho\cos\theta < 0.$$

Thus, when $\tau_1 + \tau_2 + \cdots + \tau_N = 0$, the characteristic root of (2.2) lies in the left half of the complex plane. Also, when $\lambda = 0$ $N\rho e^{\pm i\theta} = 0$, but $N\rho e^{\pm i\theta} \neq 0$ since (1.6). Thus, $\lambda = 0$ is not a characteristic root of (2.2).

If the increasing of $\tau_1 + \tau_2 + \cdots + \tau_N$ leads the zero solution of (1.1) to instability, the characteristic root of (2.2) must cross the imaginary axis and lie in the right half of the complex plane, that is, there is an $\omega \neq 0$ such that

$$p_+(i\omega) = 0 \qquad \text{for some } \tau_1, \tau_2, \cdots, \tau_N, \quad \text{or} \quad p_-(i\omega) = 0 \qquad \text{for some } \tau_1, \tau_2, \cdots, \tau_N.$$

From (2.5), $p_-(i\omega) = 0$ implies $p_+(i\omega) = 0$. Also when $-\frac{\pi}{2} < \theta \leq 0$, substituting $\theta = -\tilde{\theta}$ in $p_+(i\omega) = 0$ and $p_-(i\omega) = 0$ implies $0 \leq \tilde{\theta} < \frac{\pi}{2}$. Thus, we consider only the case $p_+(i\omega) = 0$ when $0 \leq \theta < \frac{\pi}{2}$.

Substituting $\lambda = i\omega$ in (2.3), we have

$$\begin{aligned} p_+(i\omega) &= i\omega + \rho e^{i\theta} \sum_{k=1}^{N} e^{-i\omega\tau_k} \\ &= i\omega + \rho \sum_{k=1}^{N} e^{i(\theta - \omega\tau_k)} \end{aligned}$$

$$= i\omega + \rho \sum_{k=1}^{N} (\cos(\theta - \omega\tau_k) + i\sin(\theta - \omega\tau_k))$$
$$= \rho \sum_{k=1}^{N} \cos(\theta - \omega\tau_k) + i\left(\omega + \rho\sum_{k=1}^{N}\sin(\theta - \omega\tau_k)\right).$$

Note that

$$\begin{aligned}
R(\omega) &= Re\ p_+(i\omega) \\
&= \rho \sum_{k=1}^{N} \cos(\theta - \omega\tau_k) \\
&= \frac{1}{2}\rho \sum_{k=1}^{N} (\cos(\theta - \omega\tau_k) + \cos(\theta - \omega\tau_{N-k+1})) \\
&= \frac{1}{2}\rho \sum_{k=1}^{N} 2\cos\left(\frac{(\theta - \omega\tau_k) + (\theta - \omega\tau_{N-k+1})}{2}\right)\cos\left(\frac{(\theta - \omega\tau_k) - (\theta - \omega\tau_{N-k+1})}{2}\right) \\
&= \rho \sum_{k=1}^{N} \cos\left(\frac{2\theta - \omega(\tau_k + \tau_{N-k+1})}{2}\right)\cos\left(\frac{-\omega(\tau_k - \tau_{N-k+1})}{2}\right) \\
&= \rho\cos\left(\frac{2\theta - \omega(\tau_1 + \tau_N)}{2}\right)\sum_{k=1}^{N}\cos\left(\frac{\omega(\tau_k - \tau_{N-k+1})}{2}\right),
\end{aligned}$$

and

$$\begin{aligned}
I(\omega) &= Im\ p_-(i\omega) \\
&= \omega + \rho \sum_{k=1}^{N} \sin(\theta - \omega\tau_k) \\
&= \omega + \frac{1}{2}\rho \sum_{k=1}^{N} (\sin(\theta - \omega\tau_k) + \sin(\theta - \omega\tau_{N-k+1})) \\
&= \omega + \frac{1}{2}\rho \sum_{k=1}^{N} 2\sin\left(\frac{(\theta - \omega\tau_k) + (\theta - \omega\tau_{N-k+1})}{2}\right)\cos\left(\frac{(\theta - \omega\tau_k) - (\theta - \omega\tau_{N-k+1})}{2}\right) \\
&= \omega + \rho \sum_{k=1}^{N} \sin\left(\frac{2\theta - \omega(\tau_k + \tau_{N-k+1})}{2}\right)\cos\left(\frac{-\omega(\tau_k - \tau_{N-k+1})}{2}\right) \\
&= \omega + \rho\sin\left(\frac{2\theta - \omega(\tau_1 + \tau_N)}{2}\right)\sum_{k=1}^{N}\cos\left(\frac{\omega(\tau_k - \tau_{N-k+1})}{2}\right).
\end{aligned}$$

To satisfy $p_+(i\omega) = 0$, $R(\omega) = 0$ and $I(\omega) = 0$. First, observe that $R(\omega) = 0$ if and only if

$$\cos\left(\frac{2\theta - \omega(\tau_1 + \tau_N)}{2}\right) = 0 \qquad \text{or} \qquad \sum_{k=1}^{N}\cos\left(\frac{\omega(\tau_k - \tau_{N-k+1})}{2}\right) = 0$$

If the latter is true, then $I(\omega) = 0$ would imply $\omega = 0$ which contradicts the assumption. Hence, we must have, for $n = 0, 1, 2, \cdots$,

$$\text{(a)}\ \frac{2\theta - \omega(\tau_1 + \tau_N)}{2} = \frac{\pi}{2} + n\pi \qquad \text{or} \qquad \text{(b)}\ \frac{2\theta - \omega(\tau_1 + \tau_N)}{2} = -\frac{\pi}{2} - n\pi.$$

In case (a),

$$\omega = \frac{-(2n+1)\pi + 2\theta}{\tau_1 + \tau_N}, \tag{2.6}$$

and when $0 \leq \theta < \frac{\pi}{2}$, $\omega < 0$ for $n = 0, 1, 2, \cdots$.

In case (b),

$$\omega = \frac{(2n+1)\pi + 2\theta}{\tau_1 + \tau_N}, \tag{2.7}$$

and when $0 \leq \theta < \frac{\pi}{2}$, $\omega > 0$ for $n = 0, 1, 2, \cdots$. Also, $R(\omega) = 0$ and $I(\omega) = 0$ imply

$$\rho \cos\left(\frac{2\theta - \omega(\tau_1 + \tau_N)}{2}\right) \sum_{k=1}^{N} \cos\left(\frac{\omega(\tau_k - \tau_{N-k+1})}{2}\right) = 0$$

and

$$\rho \sin\left(\frac{2\theta - \omega(\tau_1 + \tau_N)}{2}\right) \sum_{k=1}^{N} \cos\left(\frac{\omega(\tau_k - \tau_{N-k+1})}{2}\right) = -\omega.$$

By squaring both sides of equations above and adding them together, we have

$$\rho^2 \left(\sum_{k=1}^{N} \cos\left(\frac{\omega(\tau_k - \tau_{N-k+1})}{2}\right)\right)^2 = \omega^2,$$

which implies

$$\rho \left|\sum_{k=1}^{N} \cos\left(\frac{\omega(\tau_k - \tau_{N-k+1})}{2}\right)\right| = |\omega|. \tag{2.8}$$

In case (a), (2.6) and (2.8) imply

$$\left|\frac{-(2n+1)\pi + 2\theta}{\tau_1 + \tau_N}\right| = \rho \left|\sum_{k=1}^{N} \cos\left(\frac{-(2n+1)\pi + 2\theta}{\tau_1 + \tau_N} \frac{\tau_k - \tau_{N-k+1}}{2}\right)\right|.$$

Hence,

$$\rho \left|\sum_{k=1}^{N} \cos\left(\frac{\tau_k - \tau_{N-k+1}}{\tau_1 + \tau_N} \frac{(2n+1)\pi - 2\theta}{2}\right)\right| = \frac{(2n+1)\pi - 2\theta}{\tau_1 + \tau_N},$$

or, equivalently,

$$\rho(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi-2\theta}{2}\right)\right|=(2n+1)\pi-2\theta.$$

When the equation above holds, $p_+(\lambda)=0$ has a root $\lambda=i\omega$. Hence, if, for $n=0,1,2,\cdots$,

$$\rho(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi-2\theta}{2}\right)\right|<(2n+1)\pi-2\theta, \tag{2.9}$$

then $p_+(\lambda)=0$ does not have a root $\lambda=i\omega$. However, Lemma 3.1, which will be given in section 3, shows that if (2.1) is true, then (2.9) is also true. This indicates (2.2) does not have a characteristic root $\lambda=i\omega$ and the characteristic roots of (2.2) remain in the left half of the complex plane although increasing of $\tau_1+\tau_2+\cdots+\tau_N$.

In case (b), (2.7) and (2.8) imply

$$\left|\frac{(2n+1)\pi+2\theta}{\tau_1+\tau_N}\right|=\rho\left|\sum_{k=1}^{N}\cos\left(\frac{(2n+1)\pi+2\theta}{\tau_1+\tau_N}\frac{\tau_k-\tau_{N-k+1}}{2}\right)\right|.$$

Hence,

$$\rho\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi+2\theta}{2}\right)\right|=\frac{(2n+1)\pi+2\theta}{\tau_1+\tau_N},$$

or, equivalently,

$$\rho(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi+2\theta}{2}\right)\right|=(2n+1)\pi+2\theta.$$

When the equation above holds, $p_+(\lambda)=0$ has a root $\lambda=i\omega$. Hence, if, for $n=0,1,2,\cdots$,

$$\rho(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi+2\theta}{2}\right)\right|<(2n+1)\pi+2\theta, \tag{2.10}$$

then $p_+(\lambda)=0$ does not have a root $\lambda=i\omega$. However, Lemma 3.2, which will be given in section 3, shows that if (2.1) is true, then (2.10) is also true. This indicates (2.2) does not have a characteristic root $\lambda=i\omega$ and the characteristic roots of (2.2) remain in the left half of the complex plane although increasing of $\tau_1+\tau_2+\cdots+\tau_N$.

Therefore, if (1.6) and (2.1) are true, then the zero solution of (1.1) is uniformly asymptotically stable.

(necessity) Suppose the zero solution of (1.1) is uniformly asymptotically stable and consider the following two cases:

$$\text{(A)} \quad \rho > 0 \qquad \text{and} \qquad \text{(B)} \quad \rho \leq 0.$$

For the case (A), assume, for the sake of contradiction, the zero solution of (1.1) is uniformly asymptotically stable, $\rho > 0$, and

$$\frac{\rho(\tau_1+\tau_N)}{2}\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right) \geq \frac{\pi}{2}-|\theta|. \tag{2.11}$$

By Lemma 3.1, (2.11) and there exists an integer $m \geq 0$ such that

$$\rho(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2m+1)\pi-2\theta}{2}\right)\right| \geq (2m+1)\pi-2\theta$$

are equivalent. Hence, there exists an ρ_m where $0 < \rho_m \leq \rho$ such that

$$\rho_m(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2m+1)\pi-2\theta}{2}\right)\right| = (2m+1)\pi-2\theta.$$

When the equation above holds,

$$\lambda+\rho_m e^{i\theta}\sum_{k=1}^{N}e^{-\lambda\tau_k}=0$$

has a root

$$\lambda=-i\frac{(2m+1)\pi-2\theta}{\tau_1+\tau_N}.$$

from first part of the proof.

Here, consider the movement of the zero of

$$p_+(\lambda)=\lambda+\rho e^{i\theta}\sum_{k=1}^{N}e^{-\lambda\tau_k}=0$$

with ρ as a parameter. Namely, let λ be a function of $r(0 < r \leq \rho)$ which satisfy

$$p_+(\lambda;r)=\lambda+re^{i\theta}\sum_{k=1}^{N}e^{-\lambda\tau_k}=0 \tag{2.12}$$

and investigate its movement. Note that λ is continuous of r.

From the argument above, the zero of $p_+(\lambda; r) = 0$ is on the imaginary axis when $r = \rho_m$. Also, since the zero solution of (1.1) is uniformly asymptotically stable, there exists $\rho'(\rho_m \le \rho' \le \rho)$ and real number $\omega' \neq 0$ such that $\lambda = i\omega'$ is a zero of $p_+(\lambda; r) = 0$ on the imaginary axis when $r = \rho'$ and its crossing of imaginary axis is not from left to right, that is,

$$Re\, \frac{\partial \lambda}{\partial r}\bigg|_{r=i\omega'} \le 0 \tag{2.13}$$

By the same argument of first part of the proof,

$$\omega' = \frac{-(2n+1)\pi + 2\theta}{\tau_1 + \tau_N} \neq 0, \qquad \text{for } n = 0, \pm 1, \pm 2, \cdots. \tag{2.14}$$

Taking the partial derivative of λ with r on (2.12),

$$\left(1 - re^{i\theta} \sum_{k=1}^{N} \tau_k e^{-\lambda \tau_k}\right) \frac{\partial \lambda}{\partial r} + e^{i\theta} \sum_{k=1}^{N} e^{-\lambda \tau_k} = 0,$$

or

$$\frac{\partial \lambda}{\partial r} = \frac{\lambda / r}{1 - re^{i\theta} \sum_{k=1}^{N} \tau_k e^{-\lambda \tau_k}}. \tag{2.15}$$

Let $d(\lambda; r) = 1 - re^{i\theta} \sum_{k=1}^{N} \tau_k e^{-\lambda \tau_k}$. Since $\lambda = i\omega'$ when $r = \rho'$,

$$\begin{aligned}
d(\omega'; \rho') &= 1 - \rho' e^{i\theta} \sum_{k=1}^{N} \tau_k e^{-i\omega' \tau_k} \\
&= 1 - \rho' \sum_{k=1}^{N} \tau_k e^{i(\theta - \omega' \tau_k)} \\
&= 1 - \rho' \sum_{k=1}^{N} \tau_k \left(\cos(\theta - \omega' \tau_k) + i \sin(\theta - \omega' \tau_k)\right) \\
&= \left(1 - \rho' \sum_{k=1}^{N} \tau_k \cos(\theta - \omega' \tau_k)\right) - i\rho' \sum_{k=1}^{N} \tau_k \sin(\theta - \omega' \tau_k) \\
&= \left(1 - \rho' \sum_{k=1}^{N} \tau_k \cos(\theta - \omega' \tau_k)\right) \\
&\qquad\qquad - \frac{i}{2} \rho' \sum_{k=1}^{N} \left(\tau_k \sin(\theta - \omega' \tau_k) + \tau_{N-k+1} \sin(\theta - \omega' \tau_{N-k+1})\right).
\end{aligned}$$

From (2.14), for $k = 1, 2, \cdots, N$ and $n = 0, \pm 1, \pm 2, \cdots$,

$$\theta - \omega'\tau_k = -(\theta - \omega'\tau_{N-k+1}) + (2n+1)\pi$$

which implies that

$$\sin(\theta - \omega'\tau_k) = \sin(\theta - \omega'\tau_{N-k+1}) \quad \text{for } k = 1, 2, \cdots, N. \tag{2.16}$$

Also, from $Im\ p_+(i\omega'; \rho') = 0$,

$$\omega' + \rho' \sum_{k=1}^{N} \sin(\theta - \omega'\tau_k) = 0,$$

or

$$\sum_{k=1}^{N} \sin(\theta - \omega'\tau_k) = -\frac{\omega'}{\rho'} \neq 0. \tag{2.17}$$

Thus,

$$\begin{aligned} d(\lambda; r) &= \left(1 - \rho' \sum_{k=1}^{N} \tau_k \cos(\theta - \omega'\tau_k)\right) - \frac{i}{2}\rho' \sum_{k=1}^{N} (\tau_k + \tau_{N-k+1}) \sin(\theta - \omega'\tau_k) \\ &= \left(1 - \rho' \sum_{k=1}^{N} \tau_k \cos(\theta - \omega'\tau_k)\right) - \frac{i}{2}\rho'(\tau_1 + \tau_N) \sum_{k=1}^{N} \sin(\theta - \omega'\tau_k) \\ &= \left(1 - \rho' \sum_{k=1}^{N} \tau_k \cos(\theta - \omega'\tau_k)\right) - \frac{i}{2}(\tau_1 + \tau_N)\omega' \end{aligned}$$

which implies that the denominator of (2.15) is nonzero and there exists a value of $\frac{\partial \lambda}{\partial r}$ at $\lambda = i\omega'$.

Observing the crossing of the imaginary axis by the characteristic root when $r = \rho'$ and $\lambda = i\omega'$,

$$\begin{aligned} sign\ Re \left.\frac{\partial \lambda}{\partial r}\right|_{r=i\omega'} &= sign\ Re \left.\left(\frac{\partial \lambda}{\partial r}\right)^{-1}\right|_{r=i\omega'} \\ &= sign\ Re \left.\left\{\frac{1 - re^{i\theta}\sum_{k=1}^{N} \tau_k e^{-\lambda\tau_k}}{\lambda / r}\right\}\right|_{r=i\omega'} \\ &= sign\ Re \left\{\frac{\rho'(1 - re^{i\theta}\sum_{k=1}^{N} \tau_k e^{-\lambda\tau_k})}{i\omega'}\right\} \end{aligned}$$

$$= sign\ Re \left\{ \frac{\rho'((1-\rho' \sum_{k=1}^{N} \tau_k \cos(\theta - \omega'\tau_k)) - \frac{i}{2}(\tau_1+\tau_N)\omega')}{i\omega'} \right\}$$

$$= sign \left\{ \frac{\rho'(\tau_1+\tau_N)}{2} \right\} > 0$$

which contradicts (2.13). Therefore, if $\rho > 0$ and (2.11) holds, then the zero solution of (1.1) is uniformly asymptotically stable.

In case (B), when $\rho = 0$ the zero solution of (1.1) is not uniformly asymptotically stable which contradicts the assumption. Hence, consider the case when $\rho < 0$.

We consider the following two cases when $\rho < 0$:

$$\text{(i)} \quad \frac{|\rho|(\tau_1+\tau_N)}{2} \sum_{k=1}^{N} \cos\left(\frac{\tau_k - \tau_{N-k+1}}{\tau_1+\tau_N} (\frac{\pi}{2} - |\theta|) \right) < \frac{\pi}{2} - |\theta|$$

and

$$\text{(ii)} \quad \frac{|\rho|(\tau_1+\tau_N)}{2} \sum_{k=1}^{N} \cos\left(\frac{\tau_k - \tau_{N-k+1}}{\tau_1+\tau_N} (\frac{\pi}{2} - |\theta|) \right) \geq \frac{\pi}{2} - |\theta|.$$

In case (i), assume, for the sake of contradiction, the zero solution of (1.1) is uniformly asymptotically stable, $\rho < 0$, and

$$\frac{|\rho|(\tau_1+\tau_N)}{2} \sum_{k=1}^{N} \cos\left(\frac{\tau_k - \tau_{N-k+1}}{\tau_1+\tau_N} \left(\frac{\pi}{2} - |\theta|\right) \right) < \frac{\pi}{2} - |\theta|. \tag{2.18}$$

Repeat the same argument in the proof of sufficiency using $\rho = -|\rho|$. When $\tau_1 + \tau_2 + \cdots + \tau_k = 0$, one can show that the characteristic root lies in the right half of the complex plane and remains in that plane as long as (2.18) holds. This shows the zero solution of (1.1) is not uniformly asymptotically stable which contradicts the assumption.

In case (ii), assume, for the sake of contradiction, the zero solution of (1.1) is uniformly asymptotically stable, $\rho < 0$, and

$$\frac{|\rho|(\tau_1+\tau_N)}{2} \sum_{k=1}^{N} \cos\left(\frac{\tau_k - \tau_{N-k+1}}{\tau_1+\tau_N} \left(\frac{\pi}{2} - |\theta|\right) \right) \geq \frac{\pi}{2} - |\theta|.$$

Also, in this case by repeating the same argument in the proof of neccesity of case (A), one can show the contradiction. □

3. Lemmas. In this section we give the proofs of lemmas which are used to prove the theorem.

Lemma 3.1.

$$\rho(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi-2\theta}{2}\right)\right| < (2n+1)\pi-2\theta \quad \textit{for} \ \ n=0,1,2,\cdots \tag{3.1}$$

and

$$\frac{\rho(\tau_1+\tau_N)}{2}\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right) < \frac{\pi}{2}-|\theta| \tag{3.2}$$

are equivalent where $\tau_k=\tau+(k-1)l$ $(\tau\geq 0, l>0)$ *and* $0\leq\theta<\dfrac{\pi}{2}$.

Proof. For $\phi\in[0,\frac{\pi}{2}]$ and $n=0,1,2,\cdots$, we always have that $|\sin n\phi|\leq n\sin\phi$. Denote $\frac{\pi}{2}-\phi=|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}|\frac{\pi}{2}$ for $k=1,2,\ldots,N$.

When $\phi\in[0,\frac{\pi}{2}]$, $0\leq\theta<\frac{\pi}{2}$, and $0<\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|<1$ for $k=1,2,\cdots,N$,

$$\begin{aligned}
&\left|\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi-2\theta}{2}\right)\right| \\
&=\left|\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi}{2}\right)\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\theta\right)\right. \\
&\qquad\qquad\left.+\sin\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi}{2}\right)\sin\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\theta\right)\right| \\
&\leq\left|\cos\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\frac{(2n+1)\pi}{2}\right)\cos\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right. \\
&\qquad\qquad\left.+\sin\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\frac{(2n+1)\pi}{2}\right)\sin\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right| \\
&=\left|\cos\left((2n+1)\left(\frac{\pi}{2}-\phi\right)\right)\cos\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right. \\
&\qquad\qquad\left.+\sin\left((2n+1)\left(\frac{\pi}{2}-\phi\right)\right)\sin\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right| \\
&\leq\left||\sin((2n+1)\phi)|\cos\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right. \\
&\qquad\qquad\left.+\left|\sin\left((2n+1)\left(\frac{\pi}{2}-\phi\right)\right)\right|\sin\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right| \\
&\leq\left|(2n+1)\sin\phi\cos\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)+(2n+1)\sin\left(\frac{\pi}{2}-\phi\right)\sin\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right|
\end{aligned}$$

$$
\begin{aligned}
&= (2n+1)\left|\cos\left(\frac{\pi}{2}-\phi\right)\cos\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)+\sin\left(\frac{\pi}{2}-\phi\right)\sin\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right| \\
&\le (2n+1)\left|\left(\cos\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\frac{\pi}{2}\right)\cos\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right. \\
&\qquad\qquad\qquad \left.+\sin\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\frac{\pi}{2}\right)\sin\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\theta\right)\right| \\
&= (2n+1)\left|\cos\left(\left|\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\right|\left(\frac{\pi}{2}-\theta\right)\right)\right| \\
&= (2n+1)\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right).
\end{aligned}
$$

Since

$$
\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi-2\theta}{2}\right)\right| \le \sum_{k=1}^{N}\left|\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi-2\theta}{2}\right)\right|,
$$

we have

$$
\begin{aligned}
&\rho(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi-2\theta}{2}\right)\right| \\
&\qquad\le \rho(\tau_1+\tau_N)\sum_{k=1}^{N}\left|\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi-2\theta}{2}\right)\right| \\
&\qquad\qquad\le (2n+1)\rho(\tau_1+\tau_N)\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right).
\end{aligned}
$$

Thus, to satisfy (3.1),

$$
(2n+1)\rho(\tau_1+\tau_N)\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right) < (2n+1)\pi-2\theta,
$$

or

$$
\frac{\rho(\tau_1+\tau_N)}{2}\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right) < \frac{\pi}{2}-\frac{\theta}{2n+1}.
$$

Since $\frac{\pi}{2}-|\theta| \le \frac{\pi}{2}-\frac{\theta}{2n+1}$ for $n=0,1,2,\cdots$ and $0\le\theta<\frac{\pi}{2}$, (3.2) impies (3.1). And Clearly, (3.1) implies (3.2) when $n=0$. □

Lemma 3.2. *If*

$$
\frac{\rho(\tau_1+\tau_N)}{2}\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right) < \frac{\pi}{2}-|\theta|, \tag{3.3}
$$

then

$$\rho(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi+2\theta}{2}\right)\right| < (2n+1)\pi+2\theta, \quad \textit{for}\ n=0,1,2,\cdots \tag{3.4}$$

Proof. From the same argument in the proof of Lemma 3.1, we get

$$\rho(\tau_1+\tau_N)\left|\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\frac{(2n+1)\pi+2\theta}{2}\right)\right| \le (2n+1)\rho(\tau_1+\tau_N)\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right).$$

To satisfy (3.4),

$$(2n+1)\rho(\tau_1+\tau_N)\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right) < (2n+1)\pi+2\theta,$$

or

$$\frac{\rho(\tau_1+\tau_N)}{2}\sum_{k=1}^{N}\cos\left(\frac{\tau_k-\tau_{N-k+1}}{\tau_1+\tau_N}\left(\frac{\pi}{2}-|\theta|\right)\right) < \frac{\pi}{2}+\frac{\theta}{2n+1}.$$

Since $\frac{\pi}{2}-|\theta| \le \frac{\pi}{2}+\frac{\theta}{2n+1}$ for $n=0,1,2,\cdots$ and $0\le\theta<\frac{\pi}{2}$, (3.3) implies (3.4). □

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