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# On solution semigroups of functional differential equations

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## 1 Introduction

We consider necessary conditions so that the solution operators of autonomous linear functional differential equations make a $C_0$-semigroup. In [4] Kunish and Schappacher have studied similar problem by taking the space of integrable functions as the phase space. We will attack the problem on the phase space which is as general as possible.

There are many works of the semigroup theory of several kinds of equations with delay by many authors, cf. [2]. A goal of the theory is the spectral decomposition of the phase space according to the separation of the spectrum of the infinitesimal generator of the solution semigroup by a vertical line $\Re\lambda = b$ of the complex $\lambda$ plane. In the usual works we have begun to prove the existence and the uniqueness of the solution, and then obtained the representaion of the infinitesimal generator. Using this, we have computed the point spectrum of the generator, and the generalized eigenspaces. After that we have known that the generalized eigenspaces are of finite dimension, and that, on the subspace generated by generalized eigenspaces for the eigenvalues $\lambda$ such that $\Re\lambda \geq b$, the solution are defined on $(-\infty, \infty)$. Also, on the remaininig component of the decomposition the semigroup has the exponential growth order less than $b$ as $t \to \infty$.

However, dealing with the measure of noncompactness, we have been having an impression that the decomposition theory would be valid for the general $C_0$ semigroup. Let $T(t), t \geq 0$, be a $C_0$ semigroup of bounded linear operators on a Banach space $X$, and $A$ its infinitesimal generator. It is well known [3, 2] that the spectrum $\sigma(A)$ and the point spectrum $P_\sigma(A)$ are mapped in the following manner.

**Theorem 1.1**

(i) $\exp(t\sigma(A)) \subset \sigma(T(t)), t \geq 0$.

(ii) $P_\sigma(T(t)) = \exp(tP_\sigma(A))$, *plus possibly, the point* $\mu = 0$. *If* $\mu \in P_\sigma(T(t))$ *for some fixed* $t > 0$, *where* $\mu \neq 0$, *and if* $\{\lambda_n\}$ *is the set of the roots of* $e^{t\lambda} = \mu$, *then at least one of the points* $\lambda_n$ *lies in* $P_\sigma(A)$. *Furthermore* $N((\mu I - T(t))^k), k = 1, 2, \cdots$, *is the minimal closed subspace containing the linear independent subspaces* $N((\lambda_n I - A)^k)$, *where* $n$ *ranges over all* $\lambda_n \in P_\sigma(A)$.

All the element of $\sigma(T(t))$ is not completely determined by $\sigma(A)$; we know only the correspondence between the point spectum of $T(t)$ and that of $A$. In general, $\sigma(T(t))$ may contain some points out of $\exp(t\sigma(A))$. If $T(t)$ is a compact semigroup, $\sigma(T(t)), t > 0$, consists of the point spectrum only. If we know in advance that $\sigma(T(t))$, or some subset of it, consists of the point spectrum only, then we can apply Assertion (ii) to investigate the spectra of $T(t)$ and $A$. Even in such a case, there is a remaining problem whether $\sigma(A)$ contains spectrum other than point spectrum or not. Assertion (ii) says nothing about the possibility that, for a point $\mu \in P_\sigma(T(t))$, there exists a point $\lambda \in \sigma(A) \backslash P_\sigma(A)$ such that $e^{t\lambda} = \mu$.

To deal with the problem, we have found that the method of the measure of noncompactness is effective. However, Webb has already obtained in his book [6] the same extension of the above correspondence between the spectra. In spite of importance, his results are not used so frequently in books and papers in the area of delay equations untill today. So we again demonstrate here the results.

# 2 Normal eigenvalues of $C_0$ semigroup

To begin with we recall some fundamental facts. Let $S$ be a closed linear operator with dense domain in a Banach space X. The complex number $\zeta$ is said to lie in the essential spectrum of the operator $S$ whenever at least one of the following conditions holds:

(i) $R(\zeta I - S)$, the range of $\zeta I - S$, is not closed;

(ii) $\cup_{r \geq 0} N((\zeta I - S)^r)$ is of infinite dimension,($N(U)$ being the null-space of the operator $U$);

(iii) The point $\zeta$ is a limit point of the spectrum of $S$.

Let $\rho(S)$ denote the resolvent set of $S$, $\sigma(S)$ the spectrum of $S$, $P_\sigma(S)$ the point spectrum of $S$, and $E_\sigma(S)$ the essential spectrum of $S$. We call points in $N_\sigma(S) := \sigma(S)\backslash E_\sigma(S)$ normal eigenvalues of $S$.

**Lemma 2.1** ([1]) *Let $S$ be a closed linear operator densely defined in the Banach space $X$ with finite-dimensional generalized eigenspace for the complex number $\zeta_0$. Then the point $\zeta_0$ of the spectrum of $S$ is a normal eigenvalue of $S$ if and only if the resolvent $R(\zeta, S) := (\zeta I - S)^{-1}$ is analytic in the neighborhood of $\zeta_0$ and has a pole at $\zeta_0$.*

Let $\alpha(B)$ be the Kuratowskii measure of noncompactness of a bounded set $B$ in $X$ defined by

$$\alpha(B) = \inf\{d : B \text{ has a finite cover of diameter } \leq d\}.$$

If $T$ is a bounded linear operator on $X$, define $\alpha(T)$ to be the infimum of $k > 0$ such that $\alpha(TB) \leq k\alpha(B)$ for all bounded sets $B$ in $X$. Obviously, $\alpha(T) \leq |T|$, ($|T|$ being the norm of the bounded linear operator $T$). It is well known that the spectral radius $r_\sigma(T)$ and the essential spectral radius $r_e(T)$ are given as

$$r_\sigma(T) = \lim_{n\to\infty} |T^n|^{1/n}, \qquad r_e(T) = \lim_{n\to\infty} \alpha(T^n)^{1/n}.$$

Let $T(t), t \geq 0$, be a $C_0$ semigroup of bounded linear operators on X, and $A$ its infinitesimal generator. The growth bound $\omega_s$ and the essential growth bound $\omega_e$ of $T(t)$ are defined as

$$\omega_s := \lim_{t\to\infty} \frac{\log |T(t)|}{t} = \inf_{t>0} \frac{\log |T(t)|}{t},$$

$$\omega_e := \lim_{t\to\infty} \frac{\log \alpha(T(t))}{t} = \inf_{t>0} \frac{\log \alpha(T(t))}{t}.$$

Then $\omega_e \leq \omega_s$, $r_\sigma(T(t)) = \exp(t\omega_s)$, $r_e(T(t)) = \exp(t\omega_e), t > 0$. Hence if $\mu_0 \in \sigma(T(t))$ and if $|\mu_0| > \exp(t\omega_e)$, it is a normal eigenvalue of $T(t)$; Lemma 2.1 implies that $\mu_0$ is a pole of $R(\zeta, T(t))$. Suppose that $\lambda_0 \in \sigma(A)$ and that $e^{t\lambda_0} = \mu_0$. Then we will show in Appendix that $\lambda_0$ is a pole of $R(\lambda, A)$. As a result, we have the following theorem.

**Theorem 2.2** *Suppose that* $\lambda_0 \in \sigma(A)$. *If* $e^{t\lambda_0}$ *is a normal eigenvalue of* $T(t), t > 0$, *then* $\lambda_0$ *is a normal eigenvalue of* $A$. *In other words,*

$$\exp(tE_\sigma(A)) \subset E_\sigma(T(t)) \qquad \text{for } t > 0.$$

Set

$$\beta_s = \sup\{\Re\lambda : \lambda \in \sigma(A)\},$$

$$\beta_e = \sup\{\Re\lambda : \lambda \in E_\sigma(A)\}, \qquad \beta_n = \sup\{\Re\lambda : \lambda \in N_\sigma(A)\}.$$

Then $\max\{\beta_e, \beta_n\} = \beta_s \leq \omega_s$. From Theorem 2.2, we have $\beta_e \leq \omega_e$.

**Theorem 2.3** *Suppose that* $\omega_e < \omega_s$. *Then the following results hold:*

(i) *There exists at least one point* $\lambda \in N_\sigma(A)$ *such that* $\Re\lambda = \omega_s$: *consequently,* $\beta_n = \beta_s = \omega_s$ *and* $N_\sigma(A) \neq \emptyset$.

(ii) *For any* $b, \omega_e < b < \omega_s$, *the set* $\sigma(A) \cap \{\lambda : \Re\lambda \geq b\}$ *consists of finite normal eigenvalues of* $A$, *and* $\sup\{\Re\lambda : \lambda \in \sigma(A), \Re\lambda < b\} < b$.

**Proof** Let $b$ be as in (ii), and $t$ a fixed positive number. Since $e^{t\omega_e} < e^{tb} < e^{t\omega_s}$, there exist finite points $\mu_1, \mu_2, \cdots, \mu_q$ in $\sigma(T(t)) \cap \{\mu : |\mu| \geq e^{tb}\}$, they are all normal eigenvalues of $T(t)$ and at least one point, say, $\mu_1$, satisfies $|\mu_1| = e^{t\omega_s}$. Since the generalized eigenspace for $\mu_1$ is of finite dimension, Theorem 1.1 implies that there exist finite points in $P_\sigma(A)$ such that $e^{t\lambda} = \mu_1$. Of course, they are on the line $\Re\lambda = \omega_s > \omega_e \geq \beta_e$. Thus they are all normal eigenvalues of $A$: Assertion (i) holds. Similarly, for each $j$ there are finite normal eigenvalues of $A$ such that $e^{t\lambda} = \mu_j$. Conversely, suppose that $\lambda$ lies in $\sigma(A) \cap \{\lambda : \Re\lambda \geq b\}$. Since $b > \omega_e \geq \beta_e$, $\lambda$ is a normal eigenvalue and $e^{t\lambda} = \mu_j$ for some $1 \leq j \leq q$. Summarizing these results, we have the first result in Assertion (ii). Since the same result holds for any $b' \in (\omega_e, b)$, we have the second result in Assertion (ii).

Notice that, in general, the following formula follows from the similar argument:

$$\omega_s = \max\{\omega_e, \beta_n\}.$$

If $\lambda_0$ is a pole of $R(\lambda, A)$ of order $m$, then

$$N((\lambda_0 I - A)^k) = N((\lambda_0 I - A)^m), \quad R((\lambda_0 I - A)^k) = R((\lambda_0 I - A)^m), \qquad k \geq m,$$

$$X = N((\lambda_0 I - A)^m) \oplus R((\lambda_0 I - A)^m).$$

Namely, $\lambda_0$ has a finite index, and $m$ is its index, cf. [6] p.163. The components of this direct sum are non trivial, invariant, closed subspaces of the semigroup $T(t)$. In fact, the following formula holds.

**Theorem 2.4** *Let $T(t)$ be a $C_0$ semigroup on a Banach space $X$, and $A$ its infinitesimal generator. Suppose that $(A - \lambda I)^m x = 0$. Then*

$$T(t)x = e^{\lambda t} \sum_{k=0}^{m-1} \frac{t^k}{k!}(A - \lambda I)^k x.$$

*If we read this formula for the definition of $T(t)x$ for all $t \in (-\infty, \infty)$, then $T(s)T(t)x = T(s+t)x$ for all $s, t \in (-\infty, \infty)$. Namely, $T(t), t \in (-\infty, \infty)$, becomes a group on $N((A - \lambda I)^m)$.*

**Proof** This holds clearly for the case $m = 1$. Suppose that $(A - \lambda I)^2 x = 0$. Since $(A - \lambda I)x_1 = 0$ for $x_1 := (A - \lambda I)x$, it follows that $T(t)x_1 = e^{\lambda t}x_1$, or $T(t)Ax = \lambda T(t)x + e^{\lambda t}(A - \lambda I)x$. Since $T(t)Ax = T'(t)x$, the function $y(t) := T(t)x$ satisfies the equation

$$y'(t) = \lambda y(t) + e^{\lambda t}(A - \lambda I)x.$$

Solving this equation, we have that $T(t)x = e^{\lambda t}x + te^{\lambda t}(A - \lambda I)x$: the formula in the theorem is valid for $m = 2$. Similarly, the general case is shown by induction.

If $(A - \lambda I)^m x = 0$, then $(A - \lambda I)^k x = 0$ for $k \geq m$; hence we can write

$$T(t)x = e^{\lambda t} \sum_{k=0}^{\infty} \frac{t^k}{k!}(A - \lambda I)^k x.$$

It is easy to see that $T(s)T(t)x = T(s+t)x$ for $s, t \in (-\infty, \infty)$.

In general, Webb has given the following result, Proposition 4.15 [6].

**Theorem 2.5** *Suppose that $\Lambda = \{\lambda_j : j = 1, 2, \cdots, q\}$ be the set of all normal eigenvalues of $A$ such that $\Re\lambda_j \geq b > \omega_e$. Let $m_j$ be the index of $\lambda_j$, and set $M_j = N((\lambda_j I - A)^{m_j}), j = 1, 2, \cdots q, M_0 = \cap_{j=1}^q R((\lambda_j I - A)^{m_j})$. Then $X = M \oplus M_0$, where $M = M_1 \oplus M_2 \oplus \cdots \oplus M_q$. Let $P_j$ be the projection in*

*$X$ such that $P_jX = M_j, j = 0, 1, 2, \cdots q$, and $P = P_1 + P_2 + \cdots + P_q$. Then $T(t)P_jx = P_jT(t)x$ for all $x \in X, j = 0, 1, 2, \cdots, q$. Let $c$ be a constant such that $\max\{\Re\lambda : \lambda \in \sigma(A) \setminus \Lambda\} < c < b$. Then there exists a constant $K \geq 1$ such that $|T(t)P_0x| \leq Ke^{ct}|P_0x|$ for all $x \in X, t \geq 0$. The restriction of $A$ to $M$, denoted by $A_M$, is a bounded operator with the spectrum consisting of $\Lambda, T(t)Px = \exp(tA_M)Px$ for all $x \in X, t \geq 0$, where $\exp(tA_M)$ is regarded as an exponential function of the matrix $tA_M$, and there exists a constant $K \geq 1$ such that $|\exp(tA_M)Px| \leq Ke^{ct}|Px|$ for $x \in X$ and $t \leq 0$.*

# 3 The solution semigroup of FDE

Assuming that the solution operators of a functional differential equation make a $C_0$ semigroup, we will find conditions, on the equation and the phase space, by which we are able to compute the resolvent of the generator.

Suppose that the semigroup $T(t)$ is a solution semigroup of a linear autonomous functional differential equation. More precisely, from the following equation $T(t)$ is defined on a Banach space $\mathcal{B}$ consisting of certain functions, $\phi, \psi$, e.t.c., mapping $(-\infty, 0]$ into a Banach space $E$. Let $L : \mathcal{B} \to E$ be a linear, bounded or unbounded operator, where $D(L)$, the domain of $L$, may be a proper subspace of $\mathcal{B}$. We consider the equation

$$x'(t) = L(x_t), \tag{3.1}$$

where $x_t(\theta) = x(t+\theta), \theta \in (-\infty, 0]$. Suppose that for any $\phi \in \mathcal{B}$ there exists a unique solution $x(t, \phi)$ satisfying the equation in a certain sense for $t \in [0, \infty)$ and the initial condition $x_0 = \phi$. Then the solution operator $T(t)$ is defined by $T(t)\phi = x_t(\phi), t \geq 0, \phi \in \mathcal{B}$, where $(x_t(\phi))(\theta) = x(t + \theta, \phi),\ \theta \in (-\infty, 0]$.

The fundamental assumption is the following.

**(A0)** $T(t)$ is a $C_0$ semigroup on $\mathcal{B}$, and

$$[T(t)\phi](\theta) = [T(t + \theta)\phi](0) \text{ as long as } t + \theta \geq 0.$$

Then we can derive the following result from Theorem 2.4.

**Theorem 3.1** *Let $A$ be the infinitesimal generator of the solution semigroup $T(t)$ of Equation (3.1). If $\phi$ satisfies the equation $(A - \lambda I)^m\phi = 0$ for some*

$\lambda \in C, m \geq 1$, *then*

$$\phi(\theta) = e^{\lambda\theta} \sum_{k=0}^{m-1} \frac{\theta^k}{k!}[(A - \lambda I)^k \phi](0).$$

**Proof** Set $\phi_k = (A - \lambda I)^k \phi, k = 0, 1, \cdots$. Of course, $\phi_0 = \phi$, and $\phi_k = 0$ for $k \geq m$. Applying Property (A0), we have that

$$e^{\lambda t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \phi_k(\theta) = e^{\lambda(t+\theta)} \sum_{k=0}^{\infty} \frac{(t+\theta)^k}{k!} \phi_k(0).$$

as long as $t + \theta \geq 0$. From the binomial theorem, we have

$$\sum_{k=0}^{\infty} \frac{(t+\theta)^k}{k!} \phi_k(0) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{j=0}^{\infty} \frac{\theta^j}{j!} \phi_{k+j}(0).$$

Thus we have an exact expression of $\phi_k(\theta)$ :

$$\phi_k(\theta) = e^{\lambda\theta} \sum_{j=0}^{\infty} \frac{\theta^j}{j!} \phi_{k+j}(0).$$

The expression of $\phi(\theta)$ is the case $k = 0$.

For any $\lambda \in C$ and $x \in E$, define a function $\varepsilon_\lambda \otimes x : (-\infty, 0] \to E$ by $(\varepsilon_\lambda \otimes x)(\theta) = e^{\lambda\theta} x$, $\theta \in (-\infty, 0]$. Then the solution of $(\lambda I - A)\phi = 0$ is written as $\phi = \varepsilon_\lambda \otimes x$, where $x = \phi(0)$. If $N(\lambda I - A) \neq \{0\}$, it follows that $\varepsilon_\lambda \otimes x \in D(A)$ for some $x \neq 0$. To proceed further, we make an assumption on the generator $A$.

**(A1)** If $\phi \in D(A)$, then $\phi \in D(L)$ and $(A\phi)(0) = L(\phi)$.

For $\lambda \in C$ define

$$L_\lambda(x) = L(\varepsilon_\lambda \otimes x) \quad \text{and} \quad \Delta(\lambda)x = \lambda x - L_\lambda(x)$$

as long as the right-hand side has a meaning. Namely, $L_\lambda, \Delta(\lambda)$ are linear operators on $E$ with the domain $D(L_\lambda) = \{x \in E : \varepsilon_\lambda \otimes x \in D(L)\}$.

**Theorem 3.2** *Under the assumptions* (A0,1), $\lambda$ *lies in* $P_\sigma(A)$ *if and only if* $N(\Delta(\lambda)) \neq \{0\}$.

Up to this point we need no assumption on the phase space $\mathcal{B}$ whenever the solution operators make a $C_0$ semigroup. To solve the equation $(\lambda I - A)\phi = \psi$ in the space $\mathcal{B}$, we put the following assumptions which appear in the system of axioms by Hale and Kato.

**(B1)** There exists a constant $H$ such that $|\phi(0)| \leq H\|\phi\|$ for $\phi \in \mathcal{B}$, where $|\cdot|$ is a norm in $E$, and $\|\cdot\|$ is a norm in $\mathcal{B}$.

**(B2)** If $\{\phi^n\}$ is a Cauchy sequence in $\mathcal{B}$, and if the sequence $\{\phi^n(\theta)\}$ converges to a function $\phi(\theta)$ uniformly on every compact interval of $(-\infty, 0]$, then $\phi$ lies in $\mathcal{B}$ and $\lim_{n\to\infty} \|\phi^n - \phi\| = 0$.

**(B3)** The family of operators $S(t), t \geq 0$, on $\mathcal{B}$ defined as

$$[S(t)\phi](\theta) = \begin{cases} \phi(0) & \text{for } \theta \geq -t \\ \phi(t+\theta) & \text{for } \theta < -t \end{cases}$$

is a $C_0$ semigroup on $\mathcal{B}$.

Before proceeding further, we ovserve that, from (B2), the integrals in $\mathcal{B}$ and $E$ are commutative in the following sense:

**(C1)** If $u : [a, b] \to \mathcal{B}$ is a continuous function such that $u(t)(\theta)$ is continuous for $(t, \theta) \in [a, b] \times (-\infty, 0]$, then

$$\left[\int_a^b u(t)\, dt\right](\theta) = \int_a^b u(t)(\theta)\, dt \qquad \theta \in (-\infty, 0].$$

Under these assumptions we can solve the equation $(\lambda I - A)\phi = \psi$ as follows. Let $B$ be the infinitesimal generator of $S(t)$, and $\omega_0$ the growth bound of $S(t)$. Notice that $R(\lambda, B) := (\lambda I - B)^{-1}$ is well defined for $\Re\lambda > \omega_0$. Set $\omega_0^+ = \max\{\omega_0, 0\}$.

**Theorem 3.3** *If $(\lambda I - A)\phi = \psi$ and if $\Re\lambda > \omega_0^+$, then*

$$\phi = \varepsilon_\lambda \otimes \phi(0) + R(\lambda, B)\psi - \lambda^{-1}\varepsilon_\lambda \otimes \psi(0).$$

Define an operator $M_\lambda$ on $\mathcal{B}$ by $M_\lambda\psi = R(\lambda, B)\psi - \lambda^{-1}\varepsilon_\lambda \otimes \psi(0)$, $\psi \in \mathcal{B}$, where $\Re\lambda > \omega_0^+$. It is a bounded linear operator on $\mathcal{B}$, and the solution of $(\lambda I - A)\phi = \psi$ is written as $\phi = \varepsilon_\lambda \otimes \phi(0) + M_\lambda\psi$. To determine $\phi(0)$, we require that $\varepsilon_\lambda \otimes \phi(0) \in D(L)$ as long as $\phi \in D(L)$. Namely, we assume the following property for $L$.

**(A2)** If $\Re\lambda > \omega_0^+$, then $D(L_\lambda) = \{\phi(0) : \phi \in D(L)\}$.

**Theorem 3.4** *If* $\lambda \in \rho(A)$, *and if* $\Re\lambda > \omega_0^+$, *then* $\psi(0) + L(M_\lambda\psi) \in R(\Delta(\lambda))$ *for every* $\psi \in B$, *and*

$$R(\lambda, A)\psi = \varepsilon_\lambda \otimes \Delta(\lambda)^{-1}(\psi(0) + L(M_\lambda\psi)) + M_\lambda\psi.$$

Since $A = \lambda I - R(\lambda, A)^{-1}$, we obtain a representation of $A$ as follows.

**Theorem 3.5** *The function* $\phi$ *lies in* $D(A)$ *if and only if* $\phi \in D(L)$ *and* $\phi - \lambda^{-1}\varepsilon_\lambda \otimes L(\phi) \in D(B)$ *for some* $\Re\lambda > \omega_0^+$; *and, for such a* $\phi$,

$$A\phi = \varepsilon_\lambda \otimes L(\phi) + B(\phi - \lambda^{-1}\varepsilon_\lambda \otimes L(\phi)).$$

# 4 Appendix

We present the proofs of Theorems 1.1 and 2.2. They become rather simpler than the ones in [3], [6] in several points by the frequent employment of the following operator $B_\lambda(t), \lambda \in C, t > 0$, cf. [5]:

$$B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)x\, ds \qquad x \in X.$$

**Lemma 4.1** $B_\lambda(t)$ *is a bounded linear operator on* $X$ *having the following properties:*

$$(\lambda I - A)B_\lambda(t)x = (e^{t\lambda}I - T(t))x \quad x \in X,$$
$$B_\lambda(t)(\lambda I - A)x = (e^{t\lambda}I - T(t))x \quad x \in D(A).$$

**Proof of Theorem 1.1**

(i) If $\mu = e^{t\lambda} \in \rho(T(t))$, then $(\lambda I - A)^{-1}$ is given as

$$(\lambda I - A)^{-1} = (e^{t\lambda}I - T(t))^{-1}B_\lambda(t) = B_\lambda(t)(e^{t\lambda}I - T(t))^{-1}.$$

Hence, if $\lambda \in \sigma(A)$, then $e^{t\lambda} \in \sigma(T(t)), t > 0$.

(ii) Suppose that $\mu = e^{t\lambda} \in P_\sigma(T(t))$, and that $(e^{t\lambda}I - T(t))x = 0, x \neq 0$. Then the function $S(s) = e^{-s\lambda}T(s)x$ is a continuous function with $t$ period. Its Fourier series is (C-1) summable: $S(s) = (C-1)\sum e^{s2\pi in/t}J_nx$,where $J_nx$ is given as

$$J_nx = \frac{1}{t}\int_0^t e^{-2\pi inu/t}S(u)x\, du = \frac{1}{t}\int_0^t e^{-u\lambda_n}T(u)x\, du = \frac{1}{te^{t\lambda_n}}B_{\lambda_n}(t)x,$$

where $\lambda_n = \lambda + 2\pi i n/t$. Hence we have $T(s)x = (\text{C-1})\sum e^{s\lambda_n} J_n x$; in particular, $x = (\text{C-1})\sum J_n x$. Since $x \neq 0$, there is at least an $n$ such that $J_n x \neq 0$. From Lemma 4.1 it then follows that

$$(\lambda_n I - A)J_n x = (te^{t\lambda_n})^{-1}(e^{t\lambda_n} - T(t))x = (te^{t\lambda})^{-1}(e^{t\lambda} - T(t))x = 0.$$

Hence $\lambda_n \in P_\sigma(A)$, and $J_n x \in N(\lambda_n I - A)$.

Conversely, suppose that $(\lambda_n I - A)x = 0, x \neq 0$. From Lemma 4.1 we have that $x \in N(e^{t\lambda} I - T(t))$. Thus we have proved the correspondence between the point spectra and the one between the null spaces for $k = 1$.

We set $M_k = N((\mu I - T(t))^k), k = 1, 2 \cdots$, to prove the correspondence between the null spaces for $k \geq 2$ by induction. For example, we show the proof for $k = 2$. Suppose that $x \in M_2$. Then $(\mu I - T(t))x \in M_1$ ; hence from the result above $(\mu I - T(t))x = (\text{C-1})\sum_n J_n(\mu I - T(t))x$. Since $J_n T(t) = T(t) J_n$, it follows that $(\mu I - T(t))(x - (\text{C-1})\sum_n J_n x) = 0$, that is, $x_1 := x - (\text{C-1})\sum_n J_n x \in M_1$. Next,consider each $J_n x$. Since $J_n x \in M_2$ for $x \in M_2$, we again have that $x_{n1} := J_n x - (\text{C-1})\sum_m J_m J_n x \in M_1$. Finally we consider each $J_m J_n x$ for $x \in M_2$. Observe that, by Lemma 4.1, for every $r, s > 0, \kappa, \lambda \in C$ and $x \in X$,

$$\begin{aligned}&(\kappa I - A)(\lambda I - A)B_\kappa(r)B_\lambda(s)x \\ &\quad = (\kappa I - A)B_\kappa(r)(\lambda I - A)B_\lambda(s)x = (e^{r\kappa} I - T(r))(e^{s\lambda} I - T(s))x.\end{aligned}$$

This implies that, if $x \in M_2$, then $B_{\lambda_m}(t)B_{\lambda_n}(t)x \in N((\lambda_m I - A)(\lambda_n I - A))$ for every $m, n$. Since $B_{\lambda_m}(t)B_{\lambda_n}(t)x = t^2 e^{t(\lambda_m + \lambda_n)} J_m J_n x$, we know that $J_m J_n x \in N((\lambda_m I - A)(\lambda_n I - A))$. If $m = n$, then $J_m J_n x = J_n^2 x \in N((\lambda_n I - A)^2)$. If $m \neq n$, then it is easy see that $N((\lambda_m I - A)(\lambda_n I - A)) = N((\lambda_m I - A)) \oplus N((\lambda_n I - A))$. Consequently, it follows that $x_1, x_{n1}, J_m J_n x$ are all in the space $\cup_n N((\lambda_n I - A)^2)$. Therefore, $x$ lies in the minimal closed subspace generated by $\cup_n N((\lambda_n I - A)^2)$.

It is easy to see that the converse relation holds.

**Theorem 4.2** *If $\lambda_0 \in \sigma(A)$ and if $\zeta_0 := e^{t\lambda_0}$ is a pole of $R(\zeta, T(t))$, $t > 0$, then $\lambda_0$ is a pole of $R(\lambda, A)$.*

**Proof** Since $\zeta_0$ is an isolated point in $\sigma(T(t))$ as a pole of $R(\zeta, T(t))$, and since $e^{t\sigma(A)} \subset \sigma(T(t))$, there exists an $r > 0$ such that, if $0 < |\lambda - \lambda_0| < r$, then $\lambda \in \rho(A)$ and $e^{t\lambda} \in \rho(T(t))$. As is shown in the proof of Theorem 1.1, it follows that, if $0 < |\lambda - \lambda_0| < r$, then

$$(\lambda I - A)^{-1} = B_\lambda(t)(e^{t\lambda}I - T(t))^{-1} = (e^{t\lambda}I - T(t))^{-1}B_\lambda(t).$$

Let $k$ be the order of the pole $\zeta_0$, that is,

$$(\zeta I - T(t))^{-1} = (\zeta - \zeta_0)^{-k}P_{-k} + (\zeta - \zeta_0)^{-k+1}P_{-k+1} + \cdots$$

in some neighborhood of $\zeta_0$ with bounded linear operators $P_{-k}, P_{-k+1}, \cdots, P_{-k} \neq 0$. Thus in some neighborhood of $\lambda_0$ we have

$$(\lambda I - A)^{-1} = B_\lambda(t)\left[\left(e^{t\lambda} - e^{t\lambda_0}\right)^{-k} P_{-k} + \left(e^{t\lambda} - e^{t\lambda_0}\right)^{-k+1} P_{-k+1} + \cdots\right].$$

Taking a Taylor expansion of $e^{t\lambda}$ around the $\lambda_0$ in the $\lambda$ plane, we have that

$$e^{t\lambda} - e^{t\lambda_0} = (\lambda - \lambda_0)te^{t\lambda_0} + \frac{(\lambda - \lambda_0)^2t^2}{2!}e^{t\lambda_0} + \cdots.$$

Similarly, $B_\lambda(t)$ has also a Taylor expansion around $\lambda_0$ in the $\lambda$ plane. Thus in a neighborhood of $\lambda_0$ we have a Laurent expansion of $(\lambda I - A)^{-1}$ starting from the power $(\lambda - \lambda_0)^{-k}$ apparently:

$$(\lambda I - A)^{-1} = (\lambda - \lambda_0)^{-k}\left[t^{-k}e^{-kt\lambda_0}B_{\lambda_0}(t)P_{-k}\right] + \cdots.$$

If the terms with negative power of $\lambda - \lambda_0$ are all vanishing, then $(\lambda I - A)^{-1}$ becomes analytic at $\lambda_0$, and $\lambda_0 \in \rho(A)$; it is a contradiction. Hence, $(\lambda I - A)^{-1}$ has indeed a pole at $\lambda_0$ of order $\leq k$.

**Proof of Theorem 2.2** Suppose that $\lambda_0$ is a point in $\sigma(A)$, and that $\zeta_0 := e^{t\lambda_0}$ is a normal eigenvalue of $T(t)$. Then $T(t)$ has a finite dimensional generalized eigenspace for $\zeta_0$, and from Lemma 2.1 $\zeta_0$ is a pole of $T(t)$: Since $N((\lambda_0 I - A)^k) \subset N((\zeta_0 I - T(t))^k), k = 1, 2, \cdots$, it follows that $A$ has a finite dimensional generalized eigenspace for $\lambda_0$. From Theorem 4.2 and Lemma 2.1, $\lambda_0$ is a normal eigenvalue of $A$ : Theorem 2.2 holds.

The following result is deduced from Theorem 2.2 immediately.

**Theorem 4.3** *If $\lambda \in \sigma(A)$ and $\Re\lambda > \omega_e$, then $\lambda$ is a normal eigenvalue of $A$.*

In the rest of this paper we add an elementary proof of this theorem. They consist of several parts.

**Proposition 4.4** *If $\lambda_0 \in \sigma(A)$ and $\Re\lambda_0 > \omega_e$, then $\lambda_0$ lies $P_\sigma(A)$, it is an isolated point in $\sigma(A)$ and its generalized eigenspace is of finite dimension.*

**Proof** Set $\Re\lambda_0 = b_0, \Im\lambda_0 = c_0$. Since $|e^{t\lambda_0}| = e^{tb_0} > e^{t\omega_e}$ for every $t > 0$, we have $e^{t\lambda_0} \in N_\sigma(T(t)) \subset P_\sigma(T(t))$, which implies that

$$\lambda_0 + \frac{2\pi in}{t} \in P_\sigma(A) \tag{4.1}$$

for some integer $n$. Suppose that, for every $t > 0$, there exists an $n = n(t) \neq 0$ for which Condition (4.1) holds. Let $s$ be a fixed positive constant. Then $\mu(t) := \exp s(\lambda_0 + 2\pi in(t)t^{-1}) \in P_\sigma(T(s))$ for every $t > 0$. It is clear that $|\mu(t)| = e^{sb_0} > e^{s\omega_e} = r_e(T(s))$. Since normal eigenvalues are isolated, the set $P_\sigma(T(s)) \cap \{\zeta \in C : |\zeta| = e^{sb_0}\}$ contains only finite points $e^{s\lambda_1}, e^{s\lambda_2}, \cdots, e^{s\lambda_N}$. Hence it follows that, for every $t > 0$ there exists an $m, 1 \leq m \leq N$, such that $\mu(t) = e^{s\lambda_m}$. Set $\Im\lambda_m = c_m$. Then we have

$$s\left(c_0 + \frac{2\pi n}{t}\right) = sc_m + 2\pi\ell$$

for some integer $\ell$. Thus, for every $t > 0$ there are integers $\ell, m, n, n \neq 0$, such that

$$\frac{1}{t} = \frac{1}{n}\left[\frac{c_m - c_0}{2\pi} + \frac{\ell}{s}\right].$$

However the numbers apearing in the right-hand side are at most countable. This is a contradiction. Therefore, there exists a $t > 0$ such that Condition (4.1) holds only for $n = 0$, which implies $\lambda_0 \in P_\sigma(A)$.

Since $e^{t\lambda_0}$ is a normal eigenvalue of $T(t)$, it has a finite index with a generalized eigenspace of finite dimension. Therefore, from Theorem 1.1 $\lambda_0$ has also a finite index as an eigenvalue of $A$, and its generalized eigenspace is of finite dimension.

If $\lambda_0$ is an accumulation point of $\sigma(A)$, then $e^{t\lambda_0}, t > 0$, is an accumulation point of $\sigma(T(t))$ by Theorem 1.1. This is a contradiction since $e^{t\lambda_0}$ is a normal eigenvalue of $T(t)$. Hence, $\lambda_0$ is an isolated point of $\sigma(A)$.

To show $\lambda_0$ is a normal eigenvalue of $A$, it remains to prove that $\lambda_0 I - A$ has a closed range. We devide the proof into several steps, in which Lemma 4.1 plays an importnat role.

**Lemma 4.5** *Let $T : X \to X$ be a bounded linear operator, and $\{x_n\}$ a bounded sequence in $X$ such that the sequence $\{(\lambda I - T)x_n\}$ converges. If $|\lambda| > r_e(T)$, then $\{x_n\}$ has a convergent subsequence.*

**Proof** Set $B = \{x_n : n = 1, 2, \cdots\}$, and $D = \{y_n : n = 1, 2 \cdots\}$, where $y_n = (\lambda I - T)x_n$. By induction we see that, for $k, n = 1, 2, \cdots$,

$$\lambda^k x_n = T^k x_n + T^{k-1} y_n + \lambda T^{k-2} y_n + \cdots + \lambda^{k-2} T y_n + \lambda^{k-1} y_n,$$

or

$$\alpha(\lambda^k B) \leq \alpha(T^k B) + \alpha(T^{k-1} D) + \alpha(\lambda T^{k-1} D) + \cdots + \alpha(\lambda^{k-2} T D) + \alpha(\lambda^{k-1} D).$$

Since $D$ is relatively compact, $T^i D$ is also relatively compact for every $i = 1, 2, \cdots$ : that is, $\alpha(T^i D) = 0$. It follows that

$$|\lambda|^k \alpha(B) = \alpha(\lambda^k B) \leq \alpha(T^k B) \leq \alpha(T^k)\alpha(B).$$

Suppose that $\alpha(B) > 0$. Then we have $|\lambda|^k \leq \alpha(T^k)$, or $|\lambda| \leq \alpha(T^k)^{1/k}, k = 1, 2, \cdots$. This contradicts the assumption on $\lambda$; hence, $\alpha(B) = 0$, or $B$ is relatively compact and the proof is complete.

**Lemma 4.6** *Let $A$ be the infinitesimal generator of $T(t)$. Suppose that $\{x_n\}$ is a bounded sequence in $D(A)$ such that the sequence $\{(\lambda I - A)x_n\}$ converges to $y_0$. If $\Re\lambda > \omega_e$, then $\{x_n\}$ has a convergent subsequence with a limit point $x_0 \in D(A)$, and $(\lambda I - A)x_0 = y_0$.*

**Proof** Set $y_n = (\lambda I - A)x_n$. Since $x_n \in D(A)$, we have

$$B_\lambda(t) y_n = B_\lambda(t)(\lambda I - A)x_n = (e^{t\lambda} I - T(t))x_n.$$

Since $B_\lambda(t)$ is continuous, the sequence $\{B_\lambda(t) y_n\}$ converges to $B_\lambda(t) y_0$. From Lemma 4.5 the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n(i)}\}$ with a limit point $x_0$. Since $\{y_{n(i)}\}$ converges to $y_0$, and since $(\lambda I - A)$ is a closed linear operator, it follows that $x_0 \in D(A)$ and $(\lambda I - A)x_0 = y_0$.

**Lemma 4.7** *Let $S$ be a closed linear operator on $X$. Suppose that a bounded sequence $\{x_n\}$ in $D(S)$ has a convergent subsequence whenever $\{Sx_n\}$ is a convergent sequence. Then $R(S)$ is closed.*

**Proof** Suppose that $\{y_n\}$ is a sequence in $R(S)$ convergent to a point $y_0$. Take $x_n \in D(S)$ such that $Sx_n = y_n$. If $\{x_n\}$ is itself a bounded sequence, from the assumption in Lemma it follows immediately that $y_0 \in R(S)$. In general case, set $N = \ker S, a_n = \inf\{|x_n - w| : w \in N\}$. Then there exist a $w_n \in N$ such that $a_n \leq |x_n - w_n| \leq a_n(1 + 1/n), n = 1, 2, \cdots$. Suppose that $\{a_n\}$ is unbounded. Then there exists a subsequence, denoted by $\{a_n\}$ again, such that $\lim_{n\to\infty} a_n = \infty$. Set $z_n = x_n - w_n, u_n = z_n/|z_n|$. Then $|u_n| = 1, Su_n = y_n/|x_n - w_n| \to 0$. From the assumption in Lemma, by taking a subsequence if necessary, we can assume that $u_n \to u_0 \in D(S)$ as $n \to \infty$, and $Su_0 = 0$. Since $u_0 \in N$, it follows that

$$\begin{aligned} a_n &\leq |x_n - (w_n + |x_n - w_n|u_0)| \\ &= ||x_n - w_n|(u_n - u_0)| = |x_n - w_n||u_n - u_0| \leq a_n(1 + 1/n)|u_n - u_0|, \end{aligned}$$

that is, $(1 + 1/n)^{-1} \leq |u_n - u_0|$. Since $|u_n - u_0| \to 0$, this is a contradiction. Therefore $\{x_n - w_n\}$ is a bounded sequence and $y_n = S(x_n - w_n)$; hence $y_0 \in R(S)$.

As a consequence of Lemmas 4.5,4.6 and 4.7, we obtain the following, desired result.

**Proposition 4.8** *If $\Re\lambda > \omega_e$, the ranges $R(e^{t\lambda}I - T(t)), t > 0$, and $R(\lambda I - A)$ are closed.*

# References


[1] Browder F.E., On the spectral theory of elliptic differential operators I, Math. Annalen, 142(1961), 22-130.

[2] Hale J.K. and Lunel S.M.V., Introduction to Functional Differential Equations, Springer-Verlag, 1993.

[3] Hille E. and Phillips R.S., Functional Analysis and Semi-groups, A.M.S., 1957.

[4] Kunisch K. and Schappacher W., Necessary conditions for partial differential equations with delay to generate $C_0$-semigroups, J. Differential Equations, 50(1983), 49-79.

[5] Pazy A., Semigroups of Linear Operators and Applications to Partial Differential Equations, Appl. Math. Sci. 44, Springer-Verlag, 1983.

[6] Webb G.F., Theory of Nonlinear Age-dependent Population Dynamics, Pure and Appl. Math. 89, Dekker, 1985.