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Modular varieties associated to quaternion unitary groups of degree 2

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We treat quaternion unitary groups of degree 2, which were studied by Arakawa in [1]. The purpose of this note is to report that modular varieties associated to those unitary groups with fully large levels are of general type.

1 Modular varieties

Let $\mathbf{B}$ be an indefinite division quaternion algebra over the rational number field $\mathbf{Q}$, and $^{-}: \mathbf{B} \to \mathbf{B} \quad (a \mapsto \bar{a})$ the canonical involution of $\mathbf{B}$. Since $\mathbf{B}_\infty = \mathbf{B} \otimes_{\mathbf{Q}} \mathbf{R} \cong M_2(\mathbf{R})$, we identify $\mathbf{B}_\infty$ and $M_2(\mathbf{R})$ by fixing an isomorphism. Let G be the $\mathbf{B}$-unitary group of degree 2. We put

$$G_{\mathbf{Q}} := \{g \in M_2(\mathbf{B}) \mid g \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\},$$

where ${}^t\bar{g} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $G_{\mathbf{Q}}$ is $\mathbf{Q}$-rational points of G. Let N be a natural number, and $\mathfrak{O}$ a maximal order of $\mathbf{B}$. Set

$$\Gamma(N) := \{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{Q}} \mid a-1, b, c, d-1 \in N\mathfrak{O} \}.$$

Let

$$\mathfrak{S}_2 := \{ Z \in M_2(\mathbf{C}) \mid {}^tZ = Z, \quad \mathrm{Im}(Z) > 0 \}$$

be the Siegel upper half plane of degree 2, and set

$$\mathfrak{H} := \{ Z \in M_2(\mathbf{C}) \mid ZJ^{-1} \in \mathfrak{S}_2 \}, \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For the group $G_{\mathbf{R}}$ of $\mathbf{R}$-rational points of G, we have

$$qG_{\mathbf{R}}q^{-1} = Sp_2(\mathbf{R}) := \{ g \in M_4(\mathbf{R}) \mid g \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} {}^tg = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \},$$

where $I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $q := \begin{pmatrix} I & 0 \\ 0 & J \end{pmatrix}$. The group $G_{\mathbf{R}}$ acts on $\mathfrak{H}$ by $g\langle Z\rangle = (aZ+b)(cZ+d)^{-1}$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbf{R}}, Z \in \mathfrak{H}$. Though pairs $(G_{\mathbf{R}}, \mathfrak{H})$ and $(Sp_2(\mathbf{R}), \mathfrak{S}_2)$ are the same essentially, we here consider the pair $(G_{\mathbf{R}}, \mathfrak{H})$.

Since the $\mathbf{Q}$-rank of $G_{\mathbf{Q}}$ is 1, $\Gamma(N)$ has only point cusps. Let $Y(N)$ be a toroidal compactification of $\Gamma(N) \setminus \mathfrak{H}$.

2 Modular forms

In this section, we remember modular forms with respect to $\Gamma(N)$. See Arakawa [1] and Hashimoto [2] for details. For any positive integer k, let $M_k(\Gamma(N))$ be the $\mathbf{C}$-vector space of modular forms of weight k with respect to $\Gamma(N)$. Namely, $M_k(\Gamma(N))$ is the space of holomorphic functions $f(Z)$ on $\mathfrak{H}$ satisfying

$$f(g\langle Z\rangle) = \det(cZ+d)^k f(Z) \quad \text{for all} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N).$$

An element $f(Z)$ in $M_k(\Gamma(N))$ is called a cusp form if $|f(Z)\det(\mathrm{Im}(Z))^{k/2}|$ is bounded on $\mathfrak{H}$. We denote by $S_k(\Gamma(N))$ the $\mathbf{C}$-vector space of cusp forms of weight k with respect to $\Gamma(N)$.

Let $\mathbf{B}^-$ be the set of pure quaternions in $\mathbf{B}$. We put

$$L := \mathfrak{O} \cap \mathbf{B}^-, \quad L^* := \{\, y \in \mathbf{B}^- \mid \mathrm{tr}(xy) \in \mathbf{Z} \quad \text{for all} \quad x \in L\}.$$

Then Arakawa showed the following Proposition and Theorem.

Proposition(Arakawa). *Each modular form $f(Z) \in M_k(\Gamma(N))$ has the following Fourier expansion*

$$f(Z) = a(0) + \sum_{\substack{t\in L^* \\ tJ>0}} a(t)e[\frac{1}{N}\mathrm{tr}(tZ)],$$

where $e[\,\cdot\,] = \exp(2\pi i\,\cdot\,)$. In particular, $f(Z) \in S_k(\Gamma(N))$ is equivalent to $a(0) = 0$.

Let $\mathfrak{O}^\times$ be the group of units in $\mathfrak{O}$. For any element $\epsilon \in \mathfrak{O}^\times$ and $x \in L$, we have $\epsilon x\bar{\epsilon} \in L$. The lattice L^* also has this property. The Fourier coefficients $a(t)$ in Proposition satisfy $a(\epsilon t\bar{\epsilon}) = (N\epsilon)^k a(t)$ for $\epsilon \in \mathfrak{O}^\times$.

Theorem(Arakawa). *Assume $k \geq 5, N \geq 3$. Then we have*

$$\begin{aligned} \dim_{\mathbf{C}} S_k(\Gamma(N)) &= 2^{-7}3^{-3}5^{-1}[\Gamma : \Gamma(N)](k-1)(k-\frac{3}{2})(k-2) \prod_{p|d(\mathbf{B})} (p-1)(p^2+1) \\ &+ 2^{-4}3^{-1}[\Gamma : \Gamma(N)]N^{-3} \prod_{p|d(\mathbf{B})} (p-1), \end{aligned}$$

where $d(\mathbf{B})$ is the discriminant of $\mathbf{B}$.

3 The result

Let $\begin{pmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{pmatrix}$ be the coordinates of $\mathfrak{H}$. Set $\omega := dz_1 \wedge dz_2 \wedge dz_3$. Arakawa showed that $\Gamma(N)$ is torsion-free if $N \geq 3$. We here consider the case $N \geq 3$. For any cusp form $f \in S_{3k}(\Gamma(N))$, we would like to know the extendability of a $\Gamma(N)$-invariant form $f\omega^{\otimes k}$ over the resolution of a point cusp.

We set

$$L_+^* := \{\ y \in L^* \mid yJ > 0\ \}, \quad L_+ := \{\ x \in L \mid J^{-1} > 0\ \}.$$

Put

$$\Lambda_m(\infty) := \{\ y \in L_+^* \mid \mathrm{tr}(yx) \leq m \quad \text{for some } x \in L_+\ \}, \qquad d_m(\infty) := \Lambda_m(\infty)/\sim,$$

where we write $y_1 \sim y_2$ when $y_1 = \epsilon y_2 \bar{\epsilon}$ holds for some norm 1 unit ϵ in $\mathfrak{O}^\times$. This number $d_m(\infty)$ shows us the extendability of $f\omega^{\otimes m}$.

Put $N(L_+) := \min\{\ N(x) \mid x \in L_+\ \}$. The following is the main result:

Theorem. *Assume $N \geq 3$. If*

$$3\sqrt{2}N^3[\mathfrak{O}^\times : (1+N\mathfrak{O})^\times]N(L_+)^{3/2}d(\mathbf{B}) \prod_{p|d(\mathbf{B})} (p^2+1) > 2^7 5\pi,$$

then $Y(N)$ is a modular variety of general type.

Sketch of proof: The number of cusps for $\Gamma(N)$ is $[\Gamma(1):\Gamma(N)]/[\mathfrak{O}^\times : (1+N\mathfrak{O})^\times]N^3$. Hence we get

$$P_m(Y(N)) \geq \dim S_{3m}(\Gamma(N)) - \frac{[\Gamma(1):\Gamma(N)]}{[\mathfrak{O}^\times : (1+N\mathfrak{O})^\times]N^3} \cdot d_m(\infty).$$

If $\mathrm{tr}(yx) \leq m$, then we have $N(y)N(x) \leq m^2$. Now we evaluate the cardinality of

$$\{\ y \in L_+^* \mid N(y) \leq \frac{m^2}{N(L_+)}\ \}/\sim.$$

Here $\sim$ is defined as above. Then we can show that the cardinality is not bigger than $\frac{\pi}{3\sqrt{2}d(\mathbf{B})}\prod_{p|d(\mathbf{B})}(p-1)m^3 + \epsilon m^3$ for fully big m and fully small ϵ. By using this evaluation and the dimensional formula of Arakawa, we can prove the above theorem.

References

1. T. Arakawa: The dimension of the space of cusp forms on the Siegel upper half plane of degree two related to a quaternion unitary groups, J. Math. Soc. Japan **33**, (1981), 125-145.
2. K. Hashimoto: The dimension of the spaces of cusp forms on Siegel upper half plane of degree two II, Math. Ann. **266**, (1984), 539-559.
3. F. Knöller: Beispiele dreidimensionaler Hilbertscher Modulmannigfaltigkeiten von allgemeinem Typ, Manuscripta Math. **37**, (1982), 135-161.

4. S. Tsuyumine: On the Kodaira dimensions of Hilbert modular varieties, Invent. Math. **80**, (1985), 269-281.

5. H. Yamaguchi: The parabolic contribution to the dimension of the space of cusp forms on Siegel space of degree two, Preprint, (1976).

6. T. Yamazaki: On Siegel modular forms of degree two, Amer. J. Math. **98**, (1976), 39-53.