数理解析研究所講究録
942巻 1996年 172-182

Deformation of Degenerate Curves and Automorphic Functions on the Teichmüller Space

Takashi Ichikawa (市川尚志)
Department of Mathematics
Faculty of Science and Engineering
Saga University, Saga 840, Japan
e-mail address: ichikawa@math.ms.saga-u.ac.jp

Introduction

The aim of this note is to report a recent result on arithmetic deformation of degenerate (algebraic) curves and its application to studying automorphic functions on the Teichmüller space (which we call *Teichmüller modular forms* for short). Arithmetic deformation theory is to give a higher genus version of Tate's elliptic curve over $\mathbf{Z}[[q]]$, i.e. to construct a deformation of a given degenerate curve as a stable curve over a certain "primitive" ring. The construction is done by extending Mumford's uniformization theory [M], so this theory is also called *arithmetic uniformization theory.* In this note, we only treat a deformation of a degenerate curve obtained by identifying points of the projective line in pairs. And inspired by the result of Ihara-Nakamura [I-N] giving an arithmetic deformation of a maximally degenerate curve with smooth components, we could obtain an arithmetic deformation of any degenerate curve, which will be useful to studying Teichmüller modular forms *of higher level.* The results in §2, 3 are mainly extensions of results in [I2, 3] to Teichmüller modular forms over a ring, especially over $\mathbf{Z}$, which can be obtained using results in §1.

1 Arithmetic uniformization theory

Classical Schottky uniformization theory for Riemann surfaces had been constructed by Schottky [S] about 1 century ago. Modern Schottky type uniformization theory for p-adic algebraic curves was constructed by Mumford [M] about 20 years ago. Combining these theories, we obtain arithmetic unifomization theory. This is needed to study automorphic functions on the Teichmüller space.

We review Schottky and Mumford uniformization theory. Let K be $\mathbf{C}$ or a nonarchimedean valuation field with multiplicative valuation $|\ \ |$. Let $PGL_2(K)$ (: the projective linear group of degree 2 over K) act on the projective line $\mathbf{P}^1(K) = K \cup \{\infty\}$ over K by the Möbius transformation:

$$PGL_2(K) \times \mathbf{P}^1(K) \ni \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) \mapsto \frac{az+b}{cz+d} \in \mathbf{P}^1(K).$$

Let

$$\begin{cases} D_{\pm 1}, ..., D_{\pm g} & : \text{ open domains in } \mathbf{P}^1(\mathbf{C}) \text{ bounded by Jordan curves} \\ & \quad \text{open disks in } \mathbf{P}^1(K) \text{ if } K \text{ is nonarchimedean} \\ \gamma_1, ..., \gamma_g & \in PGL_2(K) \end{cases}$$

$$\text{such that } \ \gamma_k(\mathbf{P}^1(K) - D_{-k}) = \overline{D_k} \ (\text{: the closure of } D_k).$$

Then $\gamma_k(\partial D_{-k}) = \partial D_k$ (: the boundary of D_k).
Let

$$\Gamma \stackrel{\text{def}}{=} \langle \gamma_1, ..., \gamma_g \rangle : \text{called a Schottky group over } K \text{ of rank } g.$$

Then Γ is known to be a free group of rank g consisting of hyperbolic elements except 1. Hence each γ_k has 2 fixed points $\alpha_{\pm k} \in D_{\pm k}$ and the multiplier $\beta_k \in K^\times$ with $|\beta_k| < 1$, i.e.

$$\frac{\gamma_k(z) - \alpha_k}{\gamma_k(z) - \alpha_{-k}} = \beta_k \frac{z - \alpha_k}{z - \alpha_{-k}} \quad (z \in \mathbf{P}^1(K))$$

which is equivalent to

$$\gamma_k = \begin{pmatrix} \alpha_k & \alpha_{-k} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \beta_k \end{pmatrix} \begin{pmatrix} \alpha_k & \alpha_{-k} \\ 1 & 1 \end{pmatrix}^{-1} \mod(K^\times).$$

We call $(\alpha_{\pm k}, \beta_k)_{1 \le k \le g}$ the *Koebe coordinates* of a Schottky group Γ with free generators $\gamma_1, ..., \gamma_g$.
Let C_Γ be the K-analytic space obtained from $\mathbf{P}^1(K) - \bigcup_{k=1}^g (D_k \cup D_{-k})$ identifying ∂D_k and ∂D_{-k} via γ_k:

$$C_\Gamma \stackrel{\text{def}}{=} \left(\mathbf{P}^1(K) - \bigcup_{k=1}^g (D_k \cup D_{-k}) \right) \Big/ \partial D_k \cup_{\gamma_k} \partial D_{-k}.$$

Then C_Γ is Schottky uniformized by Γ, i.e. it becomes the quotient space by Γ of $\mathbf{P}^1(K) - \{\text{limit points of } \Gamma\}$. When $K = \mathbf{C}$, C_Γ is evidently a Riemann surface of genus g, and when K is nonarchimedean, it is shown by Mumford [M] that C_Γ

has a natural structure of a proper and smooth algebraic curve over K which is called a *Mumford curve.*

Arithmetic uniformization theory is to construct a family of stable curves over

$$A \stackrel{\text{def}}{=} \mathbf{Z}\left[x_{\pm 1}, \dots, x_{\pm g}, \prod_{i \neq j} \frac{1}{x_i - x_j}\right][[y_1, \dots, y_g]]$$

($x_{\pm 1}, \dots, x_{\pm g}, y_1, \dots, y_g$: variables) which is a "universalization" of Schottky and Mumford uniformized curves, i.e. becomes those curves under specializing $x_{\pm k}, y_k$ to the associated Koebe coordinates. For this purpose, we extend Mumford's formal analytic construction [M] of stable curves for Schottky groups over complete local rings to the group generated by

$$\begin{pmatrix} x_k & x_{-k} \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & y_k \end{pmatrix}\begin{pmatrix} x_k & x_{-k} \\ 1 & 1 \end{pmatrix}^{-1} \text{ modulo center}$$

over the nonlocal ring A, and we have:

__Theorem 1.__ *There exists a stable curve C over A satisfying*

(1) *C is smooth over $A[1/y]$ $(y := y_1 \cdots y_g)$,*

(2) *$C|_{y_1 = \cdots = y_g = 0}$ becomes the degenerate curve over*

$$A_0 = \mathbf{Z}\left[x_{\pm 1}, \dots, x_{\pm g}, \prod_{i \neq j} \frac{1}{x_i - x_j}\right]$$

obtained by identifying x_k and x_{-k} $(k = 1, \dots, g)$ in $\mathbf{P}^1_{A_0}$,

(3) *for K as above and the Koebe coordinates $(\alpha_{\pm k}, \beta_k)_{1 \leq k \leq g}$ of a Schottky group $\Gamma = \langle \gamma_1, \dots, \gamma_g \rangle$ over K with sufficiently small $|\beta_k|$,*

$$C|_{x_{\pm k} = \alpha_{\pm k}, y_k = \beta_k} = C_\Gamma.$$

2 Teichmüller modular forms

We define *Teichmüller modular forms* (denoted by TMFs for short) as global sections of line bundles on the moduli space of algebraic curves, which are seen to be, over $\mathbf{C}$:

automorphic functions on the Teichmüller space, i.e. holomorphic functions on the Teichmüller space with automorphy condition under the action of the mapping class group. This naming is an analogy of

$$\begin{aligned} & \text{Siegel modular forms (denoted by SMFs for short)} \\ = \ & \text{automorphic functions on the Siegel upper half space} \end{aligned}$$

Besides the analogy of the namings, TMFs and SMFs are connected by the period map and the Torelli map. But there are TMFs not induced from SMFs. These TMFs appear in string theory, conformal field theory and soliton theory. We will study TMFs by constructing their arithmetic expansion. This is an analogy of the theory on arithmetic Fourier expansion for Siegel modular forms constructed by Shimura (over fields of characteristic 0) and by Chai and Faltings [F-C] (over rings). And these 2 expansion theories are connected by the so called "universal periods":

$$\begin{array}{ccc} \{\text{SMFs}\} & \xrightarrow{\text{Fourier expansion}} & \{\text{power series}\} \\ \downarrow & & \text{universal} \downarrow \text{periods} \\ \{\text{TMFs}\} & \xrightarrow{\text{arithmetic expansion}} & \{\text{power series}\} \end{array}$$

In what follows, we fix a natural number g. This means the genus of considering Riemann surfaces. Moreover, we assume:

$$g \geq 3.$$

Because if the genus is equal to 1 or 2, then the moduli of Riemann surfaces is an affine space, so some cusp condition is needed for the definition of TMFs. Let

$$\begin{aligned} T_g \ &: \ \text{the Teichmüller space of degree } g \\ &\stackrel{\text{def}}{=} \ \text{the moduli space of Riemann surfaces } C \text{ of genus } g \\ &\quad\ \text{with canonical generators of } \pi_1(C) \text{ modulo the conjugation.} \end{aligned}$$

Then by Teichmüller's theory, T_g is known to be diffeomorphic to $\mathbf{R}^{6g-6}$, and T_g has the natural complex structure corresponding to the deformation of Riemann surfaces. In particular, T_g becomes a simply connected complex manifold of dimension $3g-3$.
Let

$$\begin{aligned} \Gamma_g \ &: \ \text{the mapping class group of degree } g \\ &\stackrel{\text{def}}{=} \ \text{the group of orientation } (H^2(\pi_1(C), \mathbf{Z}) \tilde{\rightarrow} \mathbf{Z}) \text{ preserving} \\ &\quad\ \text{automorphisms of } \pi_1(C) \text{ modulo inner automorphisms.} \end{aligned}$$

Then Γ_g acts on canonical generators of $\pi_1(C)$, so it acts on T_g properly discontinuously. The quotient space T_g/Γ_g becomes the moduli space (as a complex orbifold) of Riemann surfaces of genus g. We denote this by M_g:

$$M_g \stackrel{\text{def}}{=} T_g/\Gamma_g : \text{the moduli space (orbifold) of Riemann surfaces of genus } g.$$

By Riemann's period relation, the period matrix of each element of T_g belongs to the Siegel upper half space H_g of degree g. Sending Riemann surfaces to their Jacobian varieties with canonical polarization, we have the Torelli map τ from M_g to the moduli space A_g of principally polarized g-dimensional abelian varieties over $\mathbf{C}$, and A_g is the quotient of H_g by the integral symplectic group $Sp_{2g}(\mathbf{Z})$ of degree g. Hence we have the following commutative diagram:

$$\begin{array}{ccc} T_g & \xrightarrow{p:\ \text{period map}} & H_g \\ \downarrow\ /\Gamma_g & & \downarrow\ /Sp_{2g}(\mathbf{Z}) \\ M_g & \xrightarrow{\tau:\ \text{Torelli map}} & A_g. \end{array}$$

Since every $\gamma \in \Gamma_g$ induces an automorphism $\overline{\gamma}$ of $\pi_1(C)/[\pi_1,\pi_1] \cong H_1(C,\mathbf{Z})$ and preserves the natural intersection form on $H^1(C,\mathbf{Z})$:

$$\Gamma_g \ni \gamma \mapsto \overline{\gamma} \text{ on } H_1(C,\mathbf{Z}) = \begin{pmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{pmatrix} \in Sp_{2g}(\mathbf{Z}).$$

Let

$$\begin{aligned} \mu \quad &: \quad \text{the Hodge line bundle on } A_g \\ &\stackrel{\text{def}}{=} \quad \text{the line bundle on } A_g \text{ corresponding to the automorphic factor} \\ & \qquad \det(C_G Z + D_G) \left(G = \begin{pmatrix} A_G & B_G \\ C_G & D_G \end{pmatrix} \in Sp_{2g}(\mathbf{Z}),\ Z \in H_g \right), \\ \lambda \quad &\stackrel{\text{def}}{=} \quad \tau^*(\mu) : \text{the line bundle on } M_g \text{ corresponding to } \det(C_\gamma p(t) + D_\gamma). \end{aligned}$$

Then

$$\begin{aligned} \Gamma(A_g, \mu^{\otimes h}) &= \{\varphi : H_g \to \mathbf{C} : \text{hol.} \mid \varphi(G(Z)) = \det(C_G Z + D_G)^h \varphi(Z)\}, \\ \Gamma(M_g, \lambda^{\otimes h}) &= \{f : T_g \to \mathbf{C} : \text{hol.} \mid f(\gamma(t)) = \det(C_\gamma p(t) + D_\gamma)^h f(t)\}. \end{aligned}$$

We note that it is shown by Harer [H] that the line bundles on M_g (modulo isomorphism) form a free cyclic group generated by λ. As is shown in [D-M], there exist canonical models of M_g and A_g as moduli stacks (i.e. algebraically defined orbifolds) over $\mathbf{Z}$. We denote these by

$\mathcal{M}_g$: the moduli stack $/\mathbf{Z}$ of (proper, smooth) algebraic curves of genus g,

$\mathcal{A}_g$: the moduli stack $/\mathbf{Z}$ of prin. polar. g-dim. abelian varieties.

Since λ and μ are defined over $\mathcal{M}_g$ and $\mathcal{A}_g$, for any $h \in \mathbf{Z}$ and $\mathbf{Z}$-algebra R, we can define

$$\begin{array}{lcl} T_{g,h}(R) & \stackrel{\text{def}}{=} & \Gamma(\mathcal{M}_g, \lambda^{\otimes h} \otimes R) : \text{TMFs } /R \text{ of degree } g \text{ and weight } h \\ \uparrow \tau^* & & \\ S_{g,h}(R) & \stackrel{\text{def}}{=} & \Gamma(\mathcal{A}_g, \mu^{\otimes h} \otimes R) : \text{SMFs } /R \text{ of degree } g \text{ and weight } h. \end{array}$$

Then as in the Siegel modular case, using the Satake-type compactification of $\mathcal{M}_g$ over fields and the principle of GAGA, we have

$$T_{g,h}(R) = \begin{cases} R & (\text{if } h = 0) \\ \{0\} & (\text{if } h < 0), \end{cases}$$

and

$$\Gamma(M_g, \lambda^{\otimes h}) = T_{g,h}(\mathbf{C}) = T_{g,h}(\mathbf{Z}) \otimes \mathbf{C}.$$

Let K be $\mathbf{C}$ or a nonarchimedean complete valuation field, and let

$$\begin{array}{lcl} \tilde{S}_{g/K} & : & \text{the space of Schottky groups over } K \text{ with free } g \text{ generators} \\ & & \stackrel{\text{open subset}}{\hookrightarrow} (\mathbf{P}^1(K) \times \mathbf{P}^1(K) \times K^\times)^g \text{ by the Koebe coordinates,} \\ S_{g/K} & \stackrel{\text{def}}{=} & \tilde{S}_{g/K}/(\text{conjugation by } PGL_2(K)) \\ & : & \text{called the Schottky space over } K \text{ of degree } g. \end{array}$$

By $\Gamma \mapsto C_\Gamma$, $S_{g/K}$ becomes a fiber space over the K-analytic orbispace $M_{g/K}$ associated with $\mathcal{M}_g \otimes K$, and in the case where $K = \mathbf{C}$, we have the following commutative diagram ($S_g \stackrel{\text{def}}{=} S_{g/\mathbf{C}}$) :

$$\begin{array}{ccc} T_g & \stackrel{p:\text{ period map}}{\longrightarrow} & H_g \\ \downarrow & & \downarrow \exp(2\pi\sqrt{-1}\ \cdot\ ) \\ S_g & \longrightarrow & H_g/\mathbf{Z}^{g(g+1)/2} \\ \downarrow & & \downarrow \\ M_g & \stackrel{\tau:\text{ Torelli map}}{\longrightarrow} & A_g. \end{array}$$

Then using results in [S], it is shown in [I1] that the middle rightarrow for sufficiently small $|\beta_k|$ is expressed by the *universal periods* $p_{ij} \in A$ $(i, j = 1, ..., g)$ which are seen to be the multiplicative periods of the Jacobian variety of $\mathcal{C}$ (cf. [M-D]). We note that p_{ij} are computable, for example,

$$p_{ij} = c_{ij} \left(1 + \sum_{|k| \neq i,j} \frac{(x_i - x_{-i})(x_j - x_{-j})(x_k - x_{-k})^2}{(x_i - x_k)(x_{-i} - x_k)(x_j - x_{-k})(x_{-j} - x_{-k})} y_{|k|} + \cdots \right),$$

where

$$c_{ij} = \begin{cases} \frac{(x_i-x_j)(x_{-i}-x_{-j})}{(x_i-x_{-j})(x_{-i}-x_j)} & (\text{if } i \neq j) \\ y_i & (\text{if } i = j). \end{cases}$$

By the fibration $S_{g/K} \to M_{g/K}$, any TMF f over K gives a K-analytic function on $S_{g/K}$, so we obtain this expansion $\kappa(f)$ by the Koebe coordinates $(\alpha_{\pm k}, \beta_k)_k$:

$$f \in T_{g,h}(K) \implies \exists \kappa(f) \in A[1/y]\hat{\otimes} K \text{ such that } \kappa(f)|_{x_{\pm k}=\alpha_{\pm k}, y_k=\beta_k} = f.$$

Then using Theorem 1 and the irreducibility of $\mathcal{M}_g$ proved by Deligne-Mumford [D-M], we have:

Theorem 2. *For any* $\mathbf{Z}$*-algebra* R*, the evaluation of TMFs on the universal curve* C *in Theorem* 1 *gives a functorial* R*-linear homomorphism*

$$\kappa_R : T_{g,h}(R) \longrightarrow A[1/y]\hat{\otimes} R$$

which satisfies the following:

(1) $\kappa_K =$ *the above* κ,

(2) κ_R *is injective,*

(3) $f \in T_{g,h}(R)$, $\kappa_R(f) \in A\hat{\otimes}R'$ $(R' \subset R) \Rightarrow f \in T_{g,h}(R')$,

(4) *the following diagram is commutative:*

$$\begin{array}{ccccc} S_{g,h}(R) & \xrightarrow{F:\text{ Fourier expansion}} & R[q_{ij}, \prod_{i\neq j} 1/q_{ij}][[q_{11}, \dots, q_{gg}]] & \ni & q_{ij} \\ \downarrow \tau^* & & \downarrow & & \downarrow \\ T_{g,h}(R) & \xrightarrow{\kappa_R} & A[1/y]\hat{\otimes} R & \ni & p_{ij}. \end{array}$$

By (2) and (4), we have a characterization of Siegel modular forms vanishing on the Jacobian locus (cf. [I1]):

Corollary (Schottky Problem). *For any* $\varphi \in S_{g,h}(R)$ *with Fourier expansion* $F(\varphi) = \sum_{T=(t_{ij})} a_T \prod_{i,j} q_{ij}{}^{t_{ij}}$ $(a_T \in R)$,

$$\tau^*(\varphi) = 0 \iff F(\varphi)|_{q_{ij}=p_{ij}} = 0 \text{ in } A\hat{\otimes}R$$

$$\implies \begin{cases} \displaystyle\sum_{t_{ii}=s_i} a_T \prod_{i<j} \left(\frac{(x_i - x_j)(x_{-i} - x_{-j})}{(x_i - x_{-j})(x_{-i} - x_j)} \right)^{2t_{ij}} = 0 \\ \text{for any } s_1, \dots, s_g \geq 0 \text{ with } \displaystyle\sum_{i=1}^{g} s_i = \min\{Tr(T) | a_T \neq 0\}. \end{cases}$$

Brinkmann-Gerritzen [B-G] shows $(\Rightarrow)$ and checks this for Schottky's relation in

genus 4.
Studying the behavior of TMFs at the boundary of the moduli space, we can show that any element of $T_{g,h}(\mathbf{Z})$ is (uniquely) extended to a form on the Deligne-Mumford compactification [D-M] of $\mathcal{M}_g$, and hence we have:

Theorem 3. *Each $T_{g,h}(\mathbf{Z})$ is a free $\mathbf{Z}$-module of finite rank, and the ring $T_g^*(\mathbf{Z}) = \bigoplus_{h\geq 0} T_{g,h}(\mathbf{Z})$ of Teichmüller modular forms over $\mathbf{Z}$ of degree g is a finitely generated $\mathbf{Z}$-algebra.*

3 Examples of TMFs

Let

$$\theta_g\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}(Z) \stackrel{\text{def}}{=} \sum_{\boldsymbol{n}\in\mathbf{Z}^g} \exp\left[\pi\sqrt{-1}(\boldsymbol{n}+\boldsymbol{a})Z^t(\boldsymbol{n}+\boldsymbol{a}) + 2\pi\sqrt{-1}(\boldsymbol{n}+\boldsymbol{a})^t\boldsymbol{b}\right]$$
$$(\boldsymbol{a},\boldsymbol{b} \in (\tfrac{1}{2}\mathbf{Z})^g/\mathbf{Z}^g,\ Z \in H_g)$$

be the theta constant with characteristic $(\boldsymbol{a},\boldsymbol{b})$, and let

$$\theta_g(Z) \stackrel{\text{def}}{=} \prod_{4\boldsymbol{a}^t\boldsymbol{b}:\ \text{even}} \theta_g\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}(Z)$$

be the product of theta constants with even characteristic. It is well known (cf. [Ig]) that $\theta_g \in S_{g,2h(g)}(\mathbf{C})$ ($h(g) \stackrel{\text{def}}{=} 2^{g-3}(2^g+1)$), and θ_g is defined over $\mathbf{Z}$ because it has rational Fourier coefficients. Tsuyumine [T2, 3] shows that θ_g has a root as a TMF over $\mathbf{C}$. Calculating the expansion of θ_g by Koebe coordinates, we can determine the number N_g such that θ_g/N_g has a root as a TMF over $\mathbf{Z}$ which is primitive, i.e. is not congruent to 0 modulo any prime:

Theorem 4. *Put*

$$N_g = \begin{cases} -2^{28} & (g=3) \\ 2^{2^{g-1}(2^g-1)} & (g\geq 4). \end{cases}$$

Then $\tau^*(\theta_g)/N_g$ has a root as a primitive Teichmüller modular form over $\mathbf{Z}$ of degree g and weight $h(g)$.

When $g=3$, using $f_3 = \sqrt{\tau^*(\theta_3)/N_3} \in T_{3,9}(\mathbf{Z})$, we can reduce the structure of $T_3^*(\mathbf{Z})$ to that of the ring $S_3^*(\mathbf{Z}) = \bigoplus_{h\geq 0} S_{3,h}(\mathbf{Z})$ of Siegel modular forms over $\mathbf{Z}$ of degree 3 (generators of $S_3^*(\mathbf{Q})$ are obtained by Tsuyumine [T1]). First note

$$\begin{array}{l} S_3^*(\mathbf{Z}) \stackrel{\text{in fact}}{=} \bigoplus_{h:\text{even}} S_{3,h}(\mathbf{Z}) \\ \downarrow \tau^* : \text{injection} \\ T_3^*(\mathbf{Z}), \end{array}$$

however, there exist TMFs of degree 3 with <u>odd</u> weight, for example f_3, because the Torelli map has degree 2 as a morphism between orbifolds. From a result of Igusa [Ig] and Theorem 2, we have:

Theorem 5. *The ring $T_3^*(\mathbf{Z})$ is generated by f_3 over $S_3^*(\mathbf{Z})$.*

When $g = 4$, τ^* induces an injective homomorphism

$$S_4^*(\mathbf{Z})/\langle \text{Schottky's relation} \rangle \hookrightarrow T_4^*(\mathbf{Z}),$$

but the author does not know whether this map is surjective or not.

In what follows, we consider a kind of TMF on the moduli space of marked Riemann surfaces. The partition function in conformal field theory with abelian gage is known to be (cf. [K-N-T-Y], [Kr]):

$$\begin{cases} \text{regarded as a TMF on the moduli of marked Riemann surfaces} \\ \text{expressed by the theta functions of Riemann surfaces} \\ \text{satisfying the system (hierarchy) of soliton (KP) equations} \end{cases}$$

Using arithmetic uniformization theory for algebraic curves, we can give the p-adic version of this result. Here we will construct p-adic solutions of soliton equations.

First we treat the genus 1 case. Let L be the lattice generated by π and $\pi\tau$, where τ is a complex number with positive imaginary part, and let

$$\wp(z) \stackrel{\text{def}}{=} \frac{1}{z^2} + \sum_{u \in L - \{0\}} \left(\frac{1}{(z-u)^2} - \frac{1}{u^2} \right) \quad (z \in \mathbf{C})$$

be the Weierstrass $\wp$-function for L. Then by the theory of elliptic functions, one can see that

$$u(x,t) = \wp(x + 3ct + d) + c \quad (c, d: \text{ constants})$$

satisfies the KdV equation:

$$\text{(KdV)} \qquad \frac{\partial u}{\partial t} - 3u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3} = 0$$

(and $u(x,t)$ becomes the 1-soliton solution under $\mathrm{Im}(\tau) \to +\infty$). By the rationality of $\zeta(2m)/\pi^{2m}$ (m : positive integers), $\wp(z)$ can be regarded as a Laurent power series of z and $q \stackrel{\text{def}}{=} \exp(2\pi\sqrt{-1}\tau)$ with rational coefficients. So we have a *universal solution* $u(x,t)$ of (KdV), and for any p-adic number a with $|a|_p < 1$, $u(x,t)|_{q=a}$ is a formal solution of (KdV) with coefficients in p-adic numbers.

In the genus ≥ 2 case, it is shown by Krichever [Kr] that the theta function of any Riemann surface induces a quasi-periodic solution of the KP equation:

$$\text{(KP)} \qquad \frac{3}{4}\frac{\partial^2 u}{\partial s^2} - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} - 3u\frac{\partial u}{\partial x} - \frac{1}{4}\frac{\partial^3 u}{\partial x^3}\right) = 0$$

(and that of the KP hierarchy more generally). Hence using arithmetic uniformization theory, we obtain a *universal power series* for solutions of KP from Riemann surfaces with square roots of canonical bundles. By specializing this universal solution to the Koebe coordinates of Schottky groups over p-adic fields, we have (cf. [I4]):

Theorem 6. *The p-adic theta function of any algebraic curve with splitting reduction over a p-adic field induces a solution of the KP hierachy.*

Finally we will mention "analytic curves of infinite genus". In string theory and soliton theory, it is necessary to consider Riemann surfaces of infinite genus and their theta functions. We can construct a theory on analytic curves (Riemann surfaces and Mumford curves) of infinite genus which gives p-adic solutions of the KP hierarchy as in the finite genus case.

References

[B-G] B. Brinkmann and L. Gerritzen, The lowest term of the Schottky modular form, Math. Ann. **292** (1992), 329-335

[D-M] P. Deligne and D. Mumford, The irreducibility of the space of curves of given genus, Publ. Math. I.H.E.S. **36** (1969), 75-109

[F-C] G. Faltings and C.L. Chai, Degeneration of abelian varieties, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Bd. 22, Springer-Verlag, 1990

[H] J. Harer, The second homology group of the mapping class group of an orientable surface, Invent. Math. **72** (1983), 221-239

[I1] T. Ichikawa, The universal periods of curves and the Schottky problem. Compos. Math. **85** (1993) 1-8

[I2] T. Ichikawa, On Teichmüller modular forms. Math. Ann. **299** (1994) 731-740

[I3] T. Ichikawa, Teichmüller modular forms of degree 3. Amer. J. Math. **117** (1995) 1057-1061

[I4] T. Ichikawa, P-adic theta functions and solutions of the KP hierarchy. to appear in Comm. Math. Phys.

[Ig] J. Igusa, Modular forms and projective invariants, Amer. J. Math. **89** (1967), 817-855

[I-N] Y. Ihara and H. Nakamura, On deformation of maximally degenerate stable marked curves and Oda's problem, preprint (1995)

[K-N-T-Y] N. Kawamoto, Y. Namikawa, A. Tsuchiya and Y. Yamada, Geometric realization of conformal field theory, Comm. Math. Phys. **116** (1988), 247-308

[Kr] I.M. Krichever, Methods of algebraic geometry in the theory of nonlinear equations, Russian Math. Surveys **32** (1977), 185-214

[M-D] Y. Manin and V. Drinfeld, Periods of p-adic Schottky groups, J. Reine Angew. Math. **262/263** (1973), 239-247

[M] D.Mumford, An analytic construction of degenerating curves over complete local rings, Compos. Math. **24** (1972), 129-174

[S] F. Schottky, Über eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Arguments univerändert bleibt, J. Reine Angew. Math. **101** (1887), 227-272

[T1] S. Tsuyumine, On Siegel modular forms of degree three, Amer. J. Math. **108** (1986), 755-862

[T2] S. Tsuyumine, Factorial property of a ring of automorphic forms, Trans. Amer. Math. Soc. **296** (1986), 111-123

[T3] S. Tsuyumine, Thetanullwerte on a moduli space of curves and hyperelliptic loci, Math. Z. **207** (1991), 539-568