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Numerical Calculation of Scattering State by Means of Higher Order Radiation Boundary Condition

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1 Introduction

The stationary scattering state of an acoustic wave with a time frequency k scattered by some bounded obstacle Ω in the Euclidean space $\mathbf{R^n}$ satisfies the following Helmholtz equation with the Sommerfeld radiation condition at infinity:

$$
(\mathbf{H})\left\{\begin{array}{rcll} -\Delta u-k^2u & = & 0 & \text{in } \Omega^c\equiv\mathrm{R}^n/\Omega,\ \ \Delta=\sum_{i=1}^n\partial^2/\partial x_i^2, \\ u(x) & = & -\varphi_0(x) & \text{on } \partial\Omega, \\ \sqrt{r}(\frac{\partial u}{\partial r}-iku) & \to & 0, & r=|x|\to\infty. \end{array}\right.
$$

Here Ω is a bounded obstacle with smooth boundary $\partial\Omega$ and $\varphi_0(x)$ is the boundary value of some incident wave.
In this paper, we study the case where the space dimension n is two. In section 2, modifying the Sommerfeld radiation condition, we find some higher order radiation condition. We construct in section 3 a sequence of approximate solutions to the scattering state for which the higher order radiation boundary condition is imposed on an artificial boundary. Our radiation condition is non-local and contains only bounded operators in its expression. Hence it is rather easy to calculate numerically. Applying the finite element method to these auxiliary problems in section 4, we propose an algorithm to calculate approximate numerical solutions to $(\mathbf{H})$. We give error estimates for respective approximations and show some numerical examples in section 5.

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2 Higher order radiation condition

Assume that Ω has a non-empty interior and includes the origin: $0 \in \Omega$. Choosing a number R_0 with the property: $\Omega \subset \mathbf{B}_{R_0} \equiv \{x \mid |x| \leq R_0\}$ and a smooth function $\chi_{R_0}(x)$ such that

$$\chi_{R_0}(x) = \begin{cases} 1 & (|x| \leq R_0) \\ 0 & (|x| \geq R_0 + 1), \end{cases} \tag{1}$$

we define a function $v(x) \equiv (1 - \chi_{R_0}(x))u(x)$. From $(\mathbf{H})$, the function v satisfies the equation:

$$(\mathbf{H_t}) \begin{cases} -\Delta v - k^2 v &= f, \quad \text{supp } f \subset B_{R_0+1} \backslash B_{R_0}, \\ \sqrt{r}(\frac{\partial v}{\partial r} - ikv) &\to 0, \quad r = |x| \to \infty, \end{cases} \tag{2}$$

where $f = (-\Delta - k^2)(1 - \chi_{R_0}(x))u(x)$. When $n = 2$, the zeroth order Hankel function of the first kind, $H_0^{(1)}(kr)$, satisfies the equation:

$$\begin{cases} -\Delta \left(\frac{i}{4} H_0^{(1)}(k|x - x'|)\right) - k^2 \left(\frac{i}{4} H_0^{(1)}(k|x - x'|)\right) &= \delta(x - x'), \\ \sqrt{r}(\frac{\partial H_0^{(1)}(k|x-x'|)}{\partial r} - ikH_0^{(1)}(k|x - x'|)) &\to 0, \quad r = |x| \to \infty. \end{cases} \tag{3}$$

This means that the function $\frac{i}{4} H_0^{(1)}(k|x - x'|)$, $x' \in B_{R_0+1}$, is Green's function of $(\mathbf{H_t})$. Noticing that

$$H_0^{(1)}(k|x - x'|) \simeq \frac{1}{\sqrt{r}} e^{ikr} \sum_{p=0}^{\infty} \frac{d_p(\tilde{x}, x')}{r^p}, \quad r \to \infty, \tag{4}$$

for fixed x' with $\tilde{x} \equiv \frac{x}{|x|} = e^{i\theta}$, we have the asymptotic expansion of $v(x)$ as r tends to infinity: $v(x) \simeq \frac{1}{\sqrt{r}} e^{ikr} \sum_{p=0}^{\infty} \frac{a_p(\theta)}{r^p}$, where $a_p(\theta) = \int_{B_{R_0+1}} \frac{i}{4} d_p(\tilde{x}, x') f(x') dx'$. We define $w(x) \equiv \sum_{p=0}^{\infty} a_p(\theta)/r^p$ and $\rho(r; \lambda) \equiv \sqrt{r} e^{-ikr}$, then $w(x) = \rho(r; \lambda) v(x)$. Multiplying $\rho(r; \lambda)$ to (2), we get $-\rho\Delta(\frac{1}{\sqrt{r}} e^{ikr} w) - \lambda w = \rho f$. Finally the relation (2) is rewritten as

$$\begin{cases} -\Delta w + \left(-\frac{1}{4r^2} + (\frac{1}{r} - 2ik)\frac{\partial}{\partial r}\right) w &= \rho f, \\ \frac{\partial w}{\partial r} \to 0, \quad r = |x| &\to \infty. \end{cases} \tag{5}$$

In particular, when r is sufficiently large, we have $\rho f = 0$. From the definition of w, we have the asymptotic relation:

$$\sum_{p=0}^{N} \{-\frac{p(p+1)}{r^{p+2}} a_p(\theta) + 2ikp \frac{a_p(\theta)}{r^{p+1}} + \frac{1}{4r^{p+2}} a_p(\theta) - \frac{1}{r^{p+2}} \Lambda_\theta a_p(\theta)\} = O(r^{-N-2}).$$

This leads to the recursion formula:

$$a_p(\theta) = \frac{1}{2ikp} \{\Lambda_\theta + p(p-1) - \frac{1}{4}\} a_{p-1}(\theta), \quad p = 1, 2, 3, ..., \tag{6}$$

with $\Lambda_\theta \equiv \frac{\partial^2}{\partial\theta^2}$. We put $B(0) \equiv 1$ and define the operators $L(p)$ and $B(p)$, $p = 1, 2, \ldots$, as $L(p) \equiv \frac{1}{2ikp}\{\Lambda_\theta + p(p-1) - \frac{1}{4}\}$ and $B(p) \equiv L(p)L(p-1)\ldots L(1)$. Then we have the following expression: $a_p(\theta) = B(p)a_0(\theta), \quad p = 0, 1, 2, \ldots$. Accordingly, the solution $u(r,\theta)$ of $(\mathbf{H})$ has an asymptotic expansion as r tends to infinity:

$$u(r,\theta) \simeq \frac{1}{\sqrt{r}} e^{ikr} (\sum_{p=0}^{N} \frac{B(p)}{r^p}) a_0(\theta) + O(r^{-N-1-1/2}), \tag{7}$$

and we have the asymptotic expansions for $\partial u / \partial r$:

$$\frac{\partial u}{\partial r} \simeq iku - \frac{1}{2r}u + \frac{1}{\sqrt{r}} e^{ikr} (\sum_{p=1}^{N} \frac{-p}{r^{p+1}} B(p)) a_0(\theta) + O(r^{-N-2-1/2}). \tag{8}$$

In particular, we have, for $N = 1$,

$$\frac{\partial u}{\partial r} - iku + \frac{1}{2r}u + \frac{1}{\sqrt{r}} e^{ikr} \frac{1}{r^2} B(1) a_0(\theta) = O(r^{-7/2}) \tag{9}$$

and

$$u(r,\theta) = \frac{1}{\sqrt{r}} e^{ikr} (1 + \frac{1}{r} B(1)) a_0(\theta) + O(r^{-5/2}). \tag{10}$$

Obviously, we have the estimate:

$$||(1 + \frac{1}{r}B(1))^{-1}||_{L^2(S^1) \to H^2(S^1)} < \infty. \tag{11}$$

Hence, from (10) we have the following asymptotic relation in $L^2(S^1)$:

$$||a_0 - \sqrt{r} e^{-ikr} (1 + \frac{1}{r}B(1))^{-1} u||_{H^2(S^1)} = O(r^{-2}). \tag{12}$$

Putting (12) into (9), we get the estimate:

$$||(\frac{\partial}{\partial r} - ik + \frac{1}{2r})u + \frac{1}{r^2}B(1)(1 + \frac{1}{r}B(1))^{-1}u||_{L^2(S^1)} = O(r^{-7/2}). \tag{13}$$

We define an operator T_r as $T_r \equiv \frac{1}{r}B(1)(1 + \frac{1}{r}B(1))^{-1}$. Then we have the following lemma and theorem:

Lemma2.1 The operator T_r is bounded in $L^2(S^1)$ with norm $||T_r||_{L^2(S^1)} \leq 1$.

Theorem2.2 There exists one and only one solution of the Helmholtz equation $(\mathbf{H})$ which satisfies the followings:

$$(\mathbf{K}) \begin{cases} -\Delta u(x) - k^2 u(x) &= 0 & \text{in } \Omega^c, \\ u(x) &= -\varphi_0 & \text{on } \partial\Omega, \\ ||\frac{\partial u}{\partial r} - iku + \frac{1}{2r}u + \frac{1}{r}T_r u||_{L^2(S^1)} &= O(r^{-7/2}), & r = |x| \to \infty. \end{cases} \tag{14}$$

3 Analytical approximation problem

We put $R \gg 1$, and let u_R be the solution of the boundary value problem:

$$(\mathbf{K_R})\begin{cases} -\Delta u_R - k^2 u_R &= 0 \quad \text{in } \Omega_R^c \equiv \Omega^c \cap B_R, \\ (\varphi_0 + u_R)|_{\partial\Omega} &= 0, \\ D_\lambda u_R &= 0 \qquad S_R = \partial B_R. \end{cases} \tag{15}$$

Here $D_\lambda \equiv \partial/\partial r - ik + 1/(2R) + (1/R)T_R$. Using the same function $\rho(r;\lambda) = \sqrt{r}e^{-ikr}$ as in section 2, we put, for $R_0 < R$, $v_R(x) \equiv \rho(r;\lambda)(u_R(x) + \chi_{R_0}(x)\varphi_0(x))$. Then $v_R(x)$ satisfies the following equation

$$(\mathbf{K_R'})\begin{cases} -\Delta v_R + \left(-\frac{1}{4r^2} + (\frac{1}{r} - 2ik)\frac{\partial}{\partial r}\right) v_R &= g \quad \text{in } \Omega_R^c, \\ v_R|_{\partial\Omega} &= 0, \\ (\frac{\partial v_R}{\partial r} + \frac{1}{R}T_R v_R)|_{S_R} &= 0. \end{cases} \tag{16}$$

with $g = -\Delta(\rho(r;\lambda)\chi_{R_0}(x)\varphi_0(x)) + (-1/4r^2 + (1/r - 2ik)\partial/\partial r)(\rho(r;\lambda)\chi_{R_0}(x)\varphi_0(x))$. Since the operator T_R is bounded, we can define bounded operator $e^{(r/R)T_R}$, $r \in \mathrm{R}$. Put $\varpi_R(r,\theta) \equiv e^{(r/R)T_R}v_R(r,\theta)$. Then we have

$$(\mathbf{K_R''})\begin{cases} -\Delta\varpi_R + \{-2T_R + (\frac{1}{r} - 2ik)\}\frac{\partial\varpi_R}{\partial r} - \{T_R^2 + \frac{1}{4r^2} + 2ikT_R\}\varpi_R &= f_R \quad \text{in } \Omega_R^c, \\ \varpi_R|_{\partial\Omega} &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial\varpi_R}{\partial r}|_{S_R} &= 0 \quad \text{on } S_R, \end{cases} \tag{17}$$

with $f_R \equiv e^{(r/R)T_R}g$. We introduce two operators H_R and Q_R as follows: $\mathcal{D}(H_R) \equiv \{u|\ u \in H^2(\Omega_R^c), u|_{\partial\Omega} = 0 \ \ \textit{and} \ \ \frac{\partial u}{\partial r}|_{S_R} = 0 \text{ on } S_R\}$, $H_R u = -\Delta u$, $\mathcal{D}(Q_R) = \mathcal{D}(H_R)$, $Q_R u = -2T_R\frac{\partial u}{\partial r} - T_R^2 u - 2ikT_R u - 2ik\frac{\partial u}{\partial r} - \frac{1}{4r^2}u + \frac{1}{r}\frac{\partial u}{\partial r} - u$. Then the equation (17) becomes an operator theoretical equation:

$$(H_R + 1)\varpi_R + Q_R\varpi_R = f_R. \tag{18}$$

Putting $w_R \equiv (H_R + 1)\varpi_R$, we have the equation in $L^2(\Omega_R^c)$ for w_R:

$$w_R + Q_R(H_R + 1)^{-1}w_R = f_R. \tag{19}$$

<u>**Theorem 3.1**</u> The equation (18) has a unique solution in $L^2(\Omega_R^c)$ given by

$$\varpi_R = (H_R + 1)^{-1}(1 + Q_R(H_R + 1)^{-1})^{-1}f_R. \tag{20}$$

<u>**Proof**</u>. By Rellich's compactness theorem, the operator Q_R is relatively compact with respect to $H_R + 1$ and hence $Q_R(H_R + 1)^{-1}$ is compact. In order to prove the existence of the solution of the equation (19) , we use the Fredholm alternative theorem. Hence we have

only to show that the solution w_R of equation (19) is zero when the right hand of (19) f_R is zero. Let w_R is a solution of (19) with $f_R = 0$. Put $u_R = r^{-\frac{1}{2}} e^{ikr}\{e^{-(r/R)T_R}(H_R+1)^{-1}w_R\}$. Then from (19), we have (15) and

$$\int_{\Omega_R^c} ((\Delta u_R)\overline{u_R} - u_R\overline{\Delta u_R})dx = 0. \tag{21}$$

Using Green's formula, we obtain

$$\int_{S_R} (\overline{u_R}\frac{\partial u_R}{\partial r} - u_R\overline{\frac{\partial u_R}{\partial r}})dx = 0, \tag{22}$$

and, from the boundary condition $D_\lambda u_R = 0$,

$$0 = \int_{S_R} \{2ik|u_R|^2 + \frac{1}{R}(u_R\overline{T_R u_R} - (T_R u_R)\overline{u_R})\}dS_R.$$

Hence, using Lemma 2.1, we have

$$\begin{aligned} ||u_R||^2_{L^2(S_R)} &= \int_{S_R} |u_R|^2 dS_R \\ &= \tfrac{1}{2ikR}\int_{S_R} \{\overline{(T_R u_R)}u_R - (T_R u_R)\overline{u_R}\}dS_R \\ &\leq \tfrac{1}{kR}||u_R||^2_{L^2(S_R)}. \end{aligned}$$

Then $u_R \equiv 0$ on S_R when $kR > 1$, and we also have $\frac{\partial u_R}{\partial r} = (ik - \frac{1}{2R} - \frac{1}{R}T_R)u_R = 0$ in S_R. Finally from the unique continuation property, we have $u_R \equiv 0$ in Ω_R^c. This proves the uniqueness and hence the existence of the solution of the equation (19).

∎

Next, we estimate the difference between u and u_R. Putting $e_R \equiv u - u_R$, we have the equation for $e_R(x)$:

$$\left\{ \begin{array}{rcll} -\Delta e_R - k^2 e_R &=& 0 & \text{in } \Omega_R^c, \\ D_\lambda e_R &=& D_\lambda u & \text{on } S_R. \end{array} \right. \tag{23}$$

In the same manner as in the proof of Theorem 3.1, we obtain

$$0 = \int_{S_R} \{(D_\lambda e_R)\overline{e_R} - e_R\overline{D_\lambda e_R}\}dS_R - 2ik\int_{S_R} |e_R|^2 dS_R - \frac{1}{R}\int_{S_R} (\overline{(T_R e_R)}e_R - (T_R e_R)\overline{e_R})dS_R.$$

Hence we have the estimate:

$$\begin{aligned} ||e_R||^2_{L^2(S_R)} = \int_{S_R} |e_R|^2 dS_R &\leq \tfrac{1}{2kR}\left|\int_{S_R} \{\overline{(T_R e_R)}e_R - (T_R e_R)\overline{e_R}\}dS_R\right| \\ &+ \tfrac{1}{2k}\left|\int_{S_R} \{(D_\lambda u)\overline{e_R} - e_R\overline{D_\lambda u}\}dS_R\right| \\ &\leq \tfrac{1}{kR}||T_R e_R||_{L^2(S_R)}||e_R||_{L^2(S_R)} + \tfrac{1}{k}||D_\lambda u||_{L^2(S_R)}||e_R||_{L^2(S_R)} \\ &\leq \tfrac{1}{kR}||e_R||^2_{L^2(S_R)} + \tfrac{1}{k}||D_\lambda u||_{L^2(S_R)}||e_R||_{L^2(S_R)}. \end{aligned}$$

Combining the estimate for $||D_\lambda u||_{L^2(S_R)}$ in (**K**), in section 2, we have

$$||e_R||_{L^2(S_R)} \le \frac{1}{(1-\frac{1}{kR})k}||D_\lambda u_R||_{L^2(S_R)} \le O(R^{-7/2}).$$

<u>**Theorem 3.2**</u> When $R \gg 1$, with some constant C, the estimates

$$\int_{S_R} |e_R|^2 dS_R \le CR^{-7} \tag{24}$$

and, for a fixed R_0,

$$\sup_{x\in\Omega_R^c} |e_R(x)| \le CR^{-3} \tag{25}$$

hold.

∎

4 Discrete approximation

For large enough R, we consider the weak formulation of the boundary value problem ($\mathbf{K'_R}$):

$$a_R(u,v) + b_R(u,v) = (g,v), \text{ for all } v \in \mathcal{V}_R. \tag{26}$$

Here, $a_R(u,v) = \int_{\Omega_R^c}\left(\frac{\partial u}{\partial r}\frac{\partial \overline{v}}{\partial r} + \frac{1}{r^2}\frac{\partial u}{\partial \theta}\frac{\partial \overline{v}}{\partial \theta} + u\overline{v}\right) rdrd\theta$, and $b_R(u,v) = b_R^1(u,v) + b_R^2(u,v)$, with $b_R^1(u,v) \equiv \int_{\Omega_R^c}\left(\left(\left(\frac{1}{r} - 2ik\right)\frac{\partial}{\partial r} - 1 - \frac{1}{4r^2}\right)u\right)\overline{v}rdrd\theta$, $b_R^2(u,v) \equiv \int_{S_R}\frac{1}{R}T_R u\overline{v}dS_R$, and $u \in \mathcal{V}_R \equiv H_D^1(\Omega_R^c) = \{u \mid u \in H^1(\Omega_R^c),\ u|_{\partial\Omega} = 0\}$. We consider the finite element method for this equation in the same way as in [4]. But we have to treat the term b_R^2 appropriately. For this purpose we approximate $b_R^2(u,v)$ in the following way. Consider the problem:

$$(1 + \frac{1}{2ikR}(\frac{\partial^2}{\partial\theta^2} - \frac{1}{4}))w_i(\theta) = \varphi_i(\theta,R) \tag{27}$$

$$w_i(0) = w_i(2\pi), \quad \frac{dw_i}{d\theta}|_{\theta=0} = \frac{dw_i}{d\theta}|_{\theta=2\pi}. \tag{28}$$

The weak formulation of the above equation is to find w^i such that

$$\int_0^{2\pi}(\frac{i}{2kR}\frac{\partial w_i}{\partial\theta}\overline{\frac{\partial v}{\partial\theta}} + (1+\frac{i}{8kR})w_i\overline{v}) - \varphi_i(\theta,R)\overline{v})d\theta = 0, \tag{29}$$

$$\text{for all } v \in H_\sharp^1(S_R).$$

Here, $H^1_\sharp(S_R) = \{v \mid v \in H^1(S_R),\ v(0) = v(2\pi),\ \frac{dv}{d\theta}|_{\theta=0} = \frac{dv}{d\theta}|_{\theta=2\pi}\}$. The approximation of $w_i(\theta)$ is calculated by the following set of linear equations:

$$\sum_{\ell=1}^{n(h)} \zeta_{hi}^{\ell} \int_0^{2\pi} \left(\frac{i}{2kR}\frac{\partial\psi_\ell(\theta)}{\partial\theta}\overline{\frac{\partial\psi_k(\theta)}{\partial\theta}} + (1+\frac{i}{8kR})\psi_\ell(\theta)\overline{\psi_k(\theta)} - \varphi_i(\theta,R)\overline{\psi_k(\theta)}\right)d\theta = 0 \qquad (30)$$

$$\text{for all } \psi_k(\theta) \in \mathcal{W}_h \subset H^1_\sharp(S_R),\quad k = 1,2,...,n(h).$$

Here $\mathcal{W}_h$ is a $n(h)$-dimensional subspace of $H^1_\sharp(S_R)$, whose basis functions are piecewise linear on S_R.

5 Numerical results

We set an another boundary value problem (cf [4]):

$$(\mathbf{G_R})\begin{cases} -\Delta u_R - k^2 u_R &= 0 \quad \text{in } \Omega_R^c \equiv \Omega^c \cap B_R, \\ \varphi_0 + u_R &= 0 \quad \text{on } \partial\Omega, \\ D_\lambda \equiv \frac{\partial u_R}{\partial r} - iku_R + \frac{1}{2R}u_R &= 0 \quad \text{on } S_R = \partial B_R. \end{cases} \qquad (31)$$

We express the boundary condition in (31) as A_1, and the one in $(\mathbf{K_R})$ as A_2. The weak formulation of $(\mathbf{G_R})$ is the same as (26) with $b_R^2(u,v) \equiv 0$. We have made two algorithms by FEM to calculate the solutions of the two boundary value problems and compare the two numerical results calculated by these algorithms. We consider 2D star-shaped obstacles. Then its boundary can be expressed by a function $f(\theta)$ of θ: $\partial\Omega = \{(x,y)|\ x = f(\theta)\sin(\theta),\ y = f(\theta)\cos(\theta),\ 0 \le \theta \le 2\pi\}$.

Example 1 When $f(\theta) = 1$ and $\varphi_0(r,\theta) = H_n^{(1)}(kr)\cos(n\theta)$, we put $u_R^i, i = 1,2$ to be the solutions of the boundary value problem with boundary condition A_i, and $u_{R,h}^i, i = 1,2$ to be the numerical solutions with boundary condition A_i. Suppose the following order relations:

$$||u_R^i - u_{R,h}^i||_{L^2(B_R)} \simeq C_{0,i}h^{\gamma_i}, \qquad (32)$$

and

$$\beta_i(h) = ||u_{R,\frac{h}{2}}^i - u_{R,h}^i||_{L^2(B_R)} \simeq C_i h^{\gamma_i}, \qquad (33)$$

then we have

$$\gamma_i = \frac{\log(\beta(2h)) - \log(\beta(h))}{\log(2)}. \qquad (34)$$

We get numerically the order 2γ as follows:

Table 1 Convergence order 2γ

A_2: High order boundary condition					
k \ n	1	2	3	4	5
1.	3.978	3.894	3.816	3.814	3.859
2.	3.988	3.892	3.554	3.366	3.418
3.	3.990	4.048	3.682	2.858	2.652
4.	3.982	4.011	4.573	3.416	1.819
5.	3.975	3.845	3.689	5.329	3.167
6.	3.976	3.935	3.539	3.071	4.749
7.	3.973	4.034	4.079	3.314	2.685
8.	3.960	3.912	4.084	4.246	3.340
9.	3.956	3.839	3.727	3.736	4.014
10.	3.962	3.928	3.761	3.618	3.333

Example 2 Putting $\Delta = \{(r,\theta) \mid 1 \le r \le 2\}$ and u be the solution of $(\mathbf{H})$, u_R^i the solutions of $(\mathbf{K}'_\mathbf{R})$ and $(\mathbf{G}_\mathbf{R})$ $i = 1,2$, and $u_{R,h}^i$ the solutions of the weak fomulation of $(\mathbf{K}_\mathbf{R})$ and $(\mathbf{G}_\mathbf{R})$. we get the estimate:

$$\begin{aligned} ||u - u_{R,h}^i||_{L^2_{(\Delta)}} &= ||u - u_R^i + u_R^i - u_{R,h}^i||_{L^2_{(\Delta)}} \\ &\le ||u - u_R^i||_{L^2_{(\Delta)}} + ||u_R^i - u_{R,h}^i||_{L^2_{(\Delta)}} \\ &\le \sup_{x\in\Delta} |u - u_R^i|(m(\Delta))^{\frac{1}{2}} + C_2(u_R^i)h^2 \\ &\sim C_1^i R^{-(i+1)} + C_2(u_R^i)h^2. \end{aligned} \tag{35}$$

We try to confirm it by numerical calculation. The following table shows $e_R^i = ||u - u_{R,h}^i||_{L^2_{(\Delta)}}$ when $k = 5, n = 1$ and $\varphi_0 = H_1^{(1)}(5r)\cos(\theta)$.

Table 2 Error e_R^i

A_1: Lower order boundary condition								
$(1/h)\backslash R$	2.	3.	4.	5.	10.	20.	30.	40.
8	3.45	1.55	1.38	1.49	0.917	0.868	0.921	0.903
16	3.39	0.648	0.33	0.143	0.052	0.0709	0.0608	0.0648
32	3.08	0.548	0.238	0.0618	0.0046	0.0044	0.00395	0.00442
64	2.97	0.573	0.197					

Table 3 Error e_R^i

A_2: High order boundary condition								
$(1/h)\backslash$ R	2.	3.	4.	5.	10.	20.	30.	40.
8	0.72	0.96	1.14	1.30	0.92	0.87	0.92	0.903
16	0.20	0.17	0.17	0.098	0.053	0.0703	0.0609	0.0647
32	0.106	0.098	0.094	0.0248	0.0038	0.00435	0.00395	0.0044
64	0.086	0.109	0.0695					

From the estimate (35), we get, when $R \to \infty$,

$$||u - u^i_{R,h}||_{L^2_{(\Delta)}} \to C_2(u^i_{R,h})h^2. \tag{36}$$

The numerical results are consistent with this assumption with $C_2(u^i_{R,h}) \sim C$:

$$||u - u^i_{R,h}||_{L^2_{(\Delta)}} \simeq Ch^\gamma, \quad R > 10. \tag{37}$$

Table 4 Convergence order 2γ

$2\gamma \setminus$ R	40	30	20	10
A_1	3.8744948	3.9444391	4.009577997	3.49959478
A_2	3.8770147	3.9456009	4.013786059	3.6851072

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