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Practical PTAS for Maximum Induced-Subgraph Problems on $K_{3,3}$-free or K_5-free Graphs

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Abstract. We show that for an integer $k \geq 2$ and an n-vertex graph G without a $K_{3,3}$ (resp., K_5) minor, we can compute k induced subgraphs of G with treewidth $\leq 3k-4$ (resp., $\leq 6k-7$) in $O(kn)$ (resp., $O(kn+n^2)$) time such that each vertex of G appears in exactly $k-1$ of these subgraphs. This leads to *practical* polynomial-time approximation schemes for many maximum induced-subgraph problems on graphs without a $K_{3,3}$ or K_5 minor.

1 Introduction

Let π be a property on graphs. π is *hereditary* if, whenever a graph G satisfies π, every induced subgraph of G also satisfies π. Suppose π is a hereditary property. The *maximum induced subgraph problem* associated with π (MISP(π)) is the following: Given a graph $G=(V,E)$, find a maximum subset U of V that induces a subgraph satisfying π. Yannakakis showed that many natural MISP(π)'s are NP-hard even if the input graph is restricted to a planar graph [13]. Thus, it is of interest to design efficient approximation algorithms for these MISP(π)'s.

An approximation algorithm A for an maximization problem Π achieves a *performance ratio* of ρ if for every instance I of Π, the ratio of the optimal value for I to the solution value returned by A is at most ρ. A *polynomial-time approximation scheme* (PTAS) for problem Π is an approximation algorithm which given an instance I of Π and an $\epsilon > 0$, returns a solution s within time polynomial in the size of I such that the ratio of the optimal value for I to the value of s is at most $(1+\epsilon)$. Much work has been devoted to designing PTASs for MISP(π)'s restricted to certain special instances [1, 5, 11]. Lipton and Tarjan were the first who proved that many MISP(π)'s restricted to *planar* instances have PTASs [11]. Unfortunately, their schemes are known to be nonpractical [6]. That is, to achieve a reasonable performance ratio (e.g., 2), the number of vertices in the input graph and/or the running time of the schemes has to be enormous ($\approx 2^{2^{400}}$). Later, Baker gave practical PTASs for the same problems [5]. By extending Lipton & Tarjan's approach, Alon et al. [1] showed that many MISP(π)'s restricted to graphs without an excluded minor have polynomial-time approximation schemes. Like Lipton and Tarjan's schemes, Alon et al.'s schemes have the shortage of being *very* nonpractical.

Since Alon et al.'s schemes are very nonpractical, it is natural to ask whether practical PTASs exist for MISP(π)'s restricted to graphs without an excluded minor. In this paper, we give an affirmative answer to this question when the minor is $K_{3,3}$ or K_5. Since neither a $K_{3,3}$ minor nor a K_5 minor can exist in a planar graph, our result extends Baker's result above. The basic idea behind our practical PTASs is to decompose a graph without a $K_{3,3}$ or K_5 minor into subgraphs of bounded treewidth. More precisely, we show that for an integer $k \geq 2$ and an n-vertex graph G without a $K_{3,3}$ (resp., K_5) minor, we can compute k induced subgraphs of G with treewidth $\leq 3k-4$ (resp., $\leq 6k-7$) in $O(kn)$ (resp., $O(kn+n^2)$) time such that each vertex of G appears in exactly $k-1$ of these subgraphs. Since many MISP(π)'s restricted to graphs of bounded treewidth are solvable optimally in linear time by dynamic programming [4], we obtain practical PTASs for these MISP(π)'s immediately. Moreover, our schemes have the advantage of being easy to parallelize while Alon et al.'s schemes do not. Our results heavily rely on the nice structures of graphs without a $K_{3,3}$ or K_5 minor that were developed in [2, 7, 10].

2 Preliminaries

Throughout this paper, a graph is always connected. Unless stated explicitly, a graph is always simple, i.e., has neither multiple edges nor self-loops. Let $G = (V, E)$ be a graph. For convenience, we allow $V = \emptyset$. If $V = \emptyset$, then we call G an *empty* graph. We sometimes write $V(G)$ instead of V and $E(G)$ instead of E. The *neighborhood* of a vertex v in G is the set of vertices in G adjacent to v. For $U \subseteq V$, the *subgraph of G induced by U* is the graph (U, F) with $F = \{\{u,v\} \in E : u, v \in U\}$ and is denoted by $G[U]$. When $U \subseteq V$, we sometimes write $G - U$ instead of $G[V - U]$.

A *contraction* of an edge $\{u,v\}$ in G is made by identifying u and v with a new vertex whose neighborhood is the union of the neighborhoods of u and v (resulting multiple edges and self-loops are deleted). A *contraction* of G is a graph obtained from G by a sequence of edge contractions. A graph H is a *minor* of G if H is the contraction of a subgraph of G. G is *H-free* if G has no minor isomorphic to H. In this paper, we deal with $K_{3,3}$-free graphs and K_5-free graphs. Recall that a planar graph must be both $K_{3,3}$-free and K_5-free by Kuratowski's Theorem.

A *tree-decomposition* of G is a pair $(\{X_i : i \in I\}, T = (I, F))$, where $\{X_i : i \in I\}$ is a family of subsets of V and T is a tree such that the following hold:

(a) $\cup_{i \in I} X_i = V$.

(b) For every edge $\{v, w\} \in E$, there is a subset X_i, $i \in I$ with $v \in X_i$ and $w \in X_i$.

(c) For all $i, j, k \in I$, if j lies on the path from i to k, then $X_i \cap X_k \subseteq X_j$.

The *treewidth* of a tree-decomposition $(\{X_i : i \in I\}, T)$ is $\max\{|X_i| - 1 : i \in I\}$. The *treewidth* of G, denoted by $\mathrm{tw}(G)$, is the minimum treewidth of a tree-decomposition of G, taken over all possible tree-decompositions of G. The treewidth of an empty graph is defined to be 0.

Lemma 1. Let $G = (V, E)$ be a graph, and R_1 and R_2 be two subsets of V such that (i) $R_1 \cap R_2 = \emptyset$ or $G[R_1 \cap R_2]$ is a clique and (ii) there is no $\{u_1, u_2\} \in E$ with $u_1 \in R_1 - R_2$ and $u_2 \in R_2 - R_1$. Then, $\mathrm{tw}(G[R_1 \cup R_2]) \leq \max\{\mathrm{tw}(G[R_1], \mathrm{tw}(G[R_2])\}$.

Proof. We can assume that $G[R_1 \cap R_2]$ is a clique because the lemma trivially holds when $R_1 \cap R_2 = \emptyset$. Let $(\{X_i : i \in I\}, T_1)$ be a tree-decomposition of $G[R_1]$ with treewidth $\mathrm{tw}(G[R_1])$, and $(\{Y_j : j \in J\}, T_2)$ be a tree-decomposition of $G[R_2]$ with treewidth $\mathrm{tw}(G[R_2])$. W.l.o.g., we may assume that $I \cap J = \emptyset$. Since $G[R_1 \cap R_2]$ is a clique in both $G[R_1]$ and $G[R_2]$, there are k and l such that $R_1 \cap R_2 \subseteq X_k$ and $R_1 \cap R_2 \subseteq Y_l$. Let T be the tree obtained from T_1 and T_2 by adding a new edge $\{k, l\}$. Then, it is easy to verify that $(\{X_i : i \in I\} \cup \{Y_j : j \in J\}, T)$ is a tree-decomposition of $G[R_1 \cup R_2]$ and has treewidth $\max\{\mathrm{tw}(G[R_1]), \mathrm{tw}(G[R_2])\}$. ∎

A set $S \subseteq V$ is a *cutset* if $G - S$ is disconnected. A cutset S is a *k-cut* if $|S| = k$. A k-cut is *strong* if $G - S$ has at least three connected components. A graph with at least k vertices is *k-connected* if it has no $(k-1)$-cut. A *biconnected component* of G is a maximal 2-connected subgraph of G.

Let C be a cutset of G, and G_1, ..., G_p be the connected components of $G - C$. For $1 \le i \le p$, let $G_i \cup K(C)$ be the graph obtained from $G[V(G_i) \cup C]$ by adding an edge between every pair of non-adjacent vertices in C. The graphs $G_1 \cup K(C)$, ..., $G_p \cup K(C)$ are called the *augmented components* induced by C. Clearly, if G is k-connected and C is a k-cut of G, then all the augmented components induced by C are also k-connected.

It is well known that the biconnected components of a graph are unique. Let $\mathcal{C}^1$ be the set of all 1-cuts of G, and $\mathcal{B}$ be the set of all biconnected components of G. Consider the bipartite graph $H = (\mathcal{C}^1 \cup \mathcal{B}, F)$, where $F = \{\{C, B\} : C \in \mathcal{C}^1,\ B \in \mathcal{B}, \text{ and } C \subseteq V(B)\}$. It is known that H is a tree. Suppose that $\mathcal{B} = \{B_1, ..., B_q\}$. Let $I = \{1, ..., q\}$. Root the tree H at B_1 and define $\mathcal{T}^1(G)$ to be the tree whose vertex set is I and edge set is $\{\{i, i'\} : B_i$ is the grandparent of $B_{i'}$ in the rooted tree $H\}$. (Note that $\mathcal{T}^1(G)$ is undirected.) The following fact is easy to prove.

Fact 1 $(\{V(B_i) : i \in I\}, \mathcal{T}^1(G))$ is a tree-decomposition of G and can be computed from G in $O(|V|)$ time.

Suppose that G is 2-connected. Further suppose that G contains a 2-cut. Replacing G by the augmented components induced by a 2-cut is called *splitting* G. Suppose G is split, the augmented components are split, and so on, until no more splits are possible. The graphs constructed in this way are 3-connected and the set of the graphs are called a *2-decomposition* of G. Each element of a 2-decomposition of G is called a *split component* of G. It is possible for G to have two or more 2-decompositions. A split component of G must be either a triangle or a 3-connected graph with at least 4 vertices. Let $\mathcal{D}$ be a 2-decomposition of G. We use $\mathcal{C}^2(\mathcal{D})$ to denote the set of the 2-cuts used to split G into the split components in $\mathcal{D}$. Consider the bipartite graph $H = (\mathcal{C}^2(\mathcal{D}) \cup \mathcal{D}, F)$, where $F = \{\{C, D\} : C \in \mathcal{C}^2(\mathcal{D}),\ D \in \mathcal{D}, \text{ and } C \subseteq V(D)\}$. It is known that H is a tree [12]. Suppose that $\mathcal{D} = \{D_1, ..., D_q\}$. Let $I = \{1, ..., q\}$. Root the tree H at D_1 and define $\mathcal{T}^2(G, \mathcal{D})$ to be the tree whose vertex set is I and edge set is $\{\{i, i'\} : D_i$ is the grandparent of $D_{i'}$ in the rooted tree $H\}$. (Note that $\mathcal{T}^2(G, \mathcal{D})$ is undirected.) Construct a supergraph $G^2(\mathcal{D})$ of G as follows: For each $\{u, v\} \in \mathcal{C}^2(\mathcal{D})$ with $\{u, v\} \notin E$, add the edge $\{u, v\}$ to G. Then, we have the following fact:

Fact 2 $(\{V(D_i) : i \in I\}, \mathcal{T}^2(G,\mathcal{D}))$ is a tree-decomposition of $G^2(\mathcal{D})$.

Proof. It is well known that every edge of G is contained in some split component in $\mathcal{D}$ and that if some vertex u of G is contained in two split components D_i and D_j in $\mathcal{D}$, then u is contained in every split component on the path between D_i and D_j in the tree H [12]. From this, it is easy to see the fact. ∎

3 A technical lemma

Let S be a set. For an integer $k \geq 2$, a *k-cover* of S is a list of k subsets of S such that each element of S is contained in exactly $k-1$ subsets in the list.

Lemma 2. Let $G = (V, E)$ be a graph. Let k and b be two integers with $k \geq 2$, and τ be a property on k-covers of subsets of V. Suppose that G has a tree-decomposition $(\{X_j : j \in I\}, T)$ and T has a rooted version such that the following three conditions are satisfied:

(1) For every $j' \in I$ and every child j of j' in T, $G[X_{j'} \cap X_j]$ is a clique.

(2) For the root $r \in I$ of T, we can compute a k-cover $\langle R_1, ..., R_k\rangle$ of X_r in $f(k, |X_r|)$ time such that

(2a) for every $1 \leq l \leq k$, $\mathrm{tw}(G[R_l]) \leq b$ and

(2b) for every child j'' of r in T, $\langle R_1 \cap X_{j''}, ..., R_k \cap X_{j''}\rangle$ is a k-cover of $X_r \cap X_{j''}$ satisfying τ.

(3) For every $j' \in I$ and every child j of j' in T and every k-cover $\langle Y_1, ..., Y_k\rangle$ of $X_{j'} \cap X_j$ satisfying τ, we can compute a k-cover $\langle Z_1, ..., Z_k\rangle$ of X_j in $f(k, |X_j|)$ time such that

(3a) for every $1 \leq l \leq k$, $Y_l = Z_l \cap X_{j'}$,

(3b) for every $1 \leq l \leq k$, $\mathrm{tw}(G[Z_l]) \leq b$, and

(3c) for every child j'' of j, $\langle Z_1 \cap X_{j''}, ..., Z_k \cap X_{j''}\rangle$ is a k-cover of $X_j \cap X_{j''}$ satisfying τ.

Then, we can compute a k-cover $\langle V_1, ..., V_k\rangle$ of V in $O(\sum_{j \in I} f(k, |X_j|))$ time such that for each $1 \leq l \leq k$, $\mathrm{tw}(G[V_l]) \leq b$ and $V_l \cap X_r = R_l$.

Proof. Consider the following algorithm for computing $\langle V_1, ..., V_k\rangle$:

Algorithm 1

1. Set V_1, ..., V_k to be the empty set.

2. While traversing T (starting at r) in a breadth-first manner, perform the following steps:

2.1. If the current vertex j is r, then compute a k-cover $\langle R_1, ..., R_k\rangle$ of X_r satisfying the two conditions (2a) and (2b) above, and further add the vertices in each R_l, $1 \leq l \leq k$, to V_l.

2.2. If the current vertex j is not r, then find the parent j' of j in T, set $\langle Y_1, ..., Y_k\rangle = \langle V_1 \cap (X_{j'} \cap X_j), ..., V_k \cap (X_{j'} \cap X_j)\rangle$, compute a k-cover $\langle Z_1, ..., Z_k\rangle$ of X_j satisfying the conditions (3a), (3b), and (3c) above, and add the vertices in each Z_l, $1 \leq l \leq k$, to V_l.

3. Output $\langle V_1, ..., V_k\rangle$.

Next, we prove that the output $\langle V_1, ..., V_k\rangle$ of Algorithm 1 satisfies that $\mathrm{tw}(G[V_l]) \le b$ and $V_l \cap X_r = R_l$ for each $1 \le l \le k$. First note that the while-loop in Algorithm 1 is executed $|I|$ times. W.l.o.g., we may assume that $I = \{1, ..., |I|\}$ and that $j+1$ is traversed by Algorithm 1 right after j for each $1 \le j \le |I|-1$. Then, $r = 1$. For each $1 \le j \le |I|$ and each $1 \le l \le k$, let V_l^j be the content of the variable V_l right after the jth iteration of the while-loop. We claim that for each $1 \le j \le |I|$, $\langle V_1^j, ..., V_k^j\rangle$ is a k-cover of $\cup_{1\le i\le j} X_i$ satisfying the following three conditions:

(C1) $\mathrm{tw}(G[V_l^j]) \le b$ and $V_l^j \cap X_1 = R_l$ for each $1 \le l \le k$.

(C2) For each son j'' of j in T, $\langle V_1^j \cap (X_j \cap X_{j''}), ..., V_k^j \cap (X_j \cap X_{j''})\rangle$ is a k-cover of $X_j \cap X_{j''}$ satisfying τ.

(C3) For each $1 \le i \le j$ and each child i' of i in T, $\langle V_1^j \cap (X_i \cap X_{i'}), ..., V_k^j \cap (X_i \cap X_{i'})\rangle = \langle V_1^i \cap (X_i \cap X_{i'}), ..., V_k^i \cap (X_i \cap X_{i'})\rangle$.

The lemma follows from the claim immediately. We prove the claim by induction on j. In case $j = 1$, the claim clearly holds. Let j be some integer with $2 \le j \le |I|$ and assume that the claim holds for all integers i with $i \le j-1$. Let j' be the parent of j in T, and let $\langle Y_1, ..., Y_k\rangle = \langle V_1^{j-1} \cap (X_{j'} \cap X_j), ..., V_k^{j-1} \cap (X_{j'} \cap X_j)\rangle$. Then, since $j' \le j-1$, we have $\langle Y_1, ..., Y_k\rangle = \langle V_1^{j'} \cap (X_{j'} \cap X_j), ..., V_k^{j'} \cap (X_{j'} \cap X_j)\rangle$ by (C3) in the inductive hypothesis. Combining this with (C2) in the inductive hypothesis, we have that $\langle Y_1, ..., Y_k\rangle$ is a k-cover of $X_{j'} \cap X_j$ satisfying τ. Thus, in the jth execution of step 2.2, we can compute a k-cover $\langle Z_1, ..., Z_k\rangle$ of X_j satisfying the conditions (3a), (3b), and (3c) above.

Firstly, we prove that $\langle V_1^j, ..., V_k^j\rangle$ is a k-cover of $\cup_{1\le i\le j} X_i$. To see this, first observe that $\langle Z_1, ..., Z_k\rangle$ is a k-cover of X_j and that $\langle V_1^j, ..., V_k^j\rangle = \langle V_1^{j-1} \cup Z_1, ..., V_k^{j-1} \cup Z_k\rangle$. Moreover, by the inductive hypothesis, $\langle V_1^{j-1}, ..., V_k^{j-1}\rangle$ is a k-cover of $\cup_{1\le i\le j-1} X_i$. Thus, each $v \in \cup_{1\le i\le j} X_i - ((\cup_{1\le i\le j-1} X_i) \cap X_j)$ appears in exactly $k-1$ sets in $\langle V_1^j, ..., V_k^j\rangle$. It remains to consider the vertices in $(\cup_{1\le i\le j-1} X_i) \cap X_j$. Since the path from j to each i, $1 \le i \le j-1$, in T must pass j', we have $(\cup_{1\le i\le j-1} X_i) \cap X_j = X_{j'} \cap X_j$ by the definition of tree-decompositions. Fix a vertex $v \in X_{j'} \cap X_j$. By the inductive hypothesis, v appears in exactly $k-1$ sets in $\langle V_1^{j-1}, ..., V_k^{j-1}\rangle$. Also, v appears in exactly $k-1$ sets in $\langle Z_1, ..., Z_k\rangle$. Moreover, for each $1 \le l \le k$, $v \in V_l^{j-1}$ if and only if $v \in Z_l$ by the condition (3a) above. Thus, v appears in exactly $k-1$ sets in $\langle V_1^j, ..., V_k^j\rangle$.

Secondly, we prove that for each $1 \le l \le k$, $\mathrm{tw}(G[V_l^j]) \le b$. Fix an integer l with $1 \le l \le k$. It suffices to prove that $\mathrm{tw}(G[V_l^j]) \le b$. This is done by applying Lemma 1. Let us be more precise. Since $(\cup_{1\le i\le j-1} X_i) \cap X_j = X_{j'} \cap X_j$, we have $V_l^{j-1} \cap Z_l \subseteq X_{j'} \cap X_j$. On the other hand, $G[X_{j'} \cap X_j]$ is a clique. Thus, $G[V_l^{j-1} \cap Z_l]$ is also a clique. Let $v_1 \in V_l^{j-1} - Z_l$ and $v_2 \in Z_l - V_l^{j-1}$. We want to show that $\{v_1, v_2\} \notin E$. Assume, on the contrary, that $\{v_1, v_2\} \in E$. Then, since the path from j to each i, $1 \le i \le j-1$, in T must pass j', we have that $v_1 \in X_{j'} \cap X_j$ or $v_2 \in X_{j'} \cap X_j$ by the definition of tree-decompositions. If $v_1 \in X_{j'} \cap X_j$, then $v_1 \in (V_l^{j-1} \cap (X_{j'} \cap X_j)) - (Z_l \cap X_{j'})$; otherwise, $v_2 \in (Z_l \cap X_{j'}) - (V_l^{j-1} \cap (X_{j'} \cap X_j))$. However, this contradicts that $V_l^{j-1} \cap (X_{j'} \cap X_j) = Y_l = Z_l \cap X_{j'}$. Therefore, $\{v_1, v_2\} \notin E$. Recall that

$G[V_l^{j-1} \cap Z_l]$ is a clique. Hence, if we set $R_1 = V_l^{j-1}$ and $R_2 = Z_l$, then R_1 and R_2 satisfy the conditions in Lemma 1. This implies that $\mathrm{tw}(G[V_l^j]) \leq \max\{\mathrm{tw}(G[V_l^{j-1}]), \mathrm{tw}(G[Z_l])\}$. By the inductive hypothesis, $\mathrm{tw}(G[V_l^{j-1}]) \leq b$. By the condition (3b) above, $\mathrm{tw}(G[Z_l]) \leq b$. Thus, we have $\mathrm{tw}(G[V_l^j]) \leq b$ by Lemma 1.

Thirdly, we prove that for each $1 \leq l \leq k$, $V_l^j \cap X_1 = R_l$. Fix an integer l with $1 \leq l \leq k$. By the inductive hypothesis, $V_l^{j-1} \cap X_1 = R_l$. Thus, to prove that $V_l^j \cap X_1 = R_l$, it suffices to prove that $Z_l \cap X_1 \subseteq V_l^{j-1} \cap X_1$. Fix a vertex $v \in Z_l \cap X_1$. Since the path from j to the root 1 in T must pass j', we have $v \in X_{j'}$ by the definition of tree-decompositions. Thus, $v \in Z_l \cap X_{j'} \cap X_1$. This together with the condition (3a) implies that $v \in Y_l \cap X_1$. Recall that $Y_l = V_l^{j-1} \cap (X_{j'} \cap X_j)$. Therefore, $v \in V_l^{j-1} \cap X_1$.

Fourthly, we prove that $\langle V_1^j, ..., V_k^j \rangle$ satisfies the condition (C2) above. Let j'' be a son of j in T. We want to show that $\langle V_1^j \cap (X_j \cap X_{j''}), ..., V_k^j \cap (X_j \cap X_{j''}) \rangle$ is a k-cover of $X_j \cap X_{j''}$ satisfying τ. Since $\langle Z_1 \cap X_{j''}, ..., Z_k \cap X_{j''} \rangle$ is a k-cover of $X_j \cap X_{j''}$ satisfying τ by the condition (3c) above, it suffices to show that for each $1 \leq l \leq k$, $V_l^j \cap X_j = Z_l$. Fix an integer l with $1 \leq l \leq k$. Since the path from j to each i, $1 \leq i \leq j-1$, in T must pass j', we have $V_l^{j-1} \cap X_j \subseteq X_{j'}$ by the definition of tree-decompositions. Thus, $V_l^{j-1} \cap X_j \subseteq V_l^{j-1} \cap (X_{j'} \cap X_j)$. On the other hand, $V_l^{j-1} \cap (X_{j'} \cap X_j) = Z_l \cap X_{j'}$ by the condition (3a) above. Hence, $V_l^{j-1} \cap X_j \subseteq Z_l$. Noting that $V_l^j = V_l^{j-1} \cup Z_l$ and $Z_l \subseteq X_j$, we see that $V_l^j \cap X_j = Z_l$ if and only if $V_l^{j-1} \cap X_j \subseteq Z_l$. Therefore, we have $V_l^j \cap X_j = Z_l$.

Finally, we prove that $\langle V_1^j, ..., V_k^j \rangle$ satisfies the condition (C3) above. Let i be an integer with $1 \leq i \leq j$, and i' be a child of i in T. We want to show that $\langle V_1^j \cap (X_i \cap X_{i'}), ..., V_k^j \cap (X_i \cap X_{i'}) \rangle = \langle V_1^i \cap (X_i \cap X_{i'}), ..., V_k^i \cap (X_i \cap X_{i'}) \rangle$. This clearly holds if $i = j$. So, we may assume that $i \leq j-1$. Then, $\langle V_1^{j-1} \cap (X_i \cap X_{i'}), ..., V_k^{j-1} \cap (X_i \cap X_{i'}) \rangle = \langle V_1^i \cap (X_i \cap X_{i'}), ..., V_k^i \cap (X_i \cap X_{i'}) \rangle$ by the inductive hypothesis. By this, we only need to show that $\langle V_1^j \cap (X_i \cap X_{i'}), ..., V_k^j \cap (X_i \cap X_{i'}) \rangle = \langle V_1^{j-1} \cap (X_i \cap X_{i'}), ..., V_k^{j-1} \cap (X_i \cap X_{i'}) \rangle$. Fix an integer l with $1 \leq l \leq k$. Since the path from j to each i, $1 \leq i \leq j-1$, in T must pass j', we have $Z_l \cap (X_i \cap X_{i'}) \subseteq X_{j'}$ by the definition of tree-decompositions. Thus, $Z_l \cap (X_i \cap X_{i'}) \subseteq Z_l \cap X_{j'} \cap (X_i \cap X_{i'})$. On the other hand, $Z_l \cap X_{j'} \subseteq V_l^{j-1}$ by the condition (3a) above. Hence, $Z_l \cap (X_i \cap X_{i'}) \subseteq V_l^{j-1} \cap (X_i \cap X_{i'})$. Noting that $V_l^j = V_l^{j-1} \cup Z_l$ and $Z_l \subseteq X_j$, we see that $V_l^j \cap (X_i \cap X_{i'}) = V_l^{j-1} \cap (X_i \cap X_{i'})$ if and only if $Z_l \cap (X_i \cap X_{i'}) \subseteq V_l^{j-1} \cap (X_i \cap X_{i'})$. Therefore, we have $V_l^j \cap (X_i \cap X_{i'}) = V_l^{j-1} \cap (X_i \cap X_{i'})$. ∎

Let us mention the property τ that will be used in the remainder of this paper. Let $G = (V, E)$ be a graph, and U be a subset of V. A k-cover L of U is ***completely unbalanced*** if exactly one set in L is empty and the others are equal to U. A k-cover L of U is ***weakly unbalanced*** if there are one vertex $u \in U$ and two sets U_1 and U_2 in L such that $U_1 = \{u\}$, $U_2 = U - \{u\}$, and all the sets in L except U_1 and U_2 are equal to U. A k-cover of U is ***unbalanced*** if it is either completely unbalanced or weakly unbalanced. Note that if $|U| \leq 2$, then every k-cover of U must be unbalanced. Hereafter, the property τ in Lemma 2 means "unbalanced", i.e., a k-cover L of U satisfies τ if and only if L is unbalanced.

4 PTASs for MISP(π)'s on $K_{3,3}$-free graphs

We start by proving a useful lemma.

Lemma 3. Let $G = (V, E)$ be a connected planar graph, and k be an integer ≥ 2. Suppose that s_1 and s_2 are two adjacent vertices in G and $\langle Y_1, ..., Y_k \rangle$ is an unbalanced k-cover of $\{s_1, s_2\}$. Then, we can compute a k-cover $\langle Z_1, ..., Z_k \rangle$ of V in $O(k|V|)$ time such that $\text{tw}(G[Z_l]) \leq 3k - 4$ and $Z_l \cap \{s_1, s_2\} = Y_l$ for each $1 \leq l \leq k$.

Proof. Let us first suppose that $\langle Y_1, ..., Y_k \rangle$ is weakly unbalanced. Then, by symmetry, we may assume that $Y_1 = \{s_1\}$, $Y_2 = \{s_2\}$, and $Y_3 = \cdots = Y_k = \{s_1, s_2\}$. We perform a breadth-first-search (BFS) on G starting at s_2 to obtain a BFS tree T. For each vertex v in G, we define $lev(v)$ to be the length of the path from s_2 to v in T. Note that $lev(s_2) = 0$ and $lev(s_1) = 1$. For each $1 \leq l \leq k$, let $Z_l = V - \{v \in V : lev(v) \equiv l - 1 \pmod k\}$. Obviously, $\langle Z_1, ..., Z_k \rangle$ is a k-cover of V. Moreover, the subgraph induced by each nonempty Z_l, $1 \leq l \leq k$, is $(k-1)$-outerplanar and hence has treewidth $\leq 3k-4$ [3]. It is also clear that $Z_l \cap \{s_1, s_2\} = Y_l$ for each $1 \leq l \leq k$.

Next, suppose that $\langle Y_1, ..., Y_k \rangle$ is completely unbalanced. Then, by symmetry, we may assume that $Y_1 = \emptyset$ and $Y_2 = \cdots = Y_k = \{s_1, s_2\}$. Let H be the graph obtained from G by replacing the edge $\{s_1, s_2\}$ with two edges $\{s_1, x\}$ and $\{x, s_2\}$, where x is a new vertex. It is clear that H is still planar. We perform a breadth-first-search (BFS) on H starting at x to obtain a BFS tree T. For each vertex v in H, we define the level number of v (denoted $lev(v)$) to be the length of the path from x to v in T. Note that only x has level number 0 and only s_1 and s_2 have level number 1. For each $1 \leq l \leq k$, let $V_l = \{v \in V : lev(v) \equiv l - 1 \pmod k\}$. Let $Z_1 = V - V_2$, $Z_2 = V - V_1$, and $Z_l = V - V_l$ for each $3 \leq l \leq k$. Obviously, $\langle Z_1, ..., Z_k \rangle$ is a k-cover of V. Moreover, the subgraph induced by each nonempty Z_l, $1 \leq l \leq k$, is $(k-1)$-outerplanar and hence has treewidth $\leq 3k - 4$ [3]. It is also clear that $Z_l \cap \{s_1, s_2\} = Y_l$ for each $1 \leq l \leq k$. ∎

The following lemma states that a 2-connected $K_{3,3}$-free graph can have very special split components.

Lemma 4. [2, 7]. Each split component of a 2-connected $K_{3,3}$-free graph is either isomorphic to K_5 or planar.

Now, we are ready to show the main lemma of this section.

Lemma 5. Let $G = (V, E)$ be a 2-connected $K_{3,3}$-free graph. Then, for any $k \geq 2$, we can compute a k-cover $\langle V_1, ..., V_k \rangle$ of V in $O(k|V|)$ time such that $\text{tw}(G[V_l]) \leq 3k - 4$ for each $1 \leq l \leq k$.

Proof. Let $\mathcal{D} = \{D_1, ..., D_q\}$ be a 2-decomposition of G, and let $I = \{1, ..., q\}$. It is known that $\mathcal{D}$ can be computed in $O(|V|)$ time [8]. Moreover, $\sum_{i \in I} |V(D_i)| = O(|V|)$ [8]. W.l.o.g., we may assume that $G^2(\mathcal{D}) = G$ because a k-cover $\langle V_1, ..., V_k \rangle$ of V such that the subgraph of $G^2(\mathcal{D})$ induced by V_l has treewidth $\leq 3k - 4$ for each $1 \leq l \leq k$ is also a k-cover $\langle V_1, ..., V_k \rangle$ of V such that $\text{tw}(G[V_l]) \leq 3k - 4$ for each $1 \leq l \leq k$. Then, by Fact 2, $(\{V(D_j) : j \in I\}, \mathcal{T}^2(G, \mathcal{D}))$ is a tree-decomposition of G. For convenience, let $T = \mathcal{T}^2(G, \mathcal{D}))$, $b = 3k - 4$, and $X_j = V(D_j)$ and $f(k, |X_j|) = O(k|X_j|)$ for each $j \in I$. We want to apply Lemma 2 to the graph G and the tree-decomposition $(\{X_j : j \in I\}, T)$. To this end, we first (arbitrarily) choose an $r \in I$ and root T at r.

Clearly, the condition (1) in Lemma 2 is satisfied by G and $(\{X_j : j \in I\}, T)$. By Lemma 4, $G[X_r] = D_r$ is either isomorphic to K_5 or planar. Let us first suppose that $G[X_r]$ is isomorphic to K_5. Then, we set $R_1 = \emptyset$ and $R_2 = \cdots = R_k = X_r$ if $k \geq 3$; otherwise ($k = 2$), we arbitrarily choose two vertices v_1 and v_2 in X_r and set $R_1 = \{v_1, v_2\}$ and $R_2 = X_r - R_1$. Obviously, $\langle R_1, ..., R_k\rangle$ is a k-cover of X_r satisfying the condition (2a) in Lemma 2. $\langle R_1, ..., R_k\rangle$ also satisfies the condition (2b) in Lemma 2 since $|X_r \cap X_{j''}| = 2$ for every child j'' of r in T. Next, suppose that $G[X_r]$ is a planar graph. Then, we arbitrarily choose an edge $\{s_1, s_2\}$ in $G[X_r]$, set $Y_1 = \emptyset$ and $Y_2 = \cdots = Y_k = \{s_1, s_2\}$, and use Lemma 3 to compute a k-cover $\langle R_1, ..., R_k\rangle$ of X_r in $O(k|X_r|)$ time such that $\mathrm{tw}(G[R_l]) \leq 3k - 4$ for each $1 \leq l \leq k$. Clearly, $\langle R_1, ..., R_k\rangle$ satisfies the condition (2a) in Lemma 2. $\langle R_1, ..., R_k\rangle$ also satisfies the condition (2b) in Lemma 2 since $|X_r \cap X_{j''}| = 2$ for every child j'' of r in T.

Fix a $j' \in I$ and a child j of j' in T. Let $\langle Y_1, ..., Y_k\rangle$ be an unbalanced k-cover of $X_{j'} \cap X_j$. W.l.o.g., we may assume that $|Y_l| \leq |Y_{l+1}|$ for each $1 \leq l \leq k - 1$. By Lemma 4, $G[X_j] = D_j$ is either isomorphic to K_5 or planar. Let us first suppose that $G[X_j]$ is isomorphic to K_5. If $k \geq 3$, then we set $Z_1 = Y_1$ and $Z_l = Y_l \cup (X_j - X_{j'})$ for each $2 \leq l \leq k$. Otherwise ($k = 2$), we arbitrarily choose a vertex $v \in X_j - X_{j'}$ and set $Z_1 = Y_1 \cup (X_j - (X_{j'} \cup \{v\}))$ and $Z_2 = Y_2 \cup \{v\}$. Then, no matter what k is, $\langle Z_1, ..., Z_k\rangle$ is a k-cover of X_j satisfying the conditions (3a), (3b), and (3c) in Lemma 2. Next, suppose that $G[X_j]$ is planar. Let $X_{j'} \cap X_j = \{s_1, s_2\}$. Note that s_1 and s_2 are adjacent in G. We use Lemma 3 to compute a k-cover $\langle Z_1, ..., Z_k\rangle$ of X_j. It should be easy to see that $\langle Z_1, ..., Z_k\rangle$ is a k-cover of X_j satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

Now, the lemma follows from Lemma 2. ∎

Theorem 6. Let $G = (V, E)$ be a $K_{3,3}$-free graph. Then, for any $k \geq 2$, we can compute a k-cover $\langle V_1, ..., V_k\rangle$ of V in $O(k|V|)$ time such that $\mathrm{tw}(G[V_l]) \leq 3k - 4$ for $1 \leq l \leq k$.

Proof. This follows from Fact 1, Lemma 2, and Lemma 5 immediately. ∎

Corollary 7. Let π be a hereditary property on graphs. Suppose that MISP(π) restricted to n-vertex graphs of treewidth $\leq k$ can be solved in $T_\pi(k, n)$ time. Then, given an integer $k \geq 2$ and a $K_{3,3}$-free graph $G = (V, E)$, we can compute a subset U of V in $O(k|V| + T_\pi(3k - 4, |V|))$ time such that $G[U]$ satisfies π and $|U|$ is at least $(k - 1)/k$ optimal.

Proof. Given an integer $k \geq 2$ and a $K_{3,3}$-free graph $G = (V, E)$, we first compute a k-cover $\langle V_1, ..., V_k\rangle$ of V such that $\mathrm{tw}(G[V_l]) \leq 3k - 4$ for each $1 \leq l \leq k$ (cf. Theorem 6). Next, we compute an optimal solution U_l in each $G[V_l]$, $1 \leq l \leq k$. Finally, we set U to be the maximum subset among U_1, ..., U_k. Obviously, $G[U]$ satisfies π. Moreover, since π is hereditary, reasoning similar to that in [5] can be used to show that $|U|$ is at least $(k - 1)/k$ optimal. ∎

For many π's, it is well known that $T_\pi(k, n) = O(c^k n)$ for some small c, and hence MISP(π) restricted to $K_{3,3}$-free graphs has a practical PTAS by Corollary 7.

5 PTASs for MISP(π)'s on K_5-free graphs

We start by giving several definitions. Suppose that G is 3-connected. Further suppose that G contains a *strong* 3-cut. Replacing G by the augmented components induced by a strong 3-cut is called *strongly splitting* G. Suppose G is strongly split, the augmented components are strongly split, and so on, until no more strong splits are possible. The set of the graphs constructed in this way are called a *strong 3-decomposition* of G.

Definition 8. We define W to be the graph obtained from a 8-cycle by adding 4 crossing edges. More precisely, $W = (\{1, ..., 8\}, E_1 \cup E_2)$, where $E_1 = \{\{i, i+1\} : 1 \leq i \leq 7\} \cup \{\{8, 1\}\}$ and $E_2 = \{\{i, i+4\} : 1 \leq i \leq 4\}$. A K_5-free graph G is said to be *nice* if G is 3-connected, nonplanar, and is not isomorphic to $K_{3,3}$ or W.

Fact 3 [10] Suppose that G is a nice K_5-free graph. Let C be a strong 3-cut in G. Then, the augmented components induced by C are also nice K_5-free graphs. Moreover, C' is a strong 3-cut of G if and only if C' is a strong 3-cut of an augmented component of G induced by C.

Based on this fact, Kézdy and McGuinness further proved the following:

Fact 4 [10] A nice K_5-free graph has a *unique* strong 3-decomposition. Moreover, each graph in the strong 3-decomposition is planar.

Suppose that $G = (V, E)$ is a nice K_5-free graph. Let $\mathcal{D}^3(G)$ be the strong 3-decomposition of G, and $\mathcal{C}^3(G)$ be the set of all strong 3-cuts in G. Define $H(G)$ to be the bipartite graph $(\mathcal{D}^3(G) \cup \mathcal{C}^3(G), F)$, where $F = \{\{D, C\} : D \in \mathcal{D}^3(G),\ C \in \mathcal{C}^3(G), \text{ and } C \subseteq V(D)\}$.

Lemma 9. The following hold:

(1) Every edge of G is contained in some graph in $\mathcal{D}^3(G)$.

(2) If a subset S of V induces a triangle but $S \notin \mathcal{C}^3(G)$, then exactly one graph in $\mathcal{D}^3(G)$ contains the three vertices in S.

(3) $H(G)$ is a tree. Moreover, if some vertex $u \in V$ is contained in two graphs D and D' in $\mathcal{D}^3(G)$, then u is contained in every graph on the path between D and D' in $H(G)$.

Proof. We show the lemma by induction on the number of strong 3-cuts in G. The lemma clearly holds when G has no strong 3-cut. Assume that the lemma is true for every graph that has up to $p-1$ strong 3-cuts. Consider a graph G with p strong 3-cuts. Let C be a strong 3-cut in G, and $G_1 \cup K(C)$, ..., $G_k \cup K(C)$ be the augmented components induced by C. By Fact 3, each $G_i \cup K(C)$, $1 \leq i \leq k$, is a nice K_5-free graph, $\mathcal{C}^3(G) = \cup_{1 \leq i \leq k} \mathcal{C}^3(G_i \cup K(C)) \cup \{C\}$, and $\mathcal{D}^3(G) = \cup_{1 \leq i \leq k} \mathcal{D}^3(G_i \cup K(C))$.

It is clear that every edge of G is contained in at least one of the graphs $G_1 \cup K(C)$, ..., $G_k \cup K(C)$. Moreover, by the inductive hypothesis, every edge of each $G_i \cup K(C)$, $1 \leq i \leq k$, is contained in some graph in $\mathcal{D}^3(G_i \cup K(C))$. These together with the fact that $\mathcal{D}^3(G) = \cup_{1\le i\le k}\mathcal{D}^3(G_i \cup K(C))$ imply that the statement (1) in the lemma holds for G.

Suppose that $S \subseteq V$ induces a triangle but $S \notin \mathcal{C}^3(G)$. Then, it is clear that there is exactly one $G_i \cup K(C)$, $1 \le i \le k$, containing S. Moreover, S cannot be a strong 3-cut in the graph $G_i \cup K(C)$ or else S would be a strong 3-cut in G by Fact 3. Thus, by the inductive hypothesis, exactly one graph in $\mathcal{D}^3(G_i \cup K(C))$ contains the three vertices in S. Therefore, exactly one graph in $\mathcal{D}^3(G)$ contains the three vertices in S. This implies that the statement (2) in the lemma holds for G.

By the inductive hypothesis, each $H(G_i \cup K(C))$, $1 \le i \le k$, is a tree. Moreover, it is clear that in each graph $G_i \cup K(C)$, $1 \le i \le k$, C induces a triangle but is not a strong 3-cut. Thus, for each $1 \le i \le k$, exactly *one* graph (say, D_i) in $\mathcal{D}^3(G_i \cup K(C))$ contains the three vertices in C by the inductive hypothesis. On the other hand, $\mathcal{C}^3(G) = \cup_{1\le i\le k}\mathcal{C}^3(G_i \cup K(C)) \cup \{C\}$ and $\mathcal{D}^3(G) = \cup_{1\le i\le k}\mathcal{D}^3(G_i \cup K(C))$. Therefore, $H(G)$ can be obtained from C and the trees $H(G_1 \cup K(C))$, ..., $H(G_k \cup K(C))$ by adding the edges $\{C, D_1\}$, ..., $\{C, D_k\}$. This implies that $H(G)$ is a tree. Next, suppose that some vertex $u \in V$ is contained in two graphs D and D' in $\mathcal{D}^3(G)$. If the path between D and D' in $H(G)$ does not pass C, then u is contained in every graph on the path between D and D' in $H(G)$ by the inductive hypothesis. So, we may assume that the path between D and D' in $H(G)$ does pass C. Then, there are two neighbors D_i and D_j, $1 \le i, j \le k$, of C in $H(G)$ such that D_i lies on the path between D and C and D_j lies on the path between D' and C. Moreover, u must be contained in C since every vertex shared by a pair of two graphs among $G_1 \cup K(C)$, ..., $G_k \cup K(C)$ must be contained in C. Hence, u is contained in both D_i and D_j. By the inductive hypothesis, u is contained in every graph on both the path between D and D_i in $H(G_i \cup K(C))$ and the path between D' and D_j in $H(G_j \cup K(C))$. This implies that u is contained in every graph on the path between D and D' in $H(G)$. Therefore, the statement (3) in the lemma holds for G. ∎

Suppose that $\mathcal{D}^3(G) = \{D_1, ..., D_q\}$. Let $I = \{1, ..., q\}$. Root the tree $H(G)$ at D_1 and define $\mathcal{T}^3(G))$ to be the tree whose vertex set is I and edge set is $\{\{i, i'\} : D_i$ is the grandparent of $D_{i'}$ in the rooted tree $H(G)\}$. (Note that $\mathcal{T}^3(G)$ is undirected.) Construct a supergraph G^3 of G as follows: For each strong 3-cut C and each pair of nonadjacent vertices u and v in C, add the edge $\{u, v\}$ to G.

Corollary 10. $(\{V(D_i) : i \in I\}, \mathcal{T}^3(G))$ is a tree-decomposition of G^3.

Next, we proceed to considering how to decompose a K_5-free graph into induced subgraphs of small treewidth. The following lemma is useful:

Lemma 11. Let $G = (V, E)$ be a connected planar graph, and k be an integer ≥ 2. Suppose that S is a subset of V such that $G[S]$ is a triangle, and $\langle Y_1, ..., Y_k\rangle$ is an unbalanced k-cover of S. Then, we can compute a k-cover $\langle Z_1, ..., Z_k\rangle$ of V in $O(k|V|)$ time such that $\mathrm{tw}(G[Z_l]) \le 6k - 7$ and $Z_l \cap S = Y_l$ for each $1 \le l \le k$, and $\langle Z_1 \cap S', ..., Z_k \cap S'\rangle$ is an unbalanced k-cover of S' for all subsets S' of V with $G[S']$ being a triangle.

Proof. Let $S = \{s_1, s_2, s_3\}$. First, suppose that $\langle Y_1, ..., Y_k\rangle$ is weakly unbalanced. Then, by symmetry, we may assume that $Y_1 = \{s_1\}$, $Y_2 = \{s_2, s_3\}$, and $Y_3 = \cdots = Y_k = S$. We perform a breadth-first-search (BFS) on G starting at s_1 to obtain a BFS tree T. For each vertex v in G, we define $lev(v)$ to be the length of the path from s_1 to v in T. Note that $lev(s_1) = 0$ and both $lev(s_2) = lev(s_3) = 1$. For each $1 \le l \le k$, let $V_l = \{v \in V : lev(v) \equiv l - 1 \pmod{k}\}$. Let $Z_1 = V - V_2$, $Z_2 = V - V_1$, and $Z_l = V - V_l$ for each $3 \le l \le k$. Then, it should be clear that $\langle Z_1, ..., Z_k\rangle$ is a k-cover of V satisfying the conditions in the lemma.

Next, suppose that $\langle Y_1, ..., Y_k\rangle$ is completely unbalanced. Then, by symmetry, we may assume that $Y_1 = \emptyset$ and $Y_2 = \cdots = Y_k = S$. Let H be the graph obtained from G by contracting the three edges in the triangle $G[S]$. That is, H is obtained from G by identifying the three vertices in S with a new vertex $x \notin V$. It is clear that H is still planar. We perform a breadth-first-search (BFS) on H starting at x to obtain a BFS tree T. For each vertex v in H, we define $lev(v)$ to be the length of the path from x to v in T. Recall that s_1, s_2, and s_3 are not in H. We define $lev(s_1) = lev(s_2) = lev(s_3) = 0$. For each $1 \le l \le k$, let $Z_l = V - \{v \in V : lev(v) \equiv l - 1 \pmod{k}\}$. Obviously, $\langle Z_1, ..., Z_k\rangle$ is a k-cover of V, $Z_l \cap S = Y_l$ for each $1 \le l \le k$, and $\langle Z_1 \cap S', ..., Z_k \cap S'\rangle$ is an unbalanced k-cover of S' for all subsets S' of V with $G[S']$ being a triangle. It remains to show that $\mathrm{tw}(G[Z_l]) \le 6k - 7$ for each $1 \le l \le k$. To this end, fix an arbitrary l, $1 \le l \le k$. Consider a planar embedding of G. In the embedding, the triangle $G[S]$ splits the plane into two regions. Exactly one of the regions is infinite and the other is finite. Let Z_l^{in} be the vertices of Z_l falling into the finite region, and $Z_l^{out} = Z_l - Z_l^{in}$. It is not difficult to see that both $G[Z_l^{in}]$ and $G[Z_l^{out}]$ are $(k-1)$-outerplanar (no matter whether $S \subseteq Z_l$ or not). From this, we observe that $G[Z_l]$ is $(2k-2)$-outerplanar. Therefore, $\mathrm{tw}(G[Z_l]) \le 6k - 7$ [3]. ∎

Now, we are ready to show two main lemmas in this section.

Lemma 12. Let $G = (V, E)$ be a nice K_5-free graph, and k be an integer ≥ 2. Suppose that s_1 and s_2 are two adjacent vertices in G and $\langle U_1, ..., U_k\rangle$ is an unbalanced k-cover of $\{s_1, s_2\}$. Then, we can compute a k-cover $\langle V_1, ..., V_k\rangle$ of V in $O(k|V| + |V|^2)$ time such that $\mathrm{tw}(G[V_l]) \le 6k - 7$ and $V_l \cap \{s_1, s_2\} = U_l$ for each $1 \le l \le k$.

Proof. Let $\mathcal{D}^3(G) = \{D_1, ..., D_q\}$ be the strong 3-decomposition of G, and let $I = \{1, ..., q\}$. It is known that $\mathcal{D}^3(G)$ can be computed in $O(|V|^2)$ time [9]. W.l.o.g., we may assume that $G^3 = G$ because a k-cover $\langle V_1, ..., V_k\rangle$ of V such that the subgraph of G^3 induced by V_l has treewidth $\le 6k - 7$ for each $1 \le l \le k$ is also a k-cover $\langle V_1, ..., V_k\rangle$ of V such that $\mathrm{tw}(G[V_l]) \le 6k - 7$ for each $1 \le l \le k$. Then, by Fact 10, $(\{V(D_j) : j \in I\}, \mathcal{T}^3(G))$ is a tree-decomposition of G. For convenience, let $T = \mathcal{T}^3(G)$, $b = 6k - 7$, and $X_j = V(D_j)$ and $f(k, |X_j|) = O(k|X_j|)$ for each $j \in I$. We want to apply Lemma 2 to the graph G and the tree-decomposition $(\{X_j : j \in I\}, T)$. To this end, we first choose an $r \in I$ with $\{s_1, s_2\} \subseteq X_r$ and root T at r. Such an r must exist because $\{s_1, s_2\}$ is an edge in G.

Clearly, the condition (1) in Lemma 2 is satisfied by G and $(\{X_j : j \in I\}, T)$. By Fact 4, $G[X_r] = D_r$ is planar. So, by Lemma 3, we can compute a k-cover $\langle R_1, ..., R_k \rangle$ of X_r such that $\mathrm{tw}(G[R_l]) \leq 3k-4$ and $R_l \cap \{s_1, s_2\} = U_l$ for each $1 \leq l \leq k$. Moreover, it is clear from the proof of Lemma 3 that for every subset S of X_r with $G[S]$ being a triangle, $\langle R_1 \cap S, ..., R_k \cap S \rangle$ is an unbalanced k-cover of S. Now, it should be easy to verify that $\langle R_1, ..., R_k \rangle$ is a k-cover of X_r satisfying the conditions (2a) and (2b) in Lemma 2.

Fix a $j' \in I$ and a child j of j' in T. Let $\langle Y_1, ..., Y_k \rangle$ be an unbalanced k-cover of $X_{j'} \cap X_j$. Let $S = X_{j'} \cap X_j$. Recall that $G[S]$ is a triangle. So, we can compute a k-cover $\langle Z_1, ..., Z_k \rangle$ of X_j satisfying the conditions in Lemma 11. It should be easy to see that $\langle Z_1, ..., Z_k \rangle$ is a k-cover of X_j satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

By the discussions above and Lemma 2, there is a k-cover $\langle V_1, ..., V_k \rangle$ of V such that $\mathrm{tw}(G[V_l]) \leq 6k-7$ and $V_l \cap X_r = R_l$ for each $1 \leq l \leq k$. Fix an l with $1 \leq l \leq k$. Recall that $R_l \cap \{s_1, s_2\} = U_l$ and that $\{s_1, s_2\} \subseteq X_r$. Thus, $V_l \cap \{s_1, s_2\} = V_l \cap (X_r \cap \{s_1, s_2\}) = R_l \cap \{s_1, s_2\} = U_l$. This establishes the lemma. ∎

Lemma 13. Let $G = (V, E)$ be a 2-connected K_5-free graph. Then, for any $k \geq 2$, we can compute a k-cover $\langle V_1, ..., V_k \rangle$ of V in $O(k|V| + |V|^2)$ time such that $\mathrm{tw}(G[V_l]) \leq 6k-7$ for each $1 \leq l \leq k$.

Proof. Let $\mathcal{D} = \{D_1, ..., D_q\}$ be a 2-decomposition of G, and let $I = \{1, ..., q\}$. It is known that $\mathcal{D}$ can be computed in $O(|V|)$ time [8]. W.l.o.g., we may assume that $G^2(\mathcal{D}) = G$ because a k-cover $\langle V_1, ..., V_k \rangle$ of V such that the subgraph of $G^2(\mathcal{D})$ induced by V_l has treewidth $\leq 6k-7$ for each $1 \leq l \leq k$ is also a k-cover $\langle V_1, ..., V_k \rangle$ of V such that $\mathrm{tw}(G[V_l]) \leq 6k-7$ for each $1 \leq l \leq k$. Then, by Fact 2, $(\{V(D_j) : j \in I\}, \mathcal{T}^2(G, \mathcal{D}))$ is a tree-decomposition of G. For convenience, let $T = \mathcal{T}^2(G, \mathcal{D})$, $b = 6k-7$, and $X_j = V(D_j)$ and $f(k, |X_j|) = O(k|X_j| + |X_j|^2)$ for each $j \in I$. We want to apply Lemma 2 to the graph G and the tree-decomposition $(\{X_j : j \in I\}, T)$. To this end, we first (arbitrarily) choose an $r \in I$ and root T at r.

Clearly, the condition (1) in Lemma 2 is satisfied by G and $(\{X_j : j \in I\}, T)$. To see that the condition (2) in Lemma 2 is also satisfied, we distinguish four cases as follows:

Case 1: $G[X_r]$ is planar. Then, as stated in the proof of Lemma 5, we can compute a k-cover $\langle R_1, ..., R_k \rangle$ of X_r in $O(k|X_r|)$ time satisfying the conditions (2a) and (2b) in Lemma 2.

Case 2: $G[X_r]$ is isomorphic to $K_{3,3}$. Then, we set $R_1 = \emptyset$ and $R_2 = \cdots = R_k = X_r$. Obviously, $\langle R_1, ..., R_k \rangle$ is a k-cover of X_r satisfying the conditions (2a) and (2b) in Lemma 2.

Case 3: $G[X_r]$ is isomorphic to the graph W (see Definition 8). Then, we set $R_1 = \emptyset$ and $R_2 = \cdots = R_k = X_r$ if $k \geq 3$; otherwise ($k = 2$), we (arbitrarily) choose four vertices from X_r and set R_1 to be the set of the four vertices and R_2 to be $X_r - R_1$. Obviously, $\langle R_1, ..., R_k \rangle$ is a k-cover of X_r satisfying the conditions (2a) and (2b) in Lemma 2.

Case 4: $G[X_r]$ is a nice K_5-free graph. Then, we arbitrarily choose an edge $\{s_1, s_2\}$ in $G[X_r]$ and set $U_1 = \emptyset$ and $U_2 = \cdots = U_k = \{s_1, s_2\}$. By Lemma 12, we can compute a k-cover $\langle R_1, ..., R_k \rangle$ of X_r in $O(k|X_r| + |X_r|^2)$ time such that $\mathrm{tw}(G[R_l]) \leq 6k-7$ for each $1 \leq l \leq k$. Clearly, $\langle R_1, ..., R_k \rangle$ satisfies the condition (2a) in Lemma 2. $\langle R_1, ..., R_k \rangle$ also satisfies the condition (2b) in Lemma 2 since $|X_r \cap X_{j''}| = 2$ for every child j'' of r in T.

Note that one of the above four cases must occur. Thus, the condition (2) in Lemma 2 is satisfied by G and $(\{X_j \ : \ j \in I\}, T)$. To see that the condition (3) in Lemma 2 is also satisfied, fix a $j' \in I$ and a child j of j' in T. Let $\langle Y_1, ..., Y_k \rangle$ be an unbalanced k-cover of $X_{j'} \cap X_j$, and let $X_{j'} \cap X_j = \{s_1, s_2\}$. Recall that $\{s_1, s_2\}$ is an edge in both $G[X_{j'}]$ and $G[X_j]$. Moreover, by symmetry, we may assume that $|Y_l| \leq |Y_{l+1}|$ for all $1 \leq l \leq k-1$. We distinguish four cases as follows:

Case 1': $G[X_j]$ is planar. Then, as stated in the proof of Lemma 5, we can compute a k-cover $\langle Z_1, ..., Z_k \rangle$ of X_j in $O(k|X_j|)$ time satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

Case 2': $G[X_j]$ is isomorphic to $K_{3,3}$. Then, we set $Z_1 = Y_1$ and $Z_l = Y_l \cup (X_j - X_{j'})$ for each $2 \leq l \leq k$. Clearly, $\langle Z_1, ..., Z_k \rangle$ is a k-cover of X_j satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

Case 3': $G[X_j]$ is isomorphic to the graph W (see Definition 8). If $k \geq 3$, then we set $Z_1 = Y_1$ and $Z_l = Y_l \cup (X_j - X_{j'})$ for each $2 \leq l \leq k$; otherwise ($k = 2$), we (arbitrarily) choose a subset A of $X_j - X_{j'}$ with $|A| = 3$ and set $Z_1 = Y_1 \cup A$ and $Z_2 = X_j - Z_1$. Then, it is easy to verify that $\langle Z_1, ..., Z_k \rangle$ is a k-cover of X_j satisfying the conditions (3a), (3b), and (3c) in Lemma 2.

Case 4': $G[X_j]$ is a nice K_5-free graph. Then, by Lemma 12, we can compute a k-cover $\langle Z_1, ..., Z_k \rangle$ of X_j in $O(k|X_j| + |X_j|^2)$ time such that $\mathrm{tw}(G[Z_l]) \leq 6k-7$ and $Z_l \cap \{s_1, s_2\} = Y_l$ for each $1 \leq l \leq k$. From this, it should be clear that $\langle Z_1, ..., Z_k \rangle$ satisfies the conditions (3a), (3b), and (3c) in Lemma 2.

Note that one of the four cases must occur. Thus, by the discussions above and Lemma 2, we have the lemma. ∎

Theorem 14. Let $G = (V, E)$ be a K_5-free graph. Then, for any $k \geq 2$, we can compute a k-cover $\langle V_1, ..., V_k \rangle$ of V in $O(k|V| + |V|^2)$ time such that $\mathrm{tw}(G[V_l]) \leq 6k-7$ for each $1 \leq l \leq k$.

Proof. This follows from Fact 1, Lemma 2, and Lemma 13 immediately. ∎

Corollary 15. Let π be a hereditary property on graphs. Suppose that MISP(π) restricted to n-vertex graphs of treewidth $\leq k$ can be solved in $T_\pi(k, n)$ time. Then, given an integer $k \geq 2$ and a K_5-free graph $G = (V, E)$, we can compute a subset U of V in $O(k|V| + |V|^2 + T_\pi(6k-7, |V|))$ time such that $G[U]$ satisfies π and $|U|$ is at least $(k-1)/k$ optimal.

Proof. Similar to that of Corollary 7. ∎

For many π's, it is well known that $T_\pi(k, n) = O(c^k n)$ for some small c, and hence MISP(π) restricted to K_5-free graphs has a practical PTAS by Corollary 15.

6 Concluding remarks

We have shown that many MISP(π)'s restricted to $K_{3,3}$-free or K_5-free graphs have practical PTASs. It is worth mentioning that these PTASs are easy to parallelize. Since the details are almost trivial, we omit them here.

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