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Inflections and singularities on parametric rational cubic curves

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1 Introduction

Much attention has been recently been focused on a single- and vector-valued shape preserving interpolation. There is a considerable literature on numerical methods for generating shape preserving interpolation; see for example, Sakai & Usmani [5], Su & Liu [7] and references therein. Parametric cubic curves and useful for generating vector-valued "visually pleasing", "shape preserving" interpolants to a set of planar data points. A drawback of the parametric cubic curves is indicated the fact that unwanted inflection points or singularity may occur on these segments. A way of overcoming this problem is to consider nonlinear approximation sets, for example, exponential splines, lacunary splines, rational splines or splines with variable additional nodes. The rational splines have been of the forms: quadratic/linear, cubic/linear, quadratic/quadratic, and cubic/quadratic. For two consecutive data points $z_i\ (=(x_i,y_i)),\ i=0,1$ and assigned tangent vectors $z_i'(=((x_i',y_i'))$ at these points, we consider a "shape preserving" parametric rational cubic curve $z(t)=((x(t),y(t)),0\leq t\leq 1$, of the form with a parameter ($p\geq 0$):

$$z(t)=z_0(1-t)+z_1t+(z_0'-\Delta z)\phi(t)+(\Delta z-z_1')\phi(1-t) \tag{1}$$

with $\Delta z=z_1-z_0$ and $\phi(t)=t(1-t)^2/\{1+pt(1-t)\}$. We can easily verify that this form z satisfies the interpolation relations: $z^{(k)}(0)=z_0^{(k)}, z^{(k)}(1)=z_1^{(k)}, k=0,1$. A value of $p=0$ will let z be a cubic spline while a large value of p will let z be an unacceptable "flat" interpolant. In Section 2, much algebraic manipulation gives the complete distribution of inflections and singularity on the curve segment (1). Its use allows us to ensure that the curve z of the form (1) is free from the inflections and singularity (or simply fair) if $\lambda,\mu\geq 1/(3+p)$ or $\lambda,\mu\leq 0$ where $\lambda(=(z_0'\times\Delta z)/(z_0'\times z_1')$) and $\mu(=-(z_1'\times\Delta z)/(z_0'\times z_1')$) the relative lengths of the tangent vectors at the end points with the usual vector product '$\times$ ' where assume that $D(=z_0'\times z_1')\neq 0$, i.e., $z_i', i=0,1$ are independent.
Hence, a large value of p can always let the segment z be fair if $\lambda\mu>0$. Another use of it also enables us to find another fair curve of the form (1) if $0<\kappa\leq(3+p)\lambda$ and $0<\eta\leq(3+p)\mu$ where the tangent directions $y_i'/x_i', i=0,1$ are fixed at the two end points, and only the magnitudes of the tangents can be varied in scalar multiples η and κ , respectively.

2 On inflections and singularity

First we note Fig. 1 for $k(\lambda, \mu) = 0$ given by the following equation (2):

$$(2) \qquad k(= k(\lambda,\mu)) = 4\lambda^3\{(3+p)\mu - 1\} + 4\mu^3\{(3+p)\mu - 1\} - 3\lambda^2\mu^2 + \{(3+p)\lambda - 1\}^2\{(3+p)\mu - 1\}^2 - 6\lambda\mu\{(3+p)\lambda - 1\}\{(3+p)\mu - 1\} = 0.$$

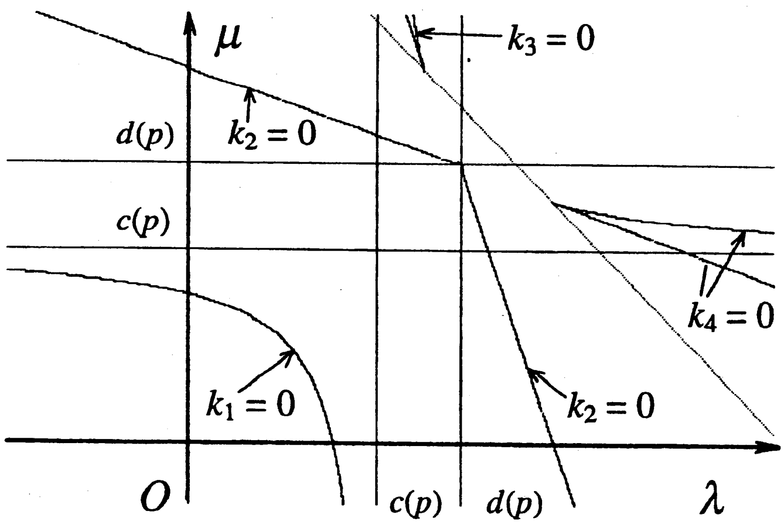

Fig. 1 Four branches $k_i = 0, 1 \leq i \leq 4$ of $k = 0$.

Inflections. We are now ready to determine the distribution of inflections on the curve (1). Since $D(= z_0' \times z_1') \neq 0$, $\Delta z (= z_1 - z_0)$ can be represented as $\Delta z = \mu z_0' + \lambda z_1'$ where $(\lambda, \mu)(= C_0/D, -C_1/D)$ with $C_i = z_i' \times \Delta z$.

Defining and doing a fairly lengthy calculation,

$$(3) \qquad w(t)(= \{\phi'(t)\phi''(u) + \phi''(t)\phi(u)\}) = 2(1 - 3tu)(1 + ptu)^3, u = 1 - t,$$

inflections of the curve (1) are determined by the equation:

$$(4) \qquad (z' \times z'')(t)/D (= \lambda\{w(t) - \phi''(t)\} + \mu\{w(u) - \phi''(u)\} - w(t)) = 0, 0 < t < 1$$

or

$$(5) \qquad \lambda(3u^2 + pu^3) + \mu(3t^2 + pt^3) + 3tu - 1 = 0, 0 < t < 1.$$

Letting t be $1/(1+t)$, the above cubic equation (5) can be equivalently rewritten as

$$(6) \qquad \{(3+p)\lambda - 1\}t^3 + 3\lambda t^2 + 3\mu t + \{(3+p)\mu - 1\} = 0, t > 0.$$

We consider several Cases (a)-(d) depending on the values of λ and μ, and for each case we count the number N of positive roots of the cubic equation (6) (= the number of inflections on (1)).

(a) $(\lambda - c(p))(\mu - c(p)) = 0$: In this case, (6) itself or (6) divided by t reduces to a quadratic equation. Hence, $N = 1$ if $\lambda = c(p), \mu < c(p)$ or $\lambda < c(p), \mu = c(p)$ and $N = 0$ if $\lambda = c(p), \mu > c(p)$ or $\lambda > c(p), \mu = c(p)$.

(b) $\lambda, \mu > c(p)$ or $\lambda, \mu \leq 0$: In this case, (6) has no positive root. Hence, $N = 0$.

For the other cases than (a)-(b), we make a Strum sequence $\{p_i(t)\}$ of the cubic equation (6) for $a^2 \neq 3b$:

$$
\begin{aligned}
& p_0(t) = t^3 + at^2 + bt + c(= -p_0'(t)) = -\ 3t^2 - 2at - b \\
(7) \qquad & p_2(t) = 2(a^2 - 3b)t + ab - 9c \\
& p_3(t) = -\ p_1\{(9c - ab)/(2a^2 - 6b)\}(= 9k\{4(a^2 - 3b)^2\}
\end{aligned}
$$

in which

$$
\begin{aligned}
& a = 3\lambda/\{(3+p)\lambda - 1\}, b = 3\mu/\{(3+p)\lambda - 1\}, c = \{(3+p)\mu - 1\}/\{(3+p)\lambda - 1\} \\
(8) \qquad & \\
& k = 4a^3c + 4b^3 + 27c^2 - 18abc - a^2b^2.
\end{aligned}
$$

Let the integer valued function $s(t)$ be the number of agreements in sign of consecutive members of the sequence $\{p_i(t)\}$. Then, the theory of Strum sequences says that the numbers of roots in the interval $a < t \leq b$ is given by $s(b) - s(a)$.

(c) $(\lambda - c(p))(\mu - c(p)) < 0$: In this case assume that $\mu < c(p) < \lambda$, the other case $\lambda < c(p) < \mu$ being similarly treated. Then, note that $a > 0$ and $c < 0$. When $b < 0$ and $k > 0$, $\epsilon = \mathrm{sign}(ab - 9c) < 0 \Rightarrow s(0) = 3$ is contradictory to $s(0) \leq s(+\infty)$, and so $\epsilon > 0$. Table 1 shows that $N(= s(+\infty) - s(0)) = 1$.

Table 1. The sequences of signs at 0 and ∞ for $\lambda - c(p))(\mu - c(p)) < 0$.

$b \backslash k$	positive	negative
positive	$(-,-,+,+),(+,-,?,+)$	$(-,-,+,-)(+,-,+^*,-)$
negative	$(-,+,\varepsilon,+),(+,-,+,+)$	$(-,+,?,-)(+,-,+,-)$

* If $a^2 - 3b < 0, s(+\infty) = 1$ is contradictory to $s(0)(= 2) \leq s(+\infty)$, and so $a^2 - 3b > 0$.

(d) $0 < \lambda, \mu < c(p)$: Note that $a < 0, b < 0$, and $c > 0$. In addition, $ab - 9c < 0$ if $k > 0$ as follows. If $ab - 9c \geq 0$, i.e., $0 < c \leq ab/9$, then we take k as a quadratic equation in c:

$$(9) \qquad k(= k(c)) = 27c^2 + c(4a^3 - 18ab) + 4b^3 - a^2b^2 < k(0) = 4b^3 - a^2b^2 < 0.$$

Table 2. The sequences of signs at 0 and ∞ for $0 < \lambda, \mu < c(p)$.

k	positive	negative
	$(+,+,-,+),(+,-,+,+)$	$(+,+,?,-),(+,-,+,-)$

Hence, $N = 0$ or 2 for $k \geq 0$ or $k < 0$ where the equation $k = 0$ in (8) is equivalent to the equation$k = 0$ defined in Lemma 1, strictly speaking, the k in (8) multiplied by $\{(3+p)\lambda - 1\}^4/27$ coincides with the k in Lemma 1. For $a^2 = 3b(\geq \Rightarrow$ no case (d)), $p_0(t) = (t+a/3)^3 + c - a^3/27$, and so $N = 1$ for the case (c). Since for $\lambda, \mu < c(p)$, $k = 0$ is equal to $k_1 = 0$, we obtain

LEMMA 1 (inflections). If $(\lambda, \mu) \in N_i, 0 \leq i \leq 2$, the curve (1) has i inflections where $N_0 = \{(\lambda,\mu)|\lambda,\mu \geq c(p)$ or $k_1(\lambda,\mu) \geq 0\}$, $N_1 = \{(\lambda,\mu)|(\lambda - c(p))(\mu - c(p)) \leq 0$ or $\lambda = c(p), \mu < c(p)$ or $\lambda < c(p), \mu = c(p)\}$ and $N_2 = \{(\lambda,\mu)|k_1(\lambda,\mu) < 0, \lambda, \mu < c(p)\}$.

Singularity. For determination of the distribution of singularity, we have to get an equation of the image of the curve (1) by eliminating the parameter t from it. Make vector product of (1) and $z_i', i = 0,1$, respectively, to obtain

$$(\lambda+\mu-1)D\phi(t) = a_0t + b_0u + c_0, (\lambda+\mu-1)D\phi(u) = a_1t + b_1u + c_1 \tag{10}$$

where

$$a_i = z_1 \times (z_{1-i}' - \Delta z), b_i = z_0 \times (z_{1-i}' - \Delta z), c_i = -z \times (z_{1-i}' - \Delta z), i = 0,1. \tag{11}$$

Two cases, $\lambda + \mu \neq 1$ and $\lambda + \mu = 1$, will be considered separately.

Case 1 ($\lambda + \mu \neq 1$). If $\lambda \neq \mu, t\phi(t) = u\phi(u)$ gives a quadratic equation:

$$(a_0 - b_0 + a_1 - b_1)t^2 + (b_0 + 2b_1 - a_1 + c_0 + c_1)t - (b_1 + c_1) = 0 \text{ or } (t+\alpha)^2 = \beta \tag{12}$$

where

$$\begin{gathered} a_0 - b_0 + a_1 - b_1 (= \Delta z(z_1' + z_0')) = (\mu - \lambda))D \\ 2\alpha = \frac{b_0 + 2b_1 - a_1 + c_0 + c_1}{a_0 - b_0 + a_1 - b_1}, \alpha^2 - \beta = -\frac{b_1 + c_1}{a_0 - b_0 + a_1 - b_1} \end{gathered} \tag{13}$$

If $\lambda = \mu$, the sum of the two equations in (10) gives a quadratic equation for (12):

$$t^2 - t + \frac{b_0 + b_1 + c_0 + c_1}{\{(2\lambda - 1)D - p(b_0 + b_1 + c_0 + c_1)\}} = 0 \tag{14}$$

which is equivalently rewritten as (12) with

$$\alpha = -1/2, \alpha^2 - \beta (= 1/4 - \beta) = \frac{b_0 + b_1 + c_0 + c_1}{\{(2\lambda - 1)D - p(b_0 + b_1 + c_0 + c_1)\}}. \tag{15}$$

Hence, let $t^* = t + \alpha$ and eliminate the parameter t^* (note $(t^*)^2 = t + \alpha$) from the second equation in (10) to obtain the equation of the image of the curve (1):

$$(16) \qquad \psi(x, y) = (A^2\beta - B^2) = 0$$

where

$$(17) \quad \begin{aligned} \text{(i)} \quad A/D &= -(\lambda + \mu - 1 + p\lambda)(\beta + 3\alpha^2 + 2\alpha) + \lambda - (p/D)(2\alpha + 1)(b_1 + c_1) \\ \text{(ii)} \quad B/D &= -(\lambda + \mu - 1 + p\lambda)(3\alpha\beta + \alpha^3 + \beta + \alpha^2) \\ &\quad + \alpha\lambda - (1/D)\{p(\alpha^2 + \beta) + p\alpha - 1\}(b_1 + c_1). \end{aligned}$$

For $\lambda \neq \mu$, since $\lambda + \mu \neq 1 \Leftrightarrow (z_0' - \Delta z) \times (z_1' - \Delta z) \neq 0, (z_{1-i} - \Delta z) \times z = c_i, i = 0, 1$ in (11) enable us to consider (x, y) as functions of (c_0, c_1), and in addition the second and third equations in (13) (i.e., $2\alpha(\mu - \lambda)D = b_0 + 2b_1 - a_1 + c_0 + c_1$ and $b_1 + c_1 = -(\mu - \lambda)(\alpha^2 - \beta)D)$) enable (c_0, c_1) as (α, β). Then, the singularity of $\psi = 0$ of (α, β) for (x, y) is determined by the system of equations

$$(18) \qquad \psi(\alpha, \beta) = \psi_\alpha(\alpha, \beta) = \psi_\beta(\alpha, \beta) = 0$$

i.e.,

$$(19) \qquad A^2\beta = B^2, 2AA_\alpha\beta = 2BB_\alpha, 2AA_\beta\beta + A^2 = 2BB_\beta.$$

where by (13), $(b_1 + c_1)/D = -(\mu - \lambda)(\alpha^2 - \beta)$ and so (17) reduce to

$$(20) \quad \begin{aligned} \text{(i)} \quad A/D &= -(\lambda + \mu - 1 + p\lambda)(\beta + 3\alpha^2 + 2\alpha) + \lambda + p(2\alpha + 1)(\mu - \lambda)(\alpha^2 - \beta) \\ \text{(ii)} \quad B/D &= -(\lambda + \mu - 1 + p\lambda)(3\alpha\beta + \alpha^3 + \beta + \alpha^2) \\ &\quad + \alpha\lambda + (\mu - \lambda)(\alpha^2 - \beta)\{p(\alpha^2 + \beta) + p\alpha - 1\}. \end{aligned}$$

For $\lambda = \mu$, we consider ψ ($= A^2\beta - B^2$) as function of (c_0, c_1) with A, B and β defined by (15) and (17). Then, the singuraity is determined by the system of equations:

$$(21) \qquad \psi(c_0, c_1) = \psi_{c_0}(c_0, c_1) = \psi_{c_1}(c_0, c_1) = 0.$$

Here note that if there were any singularity (loop or cusp) on the curve segment (1), two values of the parameter t defined by the quadratic equation $(t + \alpha)^2 = \beta$ must belong to (0, 1), i.e.

$$(22) \quad 0 < t(= t^* - \alpha) = \pm\sqrt{\beta} - \alpha < 1 \text{ or } -1 < \alpha < 0, 0 \leq \beta < \alpha^2, \beta < (1 + \alpha)^2$$

where $\beta > 0$ and $\beta = 0$ correspond to a loop and cusp, respectively For (18) (i.e., (19)) and (21), two cases $A \neq 0$ and $A = 0$, will be considered separately.

(a) $A \neq 0$: First we note $\beta \neq 0$ from (19). If $\lambda \neq \mu$, from (19) we obtain the following relations according to the sign of B/A:

(23)
$$\text{(i)}\quad B = A\sqrt{\beta}, B_\alpha = A_\alpha\sqrt{\beta}, A = 2\sqrt{\beta}(B_\beta - \sqrt{\beta}A_\beta), B = 2\beta(B_\beta - \sqrt{\beta}A_\beta)$$
$$\text{(ii)}\quad B = -A\sqrt{\beta}, B_\alpha = -A_\alpha\sqrt{\beta}, A = -2\sqrt{\beta}(B_\beta + \sqrt{\beta}A_\beta), B = -2\beta(B_\beta + \sqrt{\beta}A_\beta)$$

For 23(i) ($\Leftrightarrow B/A > 0$), by (20) $B_\alpha = A_\alpha\sqrt{\beta}$ reduces to

$$\begin{aligned} &-(\lambda + \mu - 1 + p\lambda)(3\beta + 3\alpha^2 + 2\alpha) + \lambda \\ &+ (\mu - \lambda)\{p(2\alpha + 1)(\alpha^2 - \beta) + 2p\alpha(\beta + \alpha^2) + 2\alpha(p\alpha - 1)\} \\ =\ & \sqrt{\beta}\{-(\lambda + \mu - 1 + p\lambda)(6\alpha + 2) + p(\mu - \lambda)(6\alpha^2 - 2\beta + 2\alpha)\} \end{aligned} \tag{24}$$

From $A = 2\sqrt{\beta}(B_\beta - \sqrt{\beta}A_\beta)$, by (20)

$$\begin{aligned} &-(\lambda + \mu - 1 + p\lambda)(3\beta + 3\alpha^2 + 2\alpha) + \lambda + p(\mu - \lambda)(2\alpha^3 - 2\alpha\beta + \alpha^2 - \beta) \\ &= 2\sqrt{\beta}\{-(\lambda + \mu - 1 + p\lambda)(3\alpha + 1) + p(\mu - \lambda)(\alpha^2 - \beta) - p(\alpha^2 + \beta) - p\alpha + 1\} \\ &- 2\beta\{p(\mu - \lambda)(-2\alpha - 1)\}. \end{aligned} \tag{25}$$

On subtracting from (24) to (25), we obtain a cubic equation in $\sqrt{\beta}(= u)$:

$$(u - \alpha)\{p(u - \alpha)^2 - p(u - \alpha) - 1\} = 0 \tag{26}$$

where for 23(ii) ($\Leftrightarrow B/A < 0$), we have only to change the coefficient of $\sqrt{\beta}$ in (26). These cubic equations in $\sqrt{\beta}$ have no roots satisfying the required inequalities in (22), i.e., $\beta < \alpha^2$ $(-1/2 \leq \alpha < 0)$ or $\beta < (1 + \alpha)^2$ ($-1 < \alpha < -1/2$) where for $p = 0$, (26) reduces a linear equation having no root satisfying (22). When $\lambda = \mu$, (21) give

$$2AA_{c_i}\beta + A^2\beta_{c_i} = 2BB_{c_i}, i = 0, 1 \tag{27}$$

where by (17), $A_{c_1} - A_{c_0} = -\ p(2\alpha + 1), \beta_{c_1} - \beta_{c_0} = 0, B_{c_1} - B_{c_0} = -\ p(\alpha^2 + \beta + \alpha) + 1$. Note that $\beta \neq 0$. If $\beta = 0, A = 0$ would come from (27) since $\beta = 0$ gives $B = 0$ and $\beta_{c_1}(= \beta_{c_0}) = \{(2\lambda - 1)D\}/\{(2\lambda - 1)D - p(b_0 + b_1 + c_0 + c_1)\}^2 \neq 0$ where $\lambda = \mu$ and $\lambda + \mu \neq 1$. A difference of two equations (27) gives $p(2\alpha + 1)A\beta = B\{p(\alpha^2 + \beta) + p\alpha - 1\}$. Since $B = A\sqrt{\beta}$ for $A \neq 0$, we obtain the *quadraic* equation in $u(= \sqrt{\beta}$) being equal to the *quadratic* factor in the braces in (26) where for $B = -A\sqrt{\beta}$, the coefficient of $\sqrt{\beta}$ is to be changed. That is, in the case $A \neq 0$, the curve (16) does not have a singularity.

(b) $A = 0$: Then $A^2\beta = B^2$ gives $B = 0$ by (16). First we consider the case $\lambda \neq \mu$. By

(22), while determining the singularity, we can assume that $\alpha^2-\beta\neq 0$ and $\alpha\neq 0$. On making 20(i)$\times(\beta+\alpha^2)$ $-$20(ii)$\times(2\alpha)$,

$$\alpha^2-\beta=-\frac{\lambda+2\alpha(\lambda-\mu)}{\lambda+(1+p)\mu-1}(\neq 0) \tag{28}$$

Substitute (28) into 17(i) for $\alpha^2-\beta$ to obtain

$$2\alpha=\frac{(\lambda+\mu-1)(1-2\lambda)-p(4\lambda\mu-\lambda-\mu)-p^2\lambda\mu}{(\lambda+\mu-1)^2+p(4\lambda\mu-\lambda-\mu)+p^2\lambda\mu}(\neq 0) \tag{29}$$

where the denominator of (29) is positive for $(\lambda,\mu)\neq(d(p),d(p))$. Eliminate α from (28)-(29) to obtain

$$4\beta=\frac{4\kappa(\lambda,\mu)}{\{(\lambda+\mu-1)^2+p(4\lambda\mu-\lambda-\mu)+p^2\lambda\mu\}^2} \tag{30}$$

In addition, from (28)-(29), note

$$\begin{aligned}&\text{(i)}\quad \alpha^2-\beta=\frac{\lambda^2+\mu-\lambda\mu(3+p)}{(\lambda+\mu-1)^2+p(4\lambda\mu-\lambda-\mu)+p^2\lambda\mu}\\ &\text{(ii)}\quad (1+\alpha)^2-\beta=\frac{\mu^2+\lambda-\lambda\mu(3+p)}{(\lambda+\mu-1)^2+p(4\lambda\mu-\lambda-\mu)+p^2\lambda\mu}.\end{aligned} \tag{31}$$

Here we consider the case $\alpha\in(-1/2,0)$, the other case $\alpha\in(-1,-1/2)$ similarly treated. Then, (30)-(31) imply that the singularity occurs if $\kappa(\lambda,\mu)\geq 0$ and in addition if

$$\begin{aligned}&\text{(i)}\quad (\lambda+\mu-1)(\mu-\lambda)>0\ (\Leftrightarrow -1/2<\alpha)\\ &\text{(ii)}\quad \mu\{(2+4p+p^2)\lambda-(1+p)\}>-2\lambda^2+(3+p)\lambda-1\ (\Leftrightarrow \alpha<0)\\ &\text{(iii)}\quad \lambda^2>\{(3+p)\lambda-1\}\mu\ (\Leftrightarrow \alpha^2>\beta).\end{aligned} \tag{32}$$

Next we consider the case $\lambda=\mu, \text{i.e.}, \alpha=-1/2$. Then, from 17(i), $1/4-\beta=-\lambda/\{(2+p)\lambda-1\}$ from which (i) $\beta=0$ ($\Rightarrow$ a cusp) $\Leftrightarrow$ $\lambda=1/(6+p)$ and (ii) $0<\beta<1/4$ ($\Rightarrow$ a loop) $\Leftrightarrow$ $0<\lambda<1/(6+p)$. Note that $(\lambda,\mu)=(1/(6+p),1/(6+p))$ is on the first branch $k_1=0$.

Case 2 $(\lambda+\mu=1)$ In this case, it is easy to show that neither inflection nor singularity occurs.

Summarizing the above two cases, $\lambda+\mu\neq 1$ and $\lambda+\mu=1$, we obtain a lemma concerning the distribution of the singularity on the curve (1).

LEMMA 2 (singularity). If $(\lambda,\mu)\in L$ or C,then a loop or cusp occurs on the curve (1) where $L=\{(\lambda,\mu)|k_1(\lambda,\mu)>0,\lambda^2+\mu-(3+p)\lambda\mu>0,\mu^2+\lambda-(3+p)\lambda\mu>0\}$ and

$C = \{(\lambda, \mu) | k_1(\lambda, \mu) = 0\}$.

THEOREM. (Inflections and singularity). Assume that $\Delta z = \mu z_0' + \lambda z_1'$. Then, Figure 2 gives the distribution of inflections and singularity on the curve of the form (1) with respect to (λ, μ) where $N_i, 0 \leq i \leq 2$ represent the regions for which the curve has i-inflections and no singularity.

REMARK. A larger value of p for shape control might bring a larger interpolation error and the resulting curve would be an unacceptable "flat" interpolant while a choice of $p = 0$ yields a fourth order interpolation ([9]). Therefore, Theorem 1 suggests us to take $p = 0$ for $\rho\ (= \min(\lambda, \mu)) \geq 1/3$ and $p = 1/\rho - 3$ for $0 < \rho < 1/3$, respectively.

The following technique for eliminating the unwanted inflections and singularity on the curve (1) (as is usual with the case $p = 0$) has been also used ([7]). Suppose that the tangent directions are fixed at the two end points, and only the magnitudes of the tangents are allowed to be varied in scalar multiples η and κ ($\eta, \kappa > 0$), respectively. Since $C_0 \to \eta C_0, C_1 \to \kappa C_1, D \to \eta\kappa D$, Theorem gives the following corollary concerning the existence of the region (η, κ) for which the corresponding curve (1) is fair.

COROLLARY (Theorem [7, p. 54]). Assume that λ and $\mu > 0$. Then the curve of the form (1) is fair for $0 < \kappa \leq (3 + p)\lambda$ and $0 < \eta \leq (3 + p)\mu$.

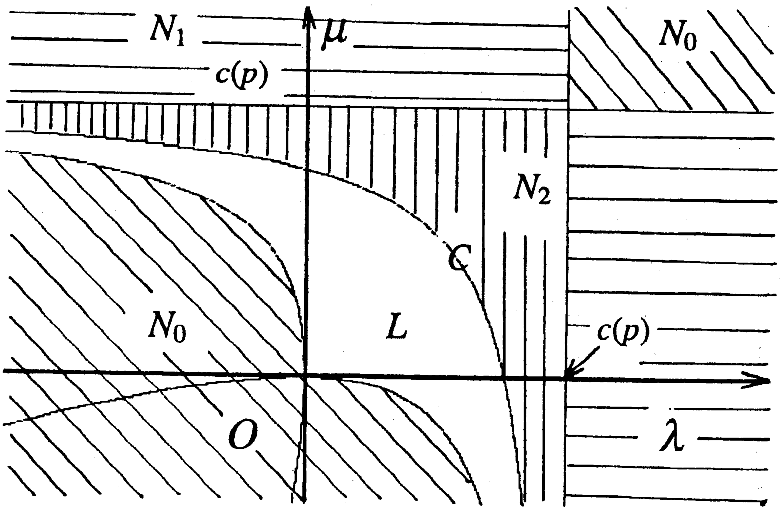

Fig. 2. Distribution of inflections and singularity.

Finally we consider the different shapes of the curves with different values of p; see Fig. 3 where the data are given by $(x_0^{(k)}, y_0^{(k)}) = (0, 1), (5, 6), (x_1^{(k)}, y_1^{(k)}) = (1, 1), (8, -4), k =$

0, 1. Since $(\lambda, \mu) = (2/17, 1/17), p = 4.21$ and $p = 14$ are an approximate value when a cusp occurs and the proposed value in Remark, respectively.

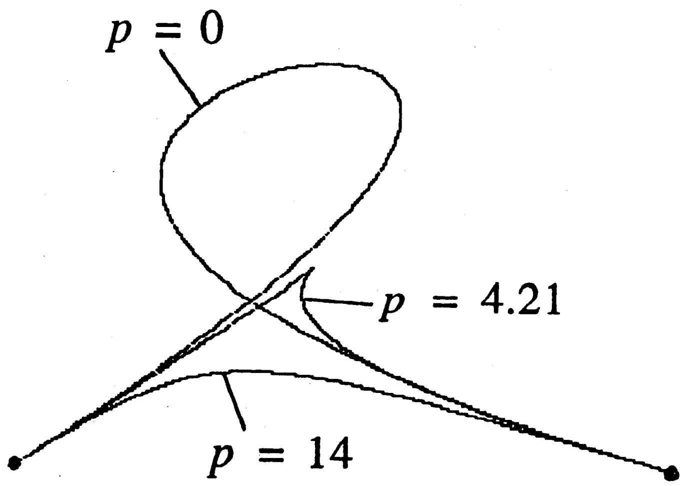

Fig. 3. Different shapes of curve segments with different values p.

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